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Convolution Inequalities in Lorentz Spaces

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Abstract In this paper we study boundedness of the convolution operator in different Lorentz spaces. We obtain the limit case of the Young-O'Neil inequality in the classical Lorentz spaces. We also investigate the convolution operator in the weighted Lorentz spaces.

Keywords Young-O'Neil inequality · Lorentz spaces · Convolution

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1 Introduction

The Young convolution inequality of the form

$$\|K * f\|_p \le \|f\|_p \, \|K\|_1$$

and in a more general form

$$\|K * f\|_q \le \|f\|_p \|K\|_r, \qquad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}, \quad 1 \le p, r, q \le \infty \tag{1.1}$$

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plays a very important role both in Harmonic Analysis and PDE (see, e.g., [1], [5, Chap. 4, Sects. 2, 4], [8], [21, Chap. V, Sect. 1]). A sharp constant in (1.1) was found in [3]. An analogue of Young's inequality (1.1) in weighted Lebesgue spaces $L_p(\omega)$ and Wiener amalgam spaces $W(L_p, L_q(\omega))$ was investigated in [9, 12] (see references therein). In [16], the authors studied sufficient and necessary conditions on ω to obtain the following convolution inequality in $L_p(\omega)$

$$\|K * f\|_{L_p(\omega)} \le c\,\|f\|_{L_p(\omega)}\|K\|_{L_p(\omega)}, \quad 1 \le p \le \infty.$$

Moreover, O'Neil [18], Hunt [13], Yap [22], and Blozinski [6] studied the boundedness of the convolution operator A given by

$$Af(y) = \int_D K(y-x)f(x)dx \tag{1.2}$$

for functions K and f belonging to suitable Lorentz spaces and where the domain D is $\mathbb{R}^n$.

In particular, the following Young-O'Neil inequality was obtained: for $1 < p, q, r < \infty$, $0 < h_1, h_2, h_3 \le \infty$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, and $\frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}$, one has

$$\|K * f\|_{L_{q,h_1}(\Omega)} \le c\,\|f\|_{L_{p,h_2}(D)}\|K\|_{L_{r,h_3}(\Omega - D)}, \tag{1.3}$$

where $\Omega - D = \{x - y : x \in \Omega, y \in D\}$. We note that inequality (1.3) unlike (1.1) gives the Hardy-Littlewood fractional integration theorem, which corresponds to the model case in which $K(x) = |x|^{-1/r}$.

The limiting case of inequality (1.3), that is, when $1 = \frac{1}{p} + \frac{1}{r}$, $1 < p < \infty$, $\frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3} < 1$, and $1 \le h_2, h_3 \le \infty$ was investigated in [8]:

$$\|K * f\|_{BW_{h_1}} \le c\,\|f\|_{L_{p,h_2}(\mathbb{R}^n)}\big(\|K\|_{L_{r,h_3}(\mathbb{R}^n)} + \|K\|_{L_1(\mathbb{R}^n)}\big), \tag{1.4}$$

where $c = c(p, h_2, h_3)$ and

$$\|\psi\|_{BW_{h_1}} = \left(\int_0^1 \left(\frac{\psi^*(t)}{1 + |\log t|}\right)^{h_1} \frac{dt}{t}\right)^{1/h_1}.$$

Young-O'Neil inequality was also studied for weighted Lorentz spaces. Kerman [15] proved the analogue of (1.3) for power weights. It is worth mentioning that in the case of non-homogeneous measures, operator (1.2) does not satisfy all requirements from [18] and needs thorough investigation.

In this paper we continue investigating the Young-type inequalities in different Lorentz spaces. The outline of the paper is as follows. In Sect. 2 we study the boundedness of the operator A from $L_{p,h_2}(\Omega)$ into $L_{p,h_1}(\Omega)$, i.e., the limit case of the Young-O'Neil inequality ($p = q$ and $r = 1$). It is known (see [7, Theorem 2]) that if $\Omega = \mathbb{R}^n$, $h_1 < h_2 \le \infty$ and $K \ge 0$, then

$$A : L_{p,h_2}(\mathbb{R}^n) \longrightarrow L_{p,h_1}(\mathbb{R}^n)$$

implies $A \equiv 0$ and in turn $K \overset{\text{a.e.}}{=} 0$. In contrast we show that in the case when Ω has finite measure, the same problem has a nontrivial solution: for $1 \le p = q < \infty$ we prove (see Theorem 2.1)

$$\|(K * f)^{**}\|_{p,h_1} \le c\,\|f^{**}\|_{p,h_2}\|K^{**}\|_{1,h_3}, \quad \frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}, \tag{1.5}$$

where

$$\psi^{**}(x) = \frac{1}{x}\int_0^x \psi^*(t)dt \tag{1.6}$$

and ψ^* is the decreasing rearrangement of ψ. Moreover, we show that the factor $\|K^{**}\|_{L_{1,h_3}}$ in (1.5) could not be changed to either $\|K\|_{L_{1,h_3}}$, or $\|K^{**}\|_{L_{1,h}}$ for any $h > h_3$. Note that $\max(\|K\|_{L_{1,h_3}}, \|K^{**}\|_{L_{1,h}}) \le c\|K^{**}\|_{1,h_3}$.

We mention that the key point in the proof of the classical estimate (1.3) is the following convolution inequality (see [18]):

$$(K * f)^{**}(t) \le tK^{**}(t)f^{**}(t) + \int_t^\infty K^*(u)f^*(u)\,du.$$

Our main idea is to use a different version of the above convolution inequality, namely with $K^{**} - K^*$ on the right-hand side in place of K^* (see Lemma 2.2).

The case $p = q = \infty$ is studied separately in Sect. 3; here the proper setting is given by $L_{\infty,q}$-spaces introduced in [4] and [2] and we obtain (see Theorem 3.1)

$$\|K * f\|_{L_{\infty,h_1}} \le 2\,\|f\|_{L_{\infty,h_2}}\|K^{**}\|_{L_{1,h_3}}.$$

Note that since $L_{\infty,h_1} \subset BW_{h_1}$ with $h_1 > 1$ (see [2, Th. 3.1]), this gives the limiting case of (1.4):

$$\|K * f\|_{BW_{h_1}} \le c\,\|f\|_{L_{\infty,h_2}}\|K^{**}\|_{L_{1,h_3}}.$$

We show (see Theorem 3.2) that the constant c can be taken as $2h_1'$.

Section 4 provides a Young-O'Neil-type inequality in Lorentz spaces with weight function $\omega(t)$, namely $\Lambda^q(\omega)$ and $\Gamma^q(\omega)$, where

$$\Lambda^q(\omega) = \left\{\|f\|_{\Lambda^q(\omega)} = \left(\int_0^\infty \left(f^*(t)\right)^q \omega(t)\,dt\right)^{1/q} < \infty\right\}$$

and $\Gamma^q(\omega)$ is defined similarly with f^{**} in place of f^* (see, e.g., [19]). Finally we mention that some results of this paper, namely Theorems 2.1 and 3.1, were announced in the note [17].

By C, C_i, c we will denote positive constants that may be different in different contexts and depend only on parameters p_i, q_i, and h_i. We will write $F \asymp G$ if $F \le C_1 G$ and $G \le C_2 F$.

2 Convolution in the Lorentz Space of Periodic Functions: the Case of $1 \le p < \infty$

Let $L_{p,q}[0,1]$ be the Lorentz space of all 1-periodic functions with the quasi-norm given by

$$\|f\|_{p,q} = \|f\|_{L_{p,q}[0,1]} = \left(\int_0^1 (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

It is known [5, p. 219] that for $1 < p < \infty$ and $1 \le q \le \infty$ we have

$$\|f\|_{L_{p,q}} \le \|f^{**}\|_{p,q} := \left(\int_0^1 (t^{\frac{1}{p}} f^{**}(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} \le p'\|f\|_{L_{p,q}}, \tag{2.1}$$

where f^{**} is defined by (1.6) and can be written as [5, p. 53]

$$f^{**}(t) = \sup_{\substack{|e|=t \\ e\subset[0,1]}} \frac{1}{|e|}\int_e |f(x)|dx. \tag{2.2}$$

Here $p' = \frac{p}{p-1}$ and $|e| := \mu e$.

Note that for $p = 1$, the norms $\|f\|_{L_{p,q}}$ and $\|f^{**}\|_{p,q}$ are not equivalent. For a fixed $f \in L_1$ the functional $\|f^{**}\|_{1,q}$ is non-decreasing as a function of $1/q \in [0, +\infty)$.

We will need the following lemmas.

Lemma 2.1 *Let $f(t)$ be a non-increasing non-negative function on $(0,\infty)$, $0 \le \phi(t) \le 1$ and $\int_0^\infty \phi(t)dt = d < \infty$. Then*

$$\int_0^d f(t)dt \ge \int_0^\infty f(t)\phi(t)dt. \tag{2.3}$$

Proof Indeed, inequality (2.3) follows from Hardy's lemma (see, e.g., [5, Prop. 3.6, p. 56]): $\int_0^x h_1\,dt \ge \int_0^x h_2\,dt$ implies $\int_0^\infty h_1 f\,dt \ge \int_0^\infty h_2 f\,dt$ for any non-increasing nonnegative function f on $(0,\infty)$. □

Lemma 2.2 *Let f, g, and K be measurable 1-periodic functions and g is non-negative. Then*

$$\int_0^1 g(t)(K*f)^{**}(t)dt \le \|g\|_{L_1}\|f\|_{L_1}\|K\|_{L_1} + \int_0^1 f^{**}(t)\big(K^{**}(t) - K^*(t)\big)\left(\int_0^t g(s)ds\right)dt. \tag{2.4}$$

Proof Let us assume the right-hand side of (2.4) is finite. From (2.2) and the Hardy-Littlewood inequality [5, p. 44], we write

$$\begin{aligned}
&\int_0^1 g(s)(K*f)^{**}(s)ds \\
&\quad\le \int_0^1 g(s) \sup_{\substack{|e|=s\\ e\subset[0,1]}} \int_0^1 |f(x)|\frac{1}{|e|}\int_e |K(y-x)|\,dy\,dx\,ds \\
&\quad\le \int_0^1 g(s) \sup_{\substack{|e|=s\\ e\subset[0,1]}} \int_0^1 f^*(t)\left(\frac{1}{|e|}\int_e |K(y-\cdot)|\,dy\right)^{**}(t)\,dt\,ds \\
&\quad= \int_0^1 g(s) \sup_{\substack{|e|=s\\ e\subset[0,1]}} \int_0^1 f^*(t) \sup_{\substack{|\omega|=t\\ \omega\subset[0,1]}} \frac{1}{|e|}\frac{1}{|\omega|}\int_e\int_\omega |K(y-x)|\,dx\,dy\,dt\,ds \\
&\quad\le \int_0^1 g(s) \int_0^1 f^*(t)\left(\sup_{\substack{|e|=s\\ e\subset[0,1]}} \sup_{\substack{|\omega|=t\\ \omega\subset[0,1]}} \frac{1}{|e|}\frac{1}{|\omega|}\int_e\int_\omega |K(y-x)|\,dxdy\right)dt\,ds.
\end{aligned}$$

We consider

$$\begin{aligned}
\Phi(s,t) &= \sup_{\substack{|e|=s\\ |\omega|=t}} \frac{1}{|e|}\frac{1}{|\omega|}\int_\omega\int_e |K(y-x)|\,dy\,dx \\
&= \sup_{\substack{|e|=s\\ |\omega|=t}} \frac{1}{|e|}\frac{1}{|w|}\int_0^1 \chi_e(x)\int_0^1 \chi_\omega(y)\,|K(y-x)|\,dy\,dx \\
&= \sup_{\substack{|e|=s\\ |w|=t}} \frac{1}{|e|}\frac{1}{|w|}\int_0^1 |K(x)|\,|e\cap(w+x)|\,dx \\
&\le \sup_{\substack{|e|=s\\ |w|=t}} \frac{1}{|e|}\frac{1}{|w|}\int_0^1 K^*(\xi)\phi(\xi)d\xi,
\end{aligned}$$

where $\phi(\xi)$ is the decreasing rearrangement of the function $|e\cap(\omega+x)|$.

Then $\phi(\xi)$ satisfies $\phi(s)\le \min(|e|,|w|)$ and

$$\int_0^1 \phi(\xi)d\xi = |e||w|.$$

We assume that $s=|e|<|w|=t$, then the function $\phi_0(\xi)=\phi(\xi)/|e|$ satisfies conditions of Lemma 2.1 with $d=|w|$. Then for $s<t$ we have

$$\begin{aligned}
\Phi(s,t) &\le \sup_{\substack{|e|=s,e\subset[0,1]\\ |w|=t,w\subset[0,1]}} \frac{1}{|w|}\int_0^1 K^*(\xi)\phi_0(\xi)d\xi \\
&\le \sup_{|w|=t} \frac{1}{|w|}\int_0^{|w|} K^*(\xi)d\xi = K^{**}(t).
\end{aligned}$$

As above, for $s \geq t$ we write

$$\Phi(s,t) \leq \sup_{|e|=s} \frac{1}{|e|} \int_0^{|e|} K^*(\xi)d\xi = K^{**}(s).$$

Combining these estimates, we have

$$\int_0^1 g(s)(K * f)^{**}(s)ds \leq \int_0^1 g(s) \int_0^1 f^*(t)K^{**}(\max\{s,t\})dt\, ds. \tag{2.5}$$

Using

$$g(s)\int_0^s f^*(t)dt + f^*(s)\int_0^s g(s)dt = \left(\int_0^s g(t)dt \int_0^s f^*(t)dt\right)',$$

and inequality (2.5), we get

$$\begin{aligned}
\int_0^1 g(s)(K * f)^{**}(s)ds &\leq \int_0^1 g(s) \int_0^1 f^*(t)K^{**}(\max(t,s))dt\, ds \\
&= \int_0^1 K^{**}(s)\left(g(s)\int_0^s f^*(t)dt + f^*(s)\int_0^s g(t)dt\right)ds \\
&= \int_0^1 K^{**}(s)\left(\int_0^s g(t)dt \int_0^s f^*(t)dt\right)' ds.
\end{aligned}$$

Integrating by parts, we get for $\varepsilon > 0$

$$\begin{aligned}
&\int_\varepsilon^1 g(s)(K * f)^{**}(s)ds \\
&\quad \leq K^{**}(s)\int_0^s f^*(t)dt \int_0^s g(t)dt\Big|_\varepsilon^1 - \int_\varepsilon^1\left(\int_0^s f^*(t)dt \int_0^s g(t)dt\right)(K^{**}(s))'ds \\
&\quad \leq K^{**}(1)\int_0^1 f^*(t)dt \int_0^1 g(t)dt + \left|\int_0^1\left(\int_0^s f^*(t)dt \int_0^s g(t)dt\right)(K^{**}(s))'ds\right|.
\end{aligned}$$

Hence, using

$$(K^{**}(s))' = -\frac{1}{s}\left(K^{**}(s) - K^*(s)\right), \tag{2.6}$$

we obtain

$$\begin{aligned}
\int_0^1 g(t)(K * f)^{**}(t)dt \leq \|g\|_{L_1}\|f\|_{L_1}\|K\|_{L_1} + \int_0^1 \left(\int_0^t g(s)ds\right) f^{**}(t)\big(K^{**}(t) \\
- K^*(t)\big)dt.
\end{aligned}$$

The proof is now complete. □

Now we give an elementary proof of the crucial convolution lemma of O'Neil-type (cf. [18, Lemma 1.5]) for the case of periodic functions.

Lemma 2.3 *Suppose f and K are 1-periodic integrable functions, then*

$$(K * f)^{**}(t) \leq f^{**}(t) \int_0^t K^*(s)ds + \int_t^1 f^{**}(s)K^*(s)ds. \tag{2.7}$$

Proof Indeed, by (2.2), we estimate

$$(K * f)^{**}(t) = (f * K)^{**}(t) \leq \sup_{|e|=t} \int_0^1 |K(x)| \frac{1}{|e|} \int_e |f(y-x)|\, dy\, dx.$$

Further, using the same calculations as in the proof of Lemma 2.2, we get

$$\begin{aligned}
&\sup_{|e|=t} \int_0^1 |K(x)| \frac{1}{|e|} \int_e |f(y-x)|\, dy\, dx \\
&\quad \leq \sup_{|e|=t} \int_0^1 K^*(s) \sup_{|w|=s} \frac{1}{|e||w|} \int_w \int_e |f(y-x)| dy\, dx\, ds \\
&\quad \leq \int_0^1 K^*(s) f^{**}(\max(t,s)) ds \\
&\quad = f^{**}(t) \int_0^t K^*(s)ds + \int_t^1 K^*(s)) f^{**}(s)ds,
\end{aligned}$$

which concludes the proof. □

Lemma 2.4 *Let $1 < p < \infty$ and $1 \leq q < \infty$. Then*

$$\|f\|_{L_{p,q}} = \sup_{g \in D_{p',q'}} \int_0^1 g(t) f^*(t) dt,$$

where

$$D_{p',q'} = \left\{ g : g \geq 0, \left(\int_0^1 \left(t^{\frac{1}{p'}} g(t) \right)^{q'} \frac{dt}{t} \right)^{1/q'} = 1 \right\}.$$

Proof Indeed,

$$\int_0^1 g(t) f^*(t) dt \leq \left(\int_0^1 \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{1/q} \left(\int_0^1 \left(t^{\frac{1}{p'}} g(t) \right)^{q'} \frac{dt}{t} \right)^{1/q'}.$$

On the other hand, we define

$$g_0(t) = \frac{t^{\frac{q}{p}-1} (f^*(t))^{q-1}}{\left(\int_0^1 (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t} \right)^{1/q'}}.$$

Hence, $g_0 \in D_{p'q'}$ and

$$\int_0^1 g_0(t) f^*(t)dt = \|f\|_{L_{p,q}}.$$

□

We are now in a position to present the limiting case of Young-O'Neil's convolution inequality.

Theorem 2.1 *Let* $1 \le p < \infty$, $1 \le h_1, h_2, h_3 \le \infty$, *and*

$$\frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}. \tag{2.8}$$

Suppose K *is a* 1*-periodic function such that* $K^{**} \in L_{1,h_3}[0,1]$.

a) *If* $1 < p < \infty$ *and* $f \in L_{p,h_2}[0,1]$, *then* $K * f \in L_{p,h_1}$ *and*

$$\|K * f\|_{L_{p,h_1}} \le c\, \|f\|_{L_{p,h_2}} \|K^{**}\|_{L_{1,h_3}}. \tag{2.9}$$

b) *If* $p = 1$ *and* $f^{**} \in L_{1,h_2}[0,1]$, *then* $(K * f)^{**} \in L_{1,h_1}$ *and*

$$\|(K * f)^{**}\|_{L_{1,h_1}} \le c\, \|f^{**}\|_{L_{1,h_2}} \|K^{**}\|_{L_{1,h_3}}.$$

Proof a). Let $1 < p < \infty$, $1 \le h_1 < \infty$. By Lemmas 2.4 and 2.2, we have

$$\begin{aligned}
&\|f * K\|_{L_{p,h_1}} \\
&\quad = \sup_{g \in D_{p',h_1'}} \int_0^1 g(t)(K * f)^*(t)dt \le \sup_{g \in D_{p',h_1'}} \int_0^1 g(t)(K * f)^{**}(t)dt \\
&\quad \le \sup_{g \in D_{p',h_1'}} \left(\|g\|_{L_1} \|f\|_{L_1} \|K\|_{L_1} + \int_0^1 \left(K^{**}(t) - K^*(t)\right) f^{**}(t) \int_0^t g(s)ds dt \right)
\end{aligned}$$

Applying Hölder's inequality (cf. (2.8)),

$$\begin{aligned}
\|f * K\|_{L_{p,h_1}} \le \sup_{g \in D_{p',h_1'}} &\Bigg(\|g\|_{L_1} \|f\|_{L_1} \|K\|_{L_1} \\
&+ \int_0^1 \left(t^{1-\frac{1}{h_3}} K^{**}(t)\right) \left(t^{\frac{1}{p}-\frac{1}{h_2}} f^{**}(t)\right) \left(t^{\frac{1}{p'}-\frac{1}{h_1'}-1} \int_0^t g(s)ds\right) dt \Bigg) \\
\le c\Bigg(&\|f\|_{L_1} \|K\|_{L_1} + \sup_{g \in D_{p',h_1'}} \|K^{**}\|_{1,h_3} \|f^{**}\|_{p,h_2} \\
&\times \left(\int_0^1 \left(t^{\frac{1}{p'}-1} \int_0^t g(s)ds\right)^{h_1'} \frac{dt}{t} \right)^{1/h_1'} \Bigg).
\end{aligned} \tag{2.10}$$

Since $1 < p < \infty$, then (2.1) yields $\|f^{**}\|_{L_{p,h_2}} \asymp \|f\|_{L_{p,h_2}}$ and Hardy's inequality gives

$$\left(\int_0^1 \left(t^{\frac{1}{p'}-1}\int_0^t g(s)ds\right)^{h_1'} \frac{dt}{t}\right)^{1/h_1'} \le c\left(\int_0^1 \left(t^{\frac{1}{p'}} g(t)\right)^{h_1'} \frac{dt}{t}\right)^{1/h_1'}.$$

Therefore, (2.10) implies

$$\|(f*K)\|_{L_{p,h_1}} \le c\,\|K^{**}\|_{L_{1,h_3}}\|f\|_{L_{p,h_2}}.$$

Let now $h_1 = \infty$. Then using Lemma 2.3 we have

$$\begin{aligned}\|f*K\|_{L_{p,\infty}} &\le \sup_t t^{1/p}(f*K)^{**}(t)\\ &\le \sup_t t^{1/p} f^{**}(t)\int_0^t K^*(s)ds + \sup_t t^{1/p}\int_t^1 f^{**}(s)K^*(s)ds\\ &\le 2\,\|K\|_1 \sup_t t^{1/p} f^{**}(t) \le 2p'\,\|K\|_1 \sup_t t^{1/p} f^*(t)\\ &= 2p'\,\|K^{**}\|_{1,\infty}\|f\|_{p,\infty}.\end{aligned}$$

b). Let $p = 1$ and $1 < h_1 < \infty$. Then using [10, Th 4.1] and periodicity of functions, we get[1]

$$\begin{aligned}\|(K*f)^{**}\|_{1,h_1} &\asymp \sup_{\|g\|_{L_{\infty,h_1'}}=1}\int_0^1 g(y)(K*f)(y)dy\\ &= \sup_{\|g\|_{L_{\infty,h_1'}}=1}\int_0^1 f(x)(g*K)(x)dx\\ &\le \sup_{\|g\|_{L_{\infty,h_1'}}=1}\int_0^1 f^*(s)(g*K)^{**}(s)ds.\end{aligned}$$

We again apply Lemma 2.2 and the Hölder inequality:

$$\begin{aligned}&\|(K*f)^{**}\|_{1,h_1}\\ &\le c \sup_{\|g\|_{L_{\infty,h_1'}}=1}\left(\|f\|_{L_1}\|g\|_{L_1}\|K\|_{L_1} + \int_0^1 tf^{**}(t)K^{**}(t)\left(g^{**}(t)-g^*(t)\right)dt\right)\\ &\le c \sup_{\|g\|_{L_{\infty,h_1'}}=1}\|f^{**}\|_{1,h_2}\|K^{**}\|_{1,h_3}\left(\|g\|_1 + \left(\int_0^1 \frac{(g^{**}-g^*)^{h_1'}}{t}dt\right)^{\frac{1}{h_1'}}\right)\\ &= c\;\|f^{**}\|_{1,h_2}\|K^{**}\|_{1,h_3}.\end{aligned}$$

[1] See the definition of the space $L_{\infty,h_1'}$ in the next section.

Let now $p = h_1 = 1$. Then we use Lemma 2.3:

$$\|(K*f)^{**}\|_{1,1} \le \int_0^1 tf^{**}(t)K^{**}(t)dt + \int_0^1 \int_t^1 f^{**}(s)K^*(s)ds\,dt$$
$$\le 2\int_0^1 tf^{**}(t)K^{**}(t)dt \le 2\|f^{**}\|_{1,h_2}\|K^{**}\|_{1,h_3},$$

where $h_3 = h_2'$. We remark that in this case if the right-hand side is finite, then $K * f$ belongs to the space $L\log L$.

Finally, if $p = 1$ and $h_1 = \infty$, then by Fubini's theorem,

$$\|(K*f)^{**}\|_{1,\infty} = \|K*f\|_1 \le \|K\|_1\|f\|_1 = \|K^{**}\|_{1,\infty}\|f^{**}\|_{1,\infty}.$$

The proof is complete. □

Let us give two examples showing the sharpness of the results of Theorem 2.1. First, we prove that in inequality (2.9), the factor $\|K^{**}\|_{L_{1,h_3}}$ could not be changed to $\|K\|_{L_{1,h_3}}$. That is, in general, for $1 \le p = q < \infty$ and $\frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}$ the inequality

$$\|K*f\|_{L_{p,h_1}} \le C\|f\|_{L_{p,h_2}}\|K\|_{L_{1,h_3}} \tag{2.11}$$

does not hold.

Example 2.1 Let $1 \le p < \infty, N \in \mathbb{N}, N > 4(e^p + 1)$. We define $f(t) = (\min(N, 1/t))^{1/p}$ and $K(t) = \min(N, 1/t)$. Then

$$\|f\|_{L_{p,h_2}} \asymp (1 + \ln N)^{1/h_2}$$

and

$$\|K\|_{L_{1,h_3}} \asymp (1 + \ln N)^{1/h_3} = (1 + \ln N)^{1/h_1 - 1/h_2}.$$

To prove that (2.11) is not true, let us first define

$$\varphi(x) = \begin{cases} 0 & \text{for } x \in [0, a) \\ (K*f)(x) & \text{for } x \in [a, 1) \end{cases}, \quad a = \frac{1}{N}\left(e^p + 1\right).$$

Then if $x \in [a, 1)$ we have

$$\varphi(x) \ge \int_{\frac{1}{N}}^{x-\frac{1}{N}} \frac{ds}{s^{\frac{1}{p}}(x-s)} \ge \frac{1}{(x-\frac{1}{N})^{\frac{1}{p}}}\int_{\frac{1}{N}}^{x-\frac{1}{N}} \frac{ds}{x-s} = \frac{N^{1/p}\ln(Nx-1)}{(Nx-1)^{1/p}}.$$

Noting $(\frac{\ln\xi}{\xi^{1/p}})' = \frac{p-\ln\xi}{p\xi^{1/p+1}} < 0$ for $\xi \in (e^p, 1]$, we obtain that the function $\frac{\ln\xi}{\xi^{1/p}}$ is decreasing on $[e^p, 1]$. Hence,

$$\varphi^*(t) \ge \frac{\ln(N(t+a)-1)}{\left(t+a-\frac{1}{N}\right)^{1/p}} \quad \text{for } t \in (0, 1-a).$$

Using this, we get

$$\begin{aligned}\|f*K\|_{L_{p,h_1}}^{h_1} &\geq \int_0^1 (t^{1/p}\varphi^*(t))^{h_1}\frac{dt}{t} \geq \int_0^{1-a}\left(\frac{t^{1/p}\ln(N(t+a)-1)}{(t+a-\frac{1}{N})^{1/p}}\right)^{h_1}\frac{dt}{t}\\ &\geq \int_a^{1-a}\left(\frac{t^{1/p}\ln(N(t+a)-1)}{(t+a-\frac{1}{N})^{1/p}}\right)^{h_1}\frac{dt}{t}\\ &\geq 2^{-\frac{h_1}{p}}\int_a^{1-a}\big(\ln(N(t+a)-1)\big)^{h_1}\frac{dt}{t}\\ &\asymp \ln^{h_1+1}\big(N(t+a)-1\big)\,\big|_a^{1-a} \asymp (1+\ln N)^{h_1+1}.\end{aligned}$$

Thus, (2.11) implies

$$\ln N \leq C.$$

This contradiction concludes the proof.

We now provide an example showing sharpness of (2.9) in the following sense: if $1 \leq p = q < \infty$ and $\frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}$, then for any $h > h_3$ the inequality

$$\|K * f\|_{L_{p,h_1}} \leq C\|f\|_{L_{p,h_2}}\|K^{**}\|_{L_{1,h}}$$

does not hold.

Example 2.2 Let $1 < p < \infty$, $1 \leq h_1, h_2 \leq \infty$ and $1/h_3 = (1/h_1 - 1/h_2)_+$. Then for the Cesàro operator of 1-periodic function f, given by

$$\sigma_N(f;x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k(f;x) = \int_0^1 f(y)F_N(x-y)dy, \qquad N \in \mathbb{N},$$

$$F_N(x) = \frac{1}{N+1}\frac{\sin^2\pi(N+1)x}{\sin^2\pi x},$$

we claim

$$\|\sigma_N\|_{L_{p,h_2}\to L_{p,h_1}} \asymp \|F_N^{**}\|_{L_{1,h_3}} \asymp (1+\ln N)^{1/h_3}.$$

As usual, $S_k(f,x)$ denotes the k-th partial sum of the Fourier series of f.

Proof of the claim By Theorem 2.1, we have

$$\|\sigma_N\|_{L_{p,h_2}\to L_{p,h_1}} \leq C\|F_N^{**}\|_{L_{1,h_3}}.$$

Further, we estimate

$$\|F_N^{**}\|_{L_{1,h_3}} \le \left(\int_0^{1/(N+1)} \left(\int_0^t F_N^*(s)ds \right)^{h_3} \frac{dt}{t} + \int_{1/(N+1)}^1 \left(\left(\int_0^{1/(N+1)} + \int_{1/(N+1)}^t \right) F_N^*(s)ds \right)^{h_3} \frac{dt}{t} \right)^{1/h_3}.$$

Using the known inequality (see, e.g., [23, Chap. 3, (3.10)])

$$F_N^*(s) \le c \min\left((N+1), \frac{1}{(N+1)s^2} \right), \quad s \in (0,1],$$

we write

$$\begin{aligned} \|F_N^{**}\|_{L_{1,h_3}} &\le c\left((N+1)^{h_3} \int_0^{1/(N+1)} t^{h_3-1}dt + 2\int_{1/(N+1)}^1 \frac{dt}{t} \right)^{1/h_3} \\ &\le c\,(1+\ln N)^{1/h_3}. \end{aligned}$$

Thus, we get

$$\|\sigma_N\|_{L_{p,h_2} \to L_{p,h_1}} \le c\,(1+\ln N)^{1/h_3}.$$

On the other hand, defining

$$f_0(x) = \sum_{k=1}^N \frac{1}{k^{1/p}} e^{2\pi i k x}$$

and using Hardy-Littlewood's theorem on monotone Fourier coefficients for the Lorentz space [20], we obtain

$$\|f_0\|_{L_{p,h_2}} \asymp \left(\sum_{k=1}^N \left(k^{1/p} \frac{1}{k^{1/p}} \right)^{h_2} \frac{1}{k} \right)^{1/h_2} \asymp (1+\ln N)^{1/h_2}$$

$$\|\sigma_N(f_0)\|_{L_{p,h_1}} \asymp \left(\sum_{k=1}^N \left(\frac{N-k}{N} \right)^{h_1} \frac{1}{k} \right)^{1/h_1} \asymp (1+\ln N)^{1/h_1}.$$

Therefore, we derive

$$\|\sigma_N\|_{L_{p,h_2} \to L_{p,h_1}} \ge \frac{\|\sigma_N(f_0)\|_{L_{p,h_1}}}{\|f_0\|_{L_{p,h_2}}} \asymp (1+\ln N)^{1/h_1 - 1/h_2} = (1+\ln N)^{1/h_3},$$

completing the proof. □

3 Convolution in the Lorentz Space of Periodic Functions: the Case of $p = \infty$

Theorem 2.1 does not encompass the limit case $p = \infty$ ($h_i < \infty$). It is clear that in this case the classical Lorentz space is trivial. We consider another scale of Lorentz spaces.

Following Bennett et al. [4] (see also [2], [10]), we define $L_{\infty,q}[0, 1]$ as follows

$$L_{\infty,q}[0,1] = \left\{ f \in L_1[0,1] : \|f\|_{L_{\infty,q}[0,1]} := \|f\|_{L_1[0,1]} + \left(\int_0^1 (f^{**} - f^*)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty \right\}.$$

We remark that for any $q, s \in [1, \infty]$ and $p \in [1, \infty)$, one has $L_{\infty,q}[0, 1] \hookrightarrow L_{p,s}[0, 1]$. What is more, the following embedding holds: for $1 \le p < \infty$ and $1 \le q < q_1 \le \infty$

$$L_\infty[0,1] = L_{\infty,1}[0,1] \hookrightarrow L_{\infty,q}[0,1] \hookrightarrow L_{\infty,q_1}[0,1] \hookrightarrow L_p[0,1]. \tag{3.1}$$

Indeed, for $1 \le q < q_1 < \infty$ we get

$$\left(\int_0^1 (f^{**} - f^*)^{q_1} \frac{dt}{t}\right)^{1/q_1} = \left(\sum_{\nu=0}^{\infty} \int_{2^{-\nu-1}}^{2^{-\nu}} (f^{**} - f^*)^{q_1} \frac{dt}{t}\right)^{1/q_1}$$
$$\le \left(\sum_{\nu=0}^{\infty} \left(\int_{2^{-\nu-1}}^{2^{-\nu}} (f^{**} - f^*)^{q_1} \frac{dt}{t}\right)^{q/q_1}\right)^{1/q}$$

and using monotonicity of $t\big(f^{**}(t) - f^*(t)\big)$,

$$\le C\left(\sum_{\nu=0}^{\infty} \Big(f^{**}(1/2^\nu) - f^*(1/2^\nu)\Big)^q\right)^{1/q}$$
$$\le C\left(f^{**}(1)^q + \sum_{\nu=1}^{\infty} \int_{2^{-\nu}}^{2^{-\nu+1}} (f^{**} - f^*)^q \frac{dt}{t}\right)^{1/q}$$
$$\le C\left(\|f\|_1 + \left(\int_0^1 (f^{**} - f^*)^q \frac{dt}{t}\right)^{1/q}\right).$$

For $q_1 = \infty$ the proof is similar. Also,

$$\|f\|_{L_{\infty,1}} = \|f\|_1 + \int_0^1 (f^{**} - f^*) \frac{dt}{t} = \|f\|_1 - \int_0^1 \big(f^{**}(t)\big)' dt$$
$$= \|f\|_1 + f^{**}(0) - f^{**}(1) = \|f\|_{L_\infty},$$

i.e., $L_{\infty,1} = L_\infty$.

The last embedding in (3.1) follows from Hölder's inequality (one can assume that $p < q_1$)

$$\|f\|_{L_p} = \|f\|_{L_{p,p}} \asymp \|f\|_{L_1} + \left(\int_0^1 (f^{**} - f^*)^p dt\right)^{1/p}$$

$$\le \|f\|_{L_1} + \left(\int_0^1 t^{\frac{p}{q_1-p}} dt\right)^{1/p-1/q_1} \left(\int_0^1 (f^{**}-f^*)^{q_1} \frac{dt}{t}\right)^{1/q_1}$$
$$\le c\,\|f\|_{L_{\infty,q_1}}.$$

Theorem 3.1 *Let* $1 \le h_1, h_2, h_3 \le \infty$ *and*

$$0 < \frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}.$$

For any 1*-periodic functions* f *and* K *such that* $f \in L_{\infty,h_2}[0,1]$ *and* $K \in L_{1,h_3}[0,1]$ *we have*

$$\|K * f\|_{L_{\infty,h_1}} \le 4\,\|f\|_{L_{\infty,h_2}} \|K^{**}\|_{L_{1,h_3}}. \tag{3.2}$$

Proof Let us suppose first that $h_1 > 1$. Let $f \in L_{\infty,h_2}[0,1]$. Without loss of generality, we may assume that K is of class C^∞ and then $h \equiv K * f$ is of C^∞.

If

$$\|h\|_1 \ge \left(\int_0^1 \left(h^{**}(t) - h^*(t)\right)^{h_1} \frac{dt}{t}\right)^{1/h_1}$$

then since

$$\|K\|_{L_1} \le (h_3)^{1/h_3}\,\|K^{**}\|_{L_{1,h_3}} \le 2\,\|K^{**}\|_{L_{1,h_3}}, \tag{3.3}$$

we have

$$\|h\|_{L_{\infty,h_1}} \le 2\,\|h\|_1 \le 4\,\|f\|_{L_{\infty,h_2}} \|K^{**}\|_{L_{1,h_3}}$$

and (3.2) is proved. Now suppose the converse holds true, then

$$\|h\|_{L_{\infty,h_1}} \le 2\left(\int_0^1 \left(h^{**}(t) - h^*(t)\right)^{h_1} \frac{dt}{t}\right)^{1/h_1} = 2\int_0^1 g(t)\frac{(h^{**}(t)-h^*(t))}{t}dt$$

where

$$g(t) = \frac{(h^{**}(t)-h^*(t))^{h_1-1}}{\left(\int_0^1 (h^{**}(s)-h^*(s))^{h_1} \frac{ds}{s}\right)^{1/h_1'}}.$$

The function g satisfies the following conditions:

1) $g(t) \ge 0$ and $\lim_{t\to+0} g(t) = 0$,
2) $g(1) \le 1$,
   since $\|h\|_1 \le \left(\int_0^1 (h^{**}(t)-h^*(t))^{h_1} \frac{dt}{t}\right)^{1/h_1}$;
3) $\left(\int_0^1 (g(t))^{h_1'} \frac{dt}{t}\right)^{1/h_1'} = 1$,
4) $g \in C^\infty$.

By A we denote a collection of all functions satisfying 1)–4). Then

$$\|h\|_{L_{\infty,h_1}} \le 2\sup_{g\in A}\int_0^1 g(t)\left(\frac{h^{**}(t)-h^*(t)}{t}\right)dt.$$

For $g \in A$, noting that h is bounded and therefore $\lim_{t\to+0} g(t)h^{**}(t) = 0$, we get

$$\begin{aligned}\int_0^1 g(t)\left(\frac{h^{**}(t)-h^*(t)}{t}\right)dt &= -\int_0^1 g(t)\big(h^{**}(t)\big)'\,dt \\ &= -g(t)h^{**}(t)\Big|_0^1 + \int_0^1 g'(t)h^{**}(t)dt \\ &= -g(1)h^{**}(1) + \int_0^1 g'(t)h^{**}(t)dt \\ &\le \int_0^1 g'(t)h^{**}(t)dt.\end{aligned}$$

Hence,

$$\|K * f\|_{L_{\infty,h_1}} \le \sup_{g\in A}\int_0^1 g'(t)h^{**}(t)\,dt = \sup_{g\in A,\, g'(t)\ge 0}\int_0^1 g'(t)h^{**}(t)\,dt. \tag{3.4}$$

We now use Lemma 2.3:

$$\begin{aligned}&\int_0^1 g'(t)(K*f)^{**}(t)dt \\ &\quad\le \int_0^1 g'(t)\left(f^{**}(t)\int_0^t K^*(s)ds + \int_t^1 f^{**}(s)K^*(s)ds\right)dt \\ &\quad= \int_0^1 \left(g'(t)f^{**}(t)\int_0^t K^*(s)ds + f^{**}(t)K^*(t)\int_0^t g'(s)ds\right)dt \\ &\quad= \int_0^1 f^{**}(t)\left(\int_0^t K^*(s)ds\int_0^t g'(s)ds\right)'dt \\ &\quad= f^{**}(t)\int_0^t K^*(s)ds\int_0^t g'(s)ds\Big|_0^1 - \int_0^1 \big(f^{**}(t)\big)'\int_0^t K^*(s)ds\int_0^t g'(s)ds\,dt \\ &\quad= g(1)\|f\|_{L_1}\|K\|_{L_1} + \int_0^1 \big(f^{**}(t)-f^*(t)\big)\,K^{**}(t)g(t)dt.\end{aligned}$$

Taking into account that $g \in A$, we use Hölder's inequality:

$$\begin{aligned}\|K*f\|_{L_{\infty,h_1}} &\le 2\sup_{g\in A}\left\{\|K\|_{L_1}\|f\|_{L_1} + \left(\int_0^1 \big(f^{**}(t)-f^*(t)\big)^{h_2}\frac{dt}{t}\right)^{1/h_2}\right. \\ &\qquad\left.\times\left(\int_0^1 \big(tK^{**}(t)\big)^{h_3}\frac{dt}{t}\right)^{1/h_3}\left(\int_0^1 (g(t))^{h_1'}\frac{dt}{t}\right)^{1/h_1'}\right\} \\ &\le 2\,\|f\|_{L_{\infty,h_2}}\|K^{**}\|_{L_{1,h_3}},\end{aligned}$$

and (3.2) follows for $h_1 > 1$ and $1 \le h_2, h_3 \le \infty$.

Finally, let $h_1 = 1$. Then using Lemma 2.3 and (2.6), we get

$$\begin{aligned}
\|f * K\|_{L_{\infty,1}} &= \|f * K\|_{L_\infty} = \sup_{t>0} (K * f)^{**}(t) \\
&\le \sup_{t>0} \left(f^{**}(t) \int_0^t K^*(s)ds + \int_t^1 f^{**}(s) K^*(s)ds \right) \\
&= \sup_{t>0} \left(f^{**}(t) \int_0^t K^*(s)ds + f^{**}(s) \left(\int_0^s K^*(\xi)d\xi \right) \Big|_t^1 \right. \\
&\quad \left. - \int_t^1 \left(f^{**}(s)\right)' \left(\int_0^s K^*(\xi)d\xi \right) ds \right) \\
&= \|f\|_{L_1} \|K\|_1 + \int_0^1 \left(f^{**}(s) - f^*(s)\right) K^{**}(s)ds.
\end{aligned}$$

Thus, Hölder's inequality completes the proof:

$$\|f * K\|_{L_{\infty,1}} \le \|f\|_{L_{\infty,h_2}} \|K^{**}\|_{L_{1,h_2'}} = \|f\|_{L_{\infty,h_2}} \|K^{**}\|_{L_{1,h_3}}. \qquad \square$$

Let us now prove the limiting case of the Brézis-Wainger inequality (1.4). We will need the following weighted inequalities of Hardy type. Denote $\|f\|_{L_r(\frac{dt}{t})} := (\int_0^1 |f(t)|^r \frac{dt}{t})^{1/r}$.

Lemma 3.1 (see, e.g., [8, Lemma 1]) *Let $1 < r < \infty$ and $t\varphi(t) \in L_r(\frac{dt}{t})$. Then*

$$\left\| (1 + |\ln t|)^{-1} \int_t^1 \varphi(s)ds \right\|_{L_r(\frac{dt}{t})} \le r' \left\| t\varphi(t) \right\|_{L_r(\frac{dt}{t})}.$$

Lemma 3.2 *Let $1 < r < \infty$ and $g \in L_r(\frac{dt}{t})$. Then*

$$\left\| \int_0^t g(s) \frac{ds}{s(1 + |\ln s|)} \right\|_{L_r(\frac{dt}{t})} \le r \left\| g \right\|_{L_r(\frac{dt}{t})}.$$

Proof Using duality arguments, we have

$$\begin{aligned}
\left\| \int_0^t g(s) \frac{ds}{s(1 + |\ln s|)} \right\|_{L_r(\frac{dt}{t})} &= \sup_{\|\varphi\|_{L_{r'}(\frac{dt}{t})}=1} \int_0^1 \left(\int_0^s g(t) \frac{dt}{t(1 + |\ln t|)} \right) \varphi(s) \frac{ds}{s} \\
&= \sup_{\|\varphi\|_{L_{r'}(\frac{dt}{t})}=1} \int_0^1 g(t) \left((1 + |\ln t|)^{-1} \int_t^1 \frac{\varphi(s)}{s} ds \right) \frac{dt}{t},
\end{aligned}$$

and, by Hölder's inequality,

$$\leq \sup_{\|\varphi\|_{L_{r'}(\frac{dt}{t})}=1} \left(\int_0^1 (g(t))^r \frac{dt}{t}\right)^{1/r} \left(\int_0^1 \left((1+|\ln t|)^{-1} \int_t^1 \frac{\varphi(s)}{s} ds\right)^{r'} \frac{dt}{t}\right)^{1/r'}.$$

Lemma 3.1 completes the proof. □

Theorem 3.2 *Let* $1 < h_1 < \infty$, $1 \leq h_2, h_3 \leq \infty$, *and* $\frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}$. *Suppose* $f \in L_{\infty,h_2}$ *and* $K^{**} \in L_{1,h_3}$; *then* $\frac{(f*K)^*(t)}{1+|\ln t|} \in L_{h_1}(\frac{dt}{t})$ *and*

$$\left\| \frac{(K*f)^*}{1+|\ln t|} \right\|_{L_{h_1}(\frac{dt}{t})} \leq 2h_1' \|f\|_{L_{\infty,h_2}} \|K^{**}\|_{L_{1,h_3}}. \tag{3.5}$$

Proof Since $\varphi^*(t) \leq \varphi^{**}(t)$, we have

$$\left\| \frac{(K*f)^*}{1+|\ln t|} \right\|_{L_{h_1}(\frac{dt}{t})} \leq \sup_{\|g\|_{L_{h_1'}(\frac{dt}{t})}=1} \int_0^1 \frac{g(t)}{t(1+|\ln t|)} (K*f)^{**}(t)dt.$$

Then Lemma 2.2 yields

$$\left\| \frac{(K*f)^*}{1+|\ln t|} \right\|_{L_{h_1}(\frac{dt}{t})} \leq \sup_{\|g\|_{L_{h_1'}(\frac{dt}{t})=1}} \left\| \frac{g(t)}{t(1+|\ln t|)} \right\|_{L_1} \|f\|_{L_1} \|K\|_{L_1}$$
$$+ \int_0^1 \left(\int_0^t \frac{g(s)}{s(1+|\ln s|)} ds\right) (f^{**}(t) - f^*(t)) K^{**}(t)dt. \tag{3.6}$$

By Hölder's inequality and Lemma 3.2,

$$\left\| \frac{(K*f)^*}{1+|\ln t|} \right\|_{L_{h_1(\frac{dt}{t})}} \leq \sup_{\|g\|_{L_{h_1'}(\frac{dt}{t})=1}} \left(\left\| \frac{g(t)}{t(1+|\ln t|)} \right\|_{L_1} \|f\|_{L_1} \|K\|_{L_1} \right.$$
$$\left. + \left\| \int_0^t \frac{g(s)}{s(1+|\ln s|)} ds \right\|_{L_{h_1'}(\frac{dt}{t})} \|f^{**} - f^*\|_{L_{h_2}(\frac{dt}{t})} \|K^{**}\|_{L_{1,h_3}} \right)$$
$$\leq \sup_{\|g\|_{L_{h_1'}(\frac{dt}{t})=1}} \left(\left\| \frac{g(t)}{t(1+|\ln t|)} \right\|_{L_1} \|f\|_{L_1} \|K\|_{L_1} \right.$$
$$\left. + h_1' \|g\|_{L_{h_1'}(\frac{dt}{t})} \|f^{**} - f^*\|_{L_{h_2}(\frac{dt}{t})} \|K^{**}\|_{L_{1,h_3}} \right).$$

Finally, taking into account

$$\left\| \frac{g(t)}{t(1+|\ln t|)} \right\|_{L_1} \le (h_1-1)^{-1/h_1} \|g\|_{L_{h_1'}(\frac{dt}{t})} \le h_1' \|g\|_{L_{h_1'}(\frac{dt}{t})}$$

and (3.3), we arrive at (3.5). □

Note that using estimate (3.6) it is easy to prove Brézis-Wainger's inequality (1.4). Indeed,

$$\begin{aligned}
&\left\| \frac{(K*f)^*}{1+|\ln t|} \right\|_{L_{h_1}(\frac{dt}{t})} \\
&\quad \le \sup_{\|g\|_{L_{h_1'}(\frac{dt}{t})}=1} \Bigg(\left\| \frac{g(t)}{t(1+|\ln t|)} \right\|_{L_1} \|f\|_{L_1} \|K\|_{L_1} \\
&\qquad + \left\| \int_0^t \frac{g(s)}{s(1+|\ln s|)} ds \right\|_{L_{h_1'}(\frac{dt}{t})} \left(\int_0^1 \left(t^{1/p} \left(f^{**}(t) - f^*(t) \right) \right)^{h_2} \frac{dt}{t} \right)^{1/h_2} \\
&\qquad \times \left(\int_0^1 \left(t^{1/p'} K^{**}(t) \right)^{h_3} \frac{dt}{t} \right)^{1/h_3} \Bigg) \\
&\quad \le \sup_{\|g\|_{L_{h_1'}(\frac{dt}{t})}=1} \left(\left\| \frac{g(t)}{t(1+|\ln t|)} \right\|_{L_1} \|f\|_{L_1} \|K\|_{L_1} + c\|g\|_{L_{h_1'}(\frac{dt}{t})} \|f\|_{L_{p,h_2}} \|K\|_{L_{p',h_3}} \right) \\
&\quad \le c\, \|f\|_{L_{p,h_2}} \|K\|_{L_{p',h_3}}.
\end{aligned}$$

4 Remarks on Convolution in Weighted Lorentz Spaces

Let us present a convolution estimate of Young-O'Neil type in the weighted Lorentz spaces $\Lambda^q(\omega)$ and $\Gamma^q(\omega)$. Since the proof is similar to the proof of Theorem 2.1, we skip the details.

First, we remark (see [19]) that $\Lambda^q(\omega) = \Gamma^q(\omega)$ if and only if the weight ω satisfies the B_q-condition, that is,

$$\int_x^\infty \frac{\omega(t)}{t^q} dt \le \frac{C}{x^q} \int_0^x \omega(t) dt$$

and we write $\omega \in B_q$.

Theorem 4.1 *Let $\omega \in B_q$ and $\int_0^\infty \omega\, dx = \infty$. We have*

$$\|K*f\|_{\Gamma^q(\omega)} \asymp \|K*f\|_{\Lambda^q(\omega)} \le C\|f\|_{\Gamma^{q_1}(\omega_1)} \|K\|_{\Gamma^{q_2}(\omega_2)} \tag{4.1}$$

for $1/q = 1/q_1 + 1/q_2$, $1 < q < \infty$, $1 \le q_1, q_2 \le \infty$, *and*

$$\int_0^t \omega(x)\,dx \le c\,\omega(t)\frac{\omega_1(t)^{1/q_1}\omega_2(t)^{1/q_2}}{\omega(t)^{1/q_1+1/q_2}} \quad t > 0. \tag{4.2}$$

Proof The proof follows the line of that of Lemma 2.2 from which we get the following inequalities

$$\begin{aligned}\int_0^\infty g^*(s)(Af)^{**}(s)ds &\le \int_0^\infty g^*(s)\int_0^\infty f^*(t)K^{**}(\max(s,t))dt\,ds \\ &\le 2\int_0^\infty tf^{**}(t)g^{**}(t)K^{**}(t)dt.\end{aligned}$$

Since (4.2) can be rewritten as $t \le c\,\widetilde{\omega}(t)^{1/q'}\omega_1(t)^{1/q_1}\omega_2(t)^{1/q_2}$, where

$$\widetilde{\omega}(t) = t^{q'}W(t)^{-q'}\omega(t), \qquad W(x) = \int_0^x \omega(t)dt,$$

Hölder's inequality implies

$$\begin{aligned}&\int_{\mathbb{R}} g(y)\left(\int_{\mathbb{R}} f(x)K(x-y)dx\right)dy \\ &\quad\le 2c\int_0^\infty \big[g^{**}(t)\widetilde{\omega}(t)^{1/q'}\big]\big[f^{**}(t)\omega_1(t)^{1/q_1}\big]\big[K^{**}(t)\omega_2(t)^{1/q_2}\big]dt \\ &\quad\le 2c\,\|g\|_{\Gamma^{q'}(\widetilde{\omega})}\,\|f\|_{\Gamma^{q_1}(\omega_1)}\,\|K\|_{\Gamma^{q_2}(\omega_2)}.\end{aligned}$$

It is known from [11, 2.4], [14], and [19, p. 147] that under the conditions $\omega \in B_q$ and $\int_0^\infty \omega(x)dx = \infty$, the dual of $\Lambda^q(\omega)$, $1 < q < \infty$, can be identified with $\Gamma^{q'}(\tilde{\omega})$. Then taking the supremum over all g such that $\|g\|_{\Gamma^{q'}(\widetilde{\omega})} = 1$, we obtain

$$\|K * f\|_{(\Gamma^{q'}(\widetilde{\omega}))'} \le 2c\,\|f\|_{\Gamma^{q_1}(\omega_1)}\,\|K\|_{\Gamma^{q_2}(\omega_2)}.$$

Finally, since $\omega \in B_p$, we have $\big(\Lambda^q(\omega)\big)'' = \Lambda^q(\omega)$ and (4.1) follows. □

Example 4.1 Let $\omega(t) = t^{q/h-1}\xi_1^q(t)$, $\omega_1(t) = t^{q_1/p-1}\xi_2^{q_1}(t)$, $\omega_2(t) = t^{q_2/r-1}\xi_3^{q_2}(t)$, where $1/q = 1/q_1 + 1/q_2$, $1 + 1/h = 1/p + 1/r$, and ξ_i are slowly oscillating functions. Then inequality (4.2) is equivalent to

$$\xi_1(x) \le C\xi_2(x)\xi_3(x),$$

i.e., in this case we obtain the Young-O'Neil-type inequality for the Lorentz-Zygmund spaces [5, p. 253].

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