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SOLITON EQUATIONS IN 2+1 DIMENSIONS AND DIFFERENTIAL GEOMETRY OF CURVES/SURFACES [1]

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Abstract

Some aspects of the relation between differential geometry of curves and surfaces and (2+1)-dimensional soliton equations are discussed. For the (2+1)-dimensional case, self-cordination of geometrical formalism with the Hirota's bilinear method is established. A connection between supersymmetry, geometry and soliton equations is also considered.

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1 Introduction

Consider the curve in 3-dimensional space. Equations of such curves, following [1] we can write in the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{1a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = G \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{1b}$$

where

$$C = \begin{pmatrix} 0 & k & 0 \\ -\beta k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta\omega_3 & 0 & \omega_1 \\ \beta\omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{2}$$

Here

$$\mathbf{e}_1^2 = \beta = \pm 1, \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1 \tag{3}$$

Note that equation (1a) is the Serret - Frenet equation (SFE). So we have

$$C_t - G_x + [C, G] = 0 \tag{4}$$

or

$$k_t - \omega_{3x} - \tau\omega_2 = 0 \tag{5a}$$

$$\omega_{2x} - \tau\omega_3 + k\omega_1 = 0 \tag{5b}$$

$$\tau_t - \omega_{1x} + \beta k\omega_2 = 0. \tag{5c}$$

We now consider the isotropic Landau-Lifshitz equation (LLE)

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{xx}. \tag{6}$$

If

$$\mathbf{e}_1 \equiv \mathbf{S} \tag{7}$$

then

$$q = \frac{k}{2} e^{i\partial_x^{-1}\tau} \tag{8}$$

satisfies the NLSE

$$iq_t + q_{xx} + 2\beta \mid q \mid^2 q = 0. \tag{9}$$

This equivalence between the LLE (6) and the NLSE (9) we call the Lakshmanan equivalence or L-equivalence [1]. These results for the case $\beta = +1$ was obtained in [2] and for the case $\beta = -1$ in [1]. Note that between these equations also take places gauge equivalence (G-equivalence) [6].

In this paper, starting from Lakshmana's idea, we will discuss some aspects of the relation between differential geometry of curves and surfaces and (2+1)-dimensional soliton equations. Before this, in [1] we proposed some approaches to this problem, namely, the A-, B-, C-, and D-approaches. Below we will work with the B-, C-, D-approaches. We will discuss the relation between geometry and the Hirota's bilinear method. Also, we will consider the connection between supersymmetry, geometry and soliton equations.

2 Curves and Solitons in 2+1

In this section, we work with the D-approach. Using this D-approach, we will establish a connection between curves and (2+1)-dimensional soliton equations.

2.1 Some 2-dimensional extensions of the SFE

According to the D-approach, to establish the connection between (2+1)-dimensional soliton equations and differential geometry of curves in [1] was constructed some two (spatial) dimensional generalizations of the SFE (1a). Here we present some of them.

2.1.1 The M-LIX equation

This equation has the form [1]

$$\alpha \mathbf{e}_{1y} = f_1 \mathbf{e}_{1x} + \sum_{j=1}^{n} b_j \mathbf{e}_1 \wedge \frac{\partial^j}{\partial x^j} \mathbf{e}_1 + c_1 \mathbf{e}_2 + d_1 \mathbf{e}_3 \tag{10a}$$

$$\alpha \mathbf{e}_{2y} = Exercise \quad N1 \tag{10b}$$

$$\alpha \mathbf{e}_{3y} = Exercise \quad N1 \tag{10c}$$

Here the finding of the explicit forms of r.h. of (10b,c) we left as the exercises (see, the section 7).

2.1.2 The M-LX equation

The M-LX equation reads as [1]

$$\alpha \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x + B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{11a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \sum_{j=0}^{n} C_j \frac{\partial^j}{\partial x^j} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{11b}$$

where A, B, C_j - some matrices.

2.1.3 The M-LXI equation

This extension has the form [1]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = D \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{12a}$$

$$C = \begin{pmatrix} 0 & k & 0 \\ -\beta k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & m_3 & -m_2 \\ -\beta m_3 & 0 & m_1 \\ \beta m_2 & -m_1 & 0 \end{pmatrix}. \tag{12b}$$

2.1.4 The modified M-LXI equation

The modified M-LXI (mM-LXI) equation usually we write in the form [1]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = C_m \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = D_m \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{13a}$$

where

$$C_m = \begin{pmatrix} 0 & k & -\sigma \\ -\beta k & 0 & \tau \\ \beta\sigma & -\tau & 0 \end{pmatrix}, \quad D_m = D = \begin{pmatrix} 0 & m_3 & -m_2 \\ -\beta m_3 & 0 & m_1 \\ \beta m_2 & -m_1 & 0 \end{pmatrix} \tag{13b}$$

and so on [1]. In this paper, we work with the M-LIX, M-LXI and mM-LXI equations. Note that the M-LXI equation is the particular case of the mM-LXI eq. as $\sigma = 0$.

2.2 The mM-LXI equation and the mM-LXII equation

Let us return to the mM-LXI equation (13), which we write in the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = C_m \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{14a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = D_m \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{14b}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = G \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{14c}$$

where

$$G = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta\omega_3 & 0 & \omega_1 \\ \beta\omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{15}$$

From (14a,b), we obtain the following mM-LXII equation [1]

$$C_y - D_x + [C, D] = 0 \tag{16a}$$

or

$$k_y - m_{3x} + \sigma m_1 - \tau m_2 = 0 \tag{16b}$$

$$\sigma_y - m_{2x} + \tau m_3 - k m_1 = 0 \tag{16c}$$

$$\tau_y - m_{1x} + \beta(k m_2 - \sigma m_3) = 0. \tag{16d}$$

As $\sigma = 0$ the mM-LXII equation reduces to the M-LXII equation [1]. The mM-LXII equation (16), we can rewrite in form

$$k_y - m_{3x} = \frac{1}{\beta}\mathbf{e}_3 \cdot (\mathbf{e}_{3x} \wedge \mathbf{e}_{3y}) \tag{17a}$$

$$\sigma_y - m_{2x} = \frac{1}{\beta}\mathbf{e}_2 \dot{(}\mathbf{e}_{2x} \wedge \mathbf{e}_{2y}) \tag{17b}$$

$$\tau_y - m_{1x} = \mathbf{e}_1 \dot{(}\mathbf{e}_{1x} \wedge \mathbf{e}_{1y}) \tag{17c}$$

Also from (14) we get

$$k_t - \omega_{3x} + \sigma\omega_1 - \tau\omega_2 = 0 \tag{18a}$$

$$\sigma_t - \omega_{2x} + \tau\omega_3 - k\omega_1 = 0 \tag{18b}$$

$$\tau_t - \omega_{1x} + \beta(k\omega_2 - \sigma\omega_3) = 0 \tag{18c}$$

and

$$m_{1t} - \omega_{1y} + \beta(m_3\omega_2 - m_2\omega_3) = 0 \tag{19a}$$

$$m_{2t} - \omega_{2y} + m_1\omega_3 - m_3\omega_1 = 0 \tag{19b}$$

$$m_{3t} - \omega_{3y} + m_2\omega_1 - m_1\omega_2 = 0. \tag{19c}$$

2.3 On the topological invariants

From the mM-LXII equation (16) follows

$$[C, D]_t + C_{ty} - D_{tx} = 0 \tag{20a}$$

or

$$(\sigma m_1 - \tau m_2)_t + k_{ty} - m_{3tx} = 0 \tag{20b}$$

$$(\tau m_3 - k m_1)_t + \sigma_{ty} - m_{2tx} = 0 \tag{20c}$$

$$\epsilon(k m_2 - \sigma m_3)_t + \tau_{ty} - m_{1tx} = 0. \tag{20d}$$

Hence we get

$$(\sigma m_1 - \tau m_2)_t + (\sigma\omega_1 - \tau\omega_2)_y - (m_2\omega_1 - m_1\omega_2)_x = 0, \tag{21a}$$

$$(\tau m_3 - k m_1)_t + (\tau\omega_3 - k\omega_1)_y - (m_1\omega_3 - m_3\omega_1)_x = 0, \tag{21b}$$

$$(k m_2 - \sigma m_3)_t + (k\omega_2 - \sigma\omega_3)_y - (m_3\omega_2 - m_2\omega_3)_x = 0. \tag{21c}$$

So we have proved the following
Teorema: The (2+1)-dimensional nonlinear evolution equations (NLEE) or dynamical curves which are given by the mM-LXI equation have the following integrals of motions

$$K_1 = \int\int(\kappa m_2 + \sigma m_3)dxdy, \quad K_2 = \int\int(\tau m_2 + \sigma m_1)dxdy, \quad K_3 = \int\int(\tau m_3 - k m_1)dxdy \tag{22a}$$

or

$$K_1 = \int\int \mathbf{e}_1(\mathbf{e}_{1x} \wedge \mathbf{e}_{1y})dxdy \tag{22b}$$

$$K_2 = \int\int \mathbf{e}_2(\mathbf{e}_{2x} \wedge \mathbf{e}_{2y})dxdy \tag{22c}$$

$$K_3 = \int\int \mathbf{e}_3(\mathbf{e}_{3x} \wedge \mathbf{e}_{3y})dxdy. \tag{22d}$$

So we have the following three topological invariants

$$Q_1 = \frac{1}{4\pi}\int\int \mathbf{e}_1 \dot{(}\mathbf{e}_{1x} \wedge \mathbf{e}_{1y})dxdy \tag{23a}$$

$$Q_2 = \frac{1}{4\pi}\int\int \mathbf{e}_2 \dot{(}\mathbf{e}_{2x} \wedge \mathbf{e}_{2y})dxdy \tag{23b}$$

$$Q_3 = \frac{1}{4\pi}\int\int \mathbf{e}_3 \dot{(}\mathbf{e}_{3x} \wedge \mathbf{e}_{3y})dxdy \tag{23c}$$

We note that may be not all of these topological invariants are independent.

2.4 The M-LXI equation and Soliton equations in 2+1

In this section we will establish the connection between the M-LXI equation (12) and soliton equations in 2+1 dimensions. Let us, we assume

$$\mathbf{e}_1 \equiv \mathbf{S} \tag{24}$$

Moreover we introduce two complex functions q, p according to the following expressions

$$q = a_1 e^{ib_1}, \quad p = a_2 e^{ib_2} \tag{25}$$

where a_j, b_j are real functions. Now we ready to consider some examples.

2.4.1 The Ishimori equation

The Ishimori equation (IE) reads as [7]

$$\mathbf{S}_t = \mathbf{S} \wedge (\mathbf{S}_{xx} + \alpha^2 \mathbf{S}_{yy}) + u_x \mathbf{S}_y + u_y \mathbf{S}_x \tag{26a}$$

$$u_{xx} - \alpha^2 u_{yy} = -2\alpha^2 \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y). \tag{26b}$$

In this case we have

$$m_1 = \partial_x^{-1}[\tau_y - \frac{\epsilon}{2\alpha^2} M_2^{Ish} u] \tag{27a}$$

$$m_2 = -\frac{1}{2\alpha^2 k} M_2^{Ish} u \tag{27b}$$

$$m_3 = \partial_x^{-1}[k_y + \frac{\tau}{2\alpha^2 k} M_2^{Ish} u] \tag{27c}$$

and

$$\omega_1 = \frac{1}{k}[-\omega_{2x} + \tau\omega_3] \tag{28a}$$

$$\omega_2 = -k_x - \alpha^2(m_{3y} + m_2 m_1) + i m_2 u_x \tag{28b}$$

$$\omega_3 = -k\tau + \alpha^2(m_{2y} - m_3 m_1) + iku_y + i m_3 u_x. \tag{28c}$$

$$M_2^{Ish} = M_2|_{a=b=-\frac{1}{2}}.$$

Functions q, p are given by (25) with

$$a_1^2 = a_1'^2 = \frac{1}{4}k^2 + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) - \frac{1}{2}\alpha_R k m_3 - \frac{1}{2}\alpha_I k m_2 \tag{29a}$$

$$b_1 = \partial_x^{-1}\{-\frac{\gamma_1}{2ia_1'^2} - (\bar{A} - A + D - \bar{D})\} \quad (29b)$$

$$a_2^2 = a_2'^2 = \frac{1}{4}k^2 + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) + \frac{1}{2}\alpha_R k m_3 - \frac{1}{2}\alpha_I k m_2 \quad (29c)$$

$$b_2 = \partial_x^{-1}\{-\frac{\gamma_2}{2ia_2'^2} - (A - \bar{A} + \bar{D} - D)\} \quad (29d)$$

where

$$\gamma_1 = i\{\frac{1}{2}k^2\tau + \frac{|\alpha|^2}{2}(m_3 k m_1 + m_2 k_y) -$$

$$\frac{1}{2}\alpha_R(k^2 m_1 + m_3 k\tau + m_2 k_x) + \frac{1}{2}\alpha_I[k(2k_y - m_{3x}) - k_x m_3]\}. \quad (30a)$$

$$\gamma_2 = -i\{\frac{1}{2}k^2\tau + \frac{|\alpha|^2}{2}(m_3 k m_1 + m_2 k_y) +$$

$$\frac{1}{2}\alpha_R(k^2 m_1 + m_3 k\tau + m_2 k_x) + \frac{1}{2}\alpha_I[k(2k_y - m_{3x}) - k_x m_3]\}. \quad (30b)$$

Here $\alpha = \alpha_R + i\alpha_I$. In this case, q, p satisfy the following DS equation

$$iq_t + q_{xx} + \alpha^2 q_{yy} + vq = 0 \quad (31a)$$

$$-ip_t + p_{xx} + \alpha^2 p_{yy} + vp = 0 \quad (31b)$$

$$v_{xx} - \alpha^2 v_{yy} + 2[(pq)_{xx} + \alpha^2(pq)_{yy}] = 0. \quad (31c)$$

So we have proved that theIE (26) and the (31) are L-equivalent to each other. As well known that these equations are G-equivalent to each other [5]. Note that the IE contains two reductions: the Ishimori I equation as $\alpha_R = 1, \alpha_I = 0$ and the Ishimori II equation as $\alpha_R = 0, \alpha_I = 1$. The corresponding versions of the DS equation (31), we obtain as the corresponding values of the parameter α [1].

2.4.2 The Myrzakulov IX equation

Now we find the connection between the Myrzakulov IX (M-IX) equation and the curves (the M-LXI equation). The M-IX equation reads as

$$\mathbf{S}_t = \mathbf{S} \wedge M_1\mathbf{S} + A_2\mathbf{S}_x + A_1\mathbf{S}_y \quad (32a)$$

$$M_2 u = 2\alpha^2\mathbf{S}(\mathbf{S}_x \wedge \mathbf{S}_y) \quad (32b)$$

where $\alpha, b, a =$ consts and

$$M_1 = \alpha^2\frac{\partial^2}{\partial y^2} + 4\alpha(b-a)\frac{\partial^2}{\partial x\partial y} + 4(a^2 - 2ab - b)\frac{\partial^2}{\partial x^2},$$

$$M_2 = \alpha^2\frac{\partial^2}{\partial y^2} - 2\alpha(2a+1)\frac{\partial^2}{\partial x\partial y} + 4a(a+1)\frac{\partial^2}{\partial x^2},$$

$$A_1 = i\{\alpha(2b+1)u_y - 2(2ab + a + b)u_x\},$$

$$A_2 = i\{4\alpha^{-1}(2a^2 b + a^2 + 2ab + b)u_x - 2(2ab + a + b)u_y\}.$$

The M-IX equation was introduced in [1] and is integrable. It admits several integrable reductions:
1) the Ishimori equation as $a=b=-\frac{1}{2}$
2) the M-VIII equation as $a=b=-1$
and so on [1]. In this case we have

$$m_1=\partial_x^{-1}[\tau_y-\frac{\beta}{2\alpha^2}M_2u] \tag{33a}$$

$$m_2=-\frac{1}{2\alpha^2k}M_2u \tag{33b}$$

$$m_3=\partial_x^{-1}[k_y+\frac{\tau}{2\alpha^2k}M_2u] \tag{33c}$$

and

$$\omega_1=\frac{1}{k}[-\omega_{2x}+\tau\omega_3], \tag{34a}$$

$$\omega_2=-4(a^2-2ab-b)k_x-4\alpha(b-a)k_y-\alpha^2(m_{3y}+m_2m_1)+m_2A_1 \tag{34b}$$

$$\omega_3=-4(a^2-2ab-b)k\tau-4\alpha(b-a)km_1+\alpha^2(m_{2y}-m_3m_1)+kA_2+m_3A_1 \tag{34c}$$

Functions q,p are given by (25) with

$$a_1^2=\frac{|a|^2}{|b|^2}a_1'^2=\frac{|a|^2}{|b|^2}\{(l+1)^2k^2+\frac{|\alpha|^2}{4}(m_3^2+m_2^2)-(l+1)\alpha_Rkm_3-(l+1)\alpha_Ikm_2\} \tag{35a}$$

$$b_1=\partial_x^{-1}\{-\frac{\gamma_1}{2ia_1'^2}-(\bar{A}-A+D-\bar{D})\} \tag{35b}$$

$$a_2^2=\frac{|b|^2}{|a|^2}a_2'^2=\frac{|b|^2}{|a|^2}\{l^2k^2+\frac{|\alpha|^2}{4}(m_3^2+m_2^2)-l\alpha_Rkm_3+l\alpha_Ikm_2\} \tag{35c}$$

$$b_2=\partial_x^{-1}\{-\frac{\gamma_2}{2ia_2'^2}-(A-\bar{A}+\bar{D}-D) \tag{2.9a}$$

where

$$\gamma_1=i\{2(l+1)^2k^2\tau+\frac{|\alpha|^2}{2}(m_3km_1+m_2k_y)-$$
$$(l+1)\alpha_R[k^2m_1+m_3k\tau+m_2k_x]+(l+1)\alpha_I[k(2k_y-m_{3x})-k_xm_3]\} \tag{36a}$$

$$\gamma_2=-i\{2l^2k^2\tau+\frac{|\alpha|^2}{2}(m_3km_1+m_2k_y)-$$
$$l\alpha_R(k^2m_1+m_3k\tau+m_2k_x)-l\alpha_I[k(2k_y-m_{3x})-k_xm_3]\}. \tag{36b}$$

Here $\alpha=\alpha_R+i\alpha_I$. In this case, q,p satisfy the following Zakharov equation [4]

$$iq_t+M_1q+vq=0 \tag{37a}$$

$$ip_t-M_1p-vp=0 \tag{37b}$$

$$M_2v=-2M_1(pq) \tag{37c}$$

As well known the M-IX equation admits several reductions: 1) the M-IXA equation as $\alpha_R=1,\alpha_I=0$; 2) the M-IXB equation as $\alpha_R=0,\alpha_I=1$; 3) the M-VIII equation as $a=b=1$ 4) the IE $a=b=-\frac{1}{2}$ and so on. The corresponding versions of the ZE (9), we obtain as the corresponding values of the parameter α.

2.5 The modified M-LXI equation and Soliton equations in 2+1

In this section we will establish the connection between the modified M-LXI equation (14) and soliton equations in 2+1 dimensions. As above we assume

$$\mathbf{e}_1 \equiv \mathbf{S} \tag{38}$$

and

$$q = a_1 e^{ib_1}, \quad p = a_2 e^{ib_2} \tag{39}$$

where a_j, b_j are as and above, real functions. Examples.

2.5.1 The Ishimori equation

Consider the IE (26). For this equation we obtain

$$m_1 = \partial_x^{-1}[\tau_y - \frac{\beta}{2\alpha^2}M_2^{Ish}u] \tag{40a}$$

$$m_2 = \frac{\sigma}{k}m_3 - \frac{1}{2\alpha^2 k}M_2^{Ish}u \tag{40b}$$

$$m_{3x} + \frac{\tau\sigma}{k}m_3 = k_y + \sigma\partial_x^{-1}[\tau_y - \frac{\epsilon}{2\alpha^2}M_2^{Ish}u] + \frac{\tau}{2\alpha^2 k}M_2^{Ish}u \tag{40c}$$

and

$$\omega_1 = \frac{1}{k}[\sigma_t - \omega_{2x} + \tau\omega_3] \tag{41a}$$

$$\omega_2 = -(k_x + \sigma\tau) - \alpha^2(m_{3y} + m_2 m_1) + i\sigma u_y + i m_2 u_x \tag{41b}$$

$$\omega_3 = (\sigma_x - k\tau) + \alpha^2(m_{2y} - m_3 m_1) + iku_y + im_3 u_x. \tag{41c}$$

Functions q, p are given by (39) with

$$a_1^2 = a_1'^2 = \frac{1}{4}(k+\sigma^2) + \frac{|\alpha|^2}{4}(m_3^2+m_2^2) - frac12\alpha_R(km_3+\sigma m_2) - \frac{1}{2}\alpha_I(km_2+\sigma m_3) \tag{42a}$$

$$b_1 = \partial_x^{-1}\{-\frac{\gamma_1}{2ia_1'^2} - (\bar{A} - A + D - \bar{D})\} \tag{42b}$$

$$a_2^2 = a_2'^2 = \frac{1}{4}(k^2+\sigma^2) + \frac{|\alpha|^2}{4}(m_3^2+m_2^2) + \frac{1}{2}\alpha_R(km_3+\sigma m_2) - \frac{1}{2}\alpha_I(km_2+\sigma m_3)\} \tag{42c}$$

$$b_2 = \partial_x^{-1}\{-\frac{\gamma_2}{2ia_2'^2} - (A - \bar{A} + \bar{D} - D)\} \tag{42d}$$

where

$$\gamma_1 = i\{\frac{1}{2}[k(k\tau - \sigma_x) + \sigma(\sigma\tau + k_x)] + \frac{|\alpha|^2}{2}[m_3(km_1 - \sigma_y) + m_2(\sigma m_1 + k_y)] -$$
$$\frac{1}{2}\alpha_R[k(km_1 - \sigma_y) + \sigma(\sigma m_1 + k_y) + m_3(k\tau - \sigma_x) + m_2(\sigma\tau + k_x)] +$$
$$\frac{1}{2}\alpha_I[k(2k_y - m_{3x}) + \sigma(2\sigma_y - m_{2x}) - k_x m_3 - \sigma_x m_2]\} \tag{43a}$$

$$\gamma_2 = -i\{\frac{1}{2}[k(k\tau - \sigma_x) + \sigma(\sigma\tau + k_x)] + \frac{|\alpha|^2}{2}[m_3(km_1 - \sigma_y) + m_2(\sigma m_1 + k_y)] +$$
$$\frac{1}{2}\alpha_R[k(km_1 - \sigma_y) + \sigma(\sigma m_1 + k_y) + m_3(k\tau - \sigma_x) + m_2(\sigma\tau + k_x)] +$$
$$\frac{1}{2}\alpha_I[k(2k_y - m_{3x}) + \sigma(2\sigma_y - m_{2x}) - k_x m_3 - \sigma_x m_2]\}. \qquad (43b)$$

Here

$$\alpha = \alpha_R + i\alpha_I, \quad A = \frac{i}{4}[u_y - \frac{2a}{\alpha}u_x], \quad D = \frac{i}{4}[\frac{(2a+1)}{\alpha}u_x - u_y].$$

In this case, q, p satisfy the DS equation (31).

The Ishimori I and DS I equations, we get as $\alpha_R = 1, \alpha_I = 0$. The Ishimori II and DS II equations we obtain from these results as $\alpha_R = 0, \alpha_I = 1$. Details, you can find in [1].

2.5.2 The Myrzakulov IX equation

Now let us establish the connection between the M-IX equation (32) and the mM-LXI equation (14). From (32) and (14) we get

$$m_1 = \partial_x^{-1}[\tau_y - \frac{\beta}{2\alpha^2}M_2 u] \qquad (44a)$$

$$m_2 = \frac{\sigma}{k}m_3 - \frac{1}{2\alpha^2 k}M_2 u \qquad (44b)$$

$$m_{3x} + \frac{\tau\sigma}{k}m_3 = k_y + \sigma\partial_x^{-1}[\tau_y - \frac{\epsilon}{2\alpha^2}M_2 u] + \frac{\tau}{2\alpha^2 k}M_2 u \qquad (44c)$$

and

$$\omega_1 = \frac{1}{k}[\sigma_t - \omega_{2x} + \tau\omega_3], \qquad (45a)$$

$$\omega_2 = -4(a^2 - 2ab - b)(k_x + \sigma\tau) - 4\alpha(b-a)(k_y + \sigma m_1) - \alpha^2(m_{3y} + m_2 m_1) + \sigma A_2 + m_2 A_1 \qquad (45b)$$

$$\omega_3 = 4(a^2 - 2ab - b)(\sigma_x - k\tau) + 4\alpha(b-a)(\sigma_y - km_1) + \alpha^2(m_{2y} - m_3 m_1) + kA_2 + m_3 A_1 \qquad (45c)$$

Functions q, p are given by (39) with

$$a_1^2 = \frac{|a|^2}{|b|^2}a_1'^2 = \frac{|a|^2}{|b|^2}\{(l+1)^2(k+\sigma^2) + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) - (l+1)\alpha_R(km_3 + \sigma m_2) - (l+1)\alpha_I(km_2 + \sigma m_3)\} \qquad (46a)$$

$$b_1 = \partial_x^{-1}\{-\frac{\gamma_1}{2ia_1'^2} - (\bar{A} - A + D - \bar{D})\} \qquad (46b)$$

$$a_2^2 = \frac{|b|^2}{|a|^2}a_2'^2 = \frac{|b|^2}{|a|^2}\{l^2(k^2 + \sigma^2) + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) - l\alpha_R(km_3 + \sigma m_2) + l\alpha_I(km_2 + \sigma m_3)\} \qquad (46c)$$

$$b_2 = \partial_x^{-1}\{-\frac{\gamma_2}{2ia_2'^2} - (A - \bar{A} + \bar{D} - D)\} \qquad (46d)$$

where

$$\gamma_1 = i\{2(l+1)^2[k(k\tau-\sigma_x)+\sigma(\sigma\tau+k_x)]+\frac{|\alpha|^2}{2}[m_3(km_1-\sigma_y)+m_2(\sigma m_1+k_y)]-$$
$$(l+1)\alpha_R[k(km_1-\sigma_y)+\sigma(\sigma m_1+k_y)+m_3(k\tau-\sigma_x)+m_2(\sigma\tau+k_x)]+$$
$$(l+1)\alpha_I[k(2k_y-m_{3x})+\sigma(2\sigma_y-m_{2x})-k_x m_3-\sigma_x m_2]\} \qquad (47a)$$

$$\gamma_2 = -i\{2l^2[k(k\tau-\sigma_x)+\sigma(\sigma\tau+k_x)]+\frac{|\alpha|^2}{2}[m_3(km_1-\sigma_y)+m_2(\sigma m_1+k_y)]-$$
$$l\alpha_R[k(km_1-\sigma_y)+\sigma(\sigma m_1+k_y)+m_3(k\tau-\sigma_x)+m_2(\sigma\tau+k_x)]-$$
$$l\alpha_I[k(2k_y-m_{3x})+\sigma(2\sigma_y-m_{2x})-k_x m_3-\sigma_x m_2]\}. \qquad (47b)$$

Directly calculation show that q, p satisfy the ZE (37).

These results gives: 1) as $\alpha_R = 1, \alpha_I = 0$ the M-IXA equation; 2) as $\alpha_R = 0, \alpha_I = 1$ the M-IXB equation; 3) as $a = b = -\frac{1}{2}, \alpha_R = 1, \alpha_I = 0$ the Ishimori I and DS I equations; 4) as $a = b = -\frac{1}{2}, \alpha_R = 0, \alpha_I = 1$ the Ishimori II and DS II equations; 5) as $a = b - 1$ the M-VIII and corresponding Zakharov equations; and so on [1].

2.6 The M-LIX equation and Soliton equations in 2+1

Now let us consider the connection between the M-LIX equation and (2+1)-dimensional soliton equations. Mention that the M-LIX equation is one of (2+1)-dimensional extensions of the SFE (1a). As example, let us consider the connection between the M-LIX equation and the M-IX equation (32). Let the M-LIX equation has the form [1]

$$\alpha \mathbf{e}_{1y} = \frac{2a+1}{2}\mathbf{e}_{1x} + \frac{i}{2}\mathbf{e}_1 \wedge \mathbf{e}_{1x} + +i(q+p)\mathbf{e}_2 + (q-p)\mathbf{e}_3 \qquad (48a)$$

$$\alpha \mathbf{e}_{2y} = Exercise \quad N1 \qquad (48b)$$

$$\alpha \mathbf{e}_{3y} = Exercise \quad N1. \qquad (48c)$$

In terms of matrix this equation we can write in the form

$$\alpha \hat{e}_{1y} = \frac{2a+1}{2}\hat{e}_{1x} + \frac{1}{4}[\hat{e}_1, \hat{e}_{1x}] + i(q+p)\hat{e}_2 + (q-p)\hat{e}_3 \qquad (49a)$$

$$\alpha \hat{e}_{2y} = Exercise \quad N1 \qquad (49b)$$

$$\alpha \hat{e}_{3y} = Exercise \quad N1 \qquad (49c)$$

where

$$\hat{e}_1 = g^{-1}\sigma_3 g, \quad \hat{e}_2 = g^{-1}\sigma_2 g, \quad \hat{e}_3 = g^{-1}\sigma_1 g \qquad (50)$$

Here σ_j are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (51)$$

So we have

$$\sigma_1\sigma_2 = i\sigma_3 = -\sigma_2\sigma_1, \quad \sigma_1\sigma_3 = -i\sigma_2 = -\sigma_3\sigma_1, \quad \sigma_3\sigma_2 = -i\sigma_1 = -\sigma_2\sigma_3 \quad (52a)$$

and

$$\sigma_j^2 = I = diag(1,1). \quad (52b)$$

Equations (49) we can rewrite in the form

$$[\sigma_3, B_0] = i(q+p)\sigma_2 + (q-p)\sigma_1 \quad (53a)$$

$$[\sigma_2, B_0] = -i(q+p)\sigma_3 \quad (53b)$$

$$[\sigma_1, B_0] = -(q-p)\sigma_3 \quad (53c)$$

where

$$B_0 = \alpha g_y g^{-1} - B_1 g_x g^{-1}, \quad B_1 = \frac{2a+1}{2}I + \frac{1}{2}\sigma_3 \quad (54)$$

Hence we get

$$B_0 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}. \quad (55)$$

Thus the matrix-function g satisfies the equations

$$\alpha g_y = B_1 g_x + B_0 g. \quad (56)$$

To find the time evolution of matrices $\hat{e}_j$ or vectors $\mathbf{e}_j$, we require that the matrix $\hat{e}_1$ saisfy the M-IX equation, i.e.

$$i\hat{e}_{1t} = \frac{1}{2}[\hat{e}_1, M_1\hat{e}_1] + A_1\hat{e}_{1y} + A_2\hat{e}_{1x} \quad (57a)$$

$$M_2 u = \frac{\alpha^2}{2i} tr(\hat{e}_1([\hat{e}_{1x}, \hat{e}_{1y}]) \quad (57b)$$

From these informations we find the time evolution of matrices $\hat{e}_2, \hat{e}_3$. So after some algebra we obtain

$$[\sigma_3, C_0] = i(c_{12} + c_{21})\sigma_2 + (c_{12} - c_{21})\sigma_1 \quad (58a)$$

$$[\sigma_2, C_0] = i(c_{11} - c_{22})\sigma_1 - i(c_{12} + c_{21})\sigma_3 \quad (58b)$$

$$[\sigma_1, C_0] = -i(c_{11} - c_{22})\sigma_2 - (c_{1@} - c_{21})\sigma_3 \quad (58c)$$

where

$$C_0 = g_t g^{-1} - 2iC_2 g_{xx} g^{-1} - C_1 g_x g^{-1}, \quad C_2 = \frac{2b+1}{2}I + \frac{1}{2}\sigma_3, \quad C_1 = iB_0. \quad (59)$$

Hence we get

$$C_0 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \quad (60)$$

with

$$c_{12} = i(2b-a+1)q_x + i\alpha q_y, \quad c_{21} = i(a-2b)q_x - i\alpha p_y \quad (61a)$$

and c_{jj} are the solutions of the following equations

$$(a+1)c_{11x} - \alpha c_{11y} = i[(2b-a+1)(pq)_x + \alpha(pq)_y] \tag{61b}$$

$$ac_{22x} - \alpha c_{22y} = i[(a-2b)(pq)_x - \alpha(pq)_y]. \tag{61c}$$

So that the matrix g satisfies the equation

$$g_t = 2C_2 g_{xx} + C_1 g_x + C_0 g. \tag{62}$$

So we have identified the curve, given by the M-LIX equation (48) with the M-IX equation (32). On the other hand, the compatibilty condition of equations (56) and (62) is equivalent to the ZE (37). So that we have also established the connection between the curve (the M-LIX equation) and the ZE. And we have shown, once more that the M-IX equation (32) and the ZE (37) are L-equivalent to each other. Finally we note as $a = b = -\frac{1}{2}$ from these results follows the corresponding connection between the M-LIX, Ishimori and DS equations [1]. And as $a = b = -1$ we get the relation between the M-VIII, M-LIX and other Zakharov equations (for details, see [1]).

2.7 Spin systems as reductions of the M-0 equation

Consider the (2+1)-dimensional M-0 equation [1]

$$\mathbf{S}_t = a_{12}\mathbf{e}_2 + a_{13}\mathbf{e}_3, \quad \mathbf{S}_x = b_{12}\mathbf{e}_2 + b_{13}\mathbf{e}_3, \quad \mathbf{S}_y = c_{12}\mathbf{e}_2 + c_{13}\mathbf{e}_3 \tag{63}$$

where

$$\mathbf{e}_2 = \frac{c_{13}}{\triangle}\mathbf{S}_x - \frac{b_{13}}{\triangle}\mathbf{S}_y, \quad \mathbf{e}_3 = -\frac{c_{12}}{\triangle}\mathbf{S}_x + \frac{b_{12}}{\triangle}\mathbf{S}_y, \quad \triangle = b_{12}c_{13} - b_{13}c_{12}. \tag{64}$$

All known spin systems (integrable and nonintegrable) in 2+1 dimensions are the particular reductions of the M-0 equation (63). In particular, the IE (26) is the integrable reduction of equation (63). In this case, we have

$$a_{12} = \omega_3, a_{13} = -\omega_2, b_{12} = k, b_{13} = -\sigma, c_{12} = m_3, c_{13} = -m_2. \tag{65}$$

Sometimes we use the following form of the M-0 equation [1]

$$\mathbf{S}_t = d_2\mathbf{S}_x + d_3\mathbf{S}_y \tag{66}$$

with

$$d_2 = \frac{a_{12}c_{13} - a_{13}c_{12}}{\triangle}, \quad d_3 = \frac{a_{12}b_{13} - a_{13}b_{12}}{\triangle}. \tag{67}$$

3 Surfaces and Solitons in 2+1

3.1 The M-LVIII equation and Soliton equationa in 2+1

In the C-approach [1], our starting point is the following (2+1)-dimensional M-LVIII equation [1]

$$\mathbf{r}_t = \Upsilon_1\mathbf{r}_x + \Upsilon_2\mathbf{r}_y + \Upsilon_3\mathbf{n} \tag{68a}$$

$$\mathbf{r}_{xx} = \Gamma^1_{11}\mathbf{r}_x + \Gamma^2_{11}\mathbf{r}_y + L\mathbf{n} \tag{68b}$$
$$\mathbf{r}_{xy} = \Gamma^1_{12}\mathbf{r}_x + \Gamma^2_{12}\mathbf{r}_y + M\mathbf{n} \tag{68c}$$
$$\mathbf{r}_{yy} = \Gamma^1_{22}\mathbf{r}_x + \Gamma^2_{22}\mathbf{r}_y + N\mathbf{n} \tag{68d}$$
$$\mathbf{n}_x = p_{11}\mathbf{r}_x + p_{12}\mathbf{r}_y \tag{68e}$$
$$\mathbf{n}_y = p_{21}\mathbf{r}_x + p_{22}\mathbf{r}_y. \tag{68f}$$

This equation admits several integrable reductions. Practically, all integrable spin systems in 2+1 dimensions are some integrable reductions of the M-LVIII equation (68).

3.2 The M-LXIII equation and Soliton equationa in 2+1

Sometimes it is convenient to work using the B-approach. In this approach the starting equation is the following M-LXIII equation [1]

$$\mathbf{r}_{tx} = \Gamma^1_{01}\mathbf{r}_x + \Gamma^2_{01}\mathbf{r}_y + \Gamma^3_{01}\mathbf{n} \tag{69a}$$
$$\mathbf{r}_{ty} = \Gamma^1_{02}\mathbf{r}_x + \Gamma^2_{02}\mathbf{r}_y + \Gamma^3_{02}\mathbf{n} \tag{69b}$$
$$\mathbf{r}_{xx} = \Gamma^1_{11}\mathbf{r}_x + \Gamma^2_{11}\mathbf{r}_y + L\mathbf{n} \tag{69c}$$
$$\mathbf{r}_{xy} = \Gamma^1_{12}\mathbf{r}_x + \Gamma^2_{12}\mathbf{r}_y + M\mathbf{n} \tag{69d}$$
$$\mathbf{r}_{yy} = \Gamma^1_{22}\mathbf{r}_x + \Gamma^2_{22}\mathbf{r}_y + N\mathbf{n} \tag{69e}$$
$$\mathbf{n}_t = p_{01}\mathbf{r}_x + p_{02}\mathbf{r}_y \tag{69f}$$
$$\mathbf{n}_x = p_{11}\mathbf{r}_x + p_{12}\mathbf{r}_y \tag{69g}$$
$$\mathbf{n}_y = p_{21}\mathbf{r}_x + p_{22}\mathbf{r}_y. \tag{69h}$$

This equation follows from the M-LVIII equation (68) under the following conditions

$$\Gamma^1_{01} = \Upsilon_{1x} + \Upsilon_1\Gamma^1_{11} + \Upsilon_2\Gamma^1_{12} + \Upsilon_3 p_{11}$$
$$\Gamma^1_{02} = \Upsilon_{2x} + \Upsilon_1\Gamma^2_{11} + \Upsilon_2\Gamma^2_{12} + \Upsilon_3 p_{12}$$
$$\Gamma^1_{03} = \Upsilon_{3x} + \Upsilon_1 L + \Upsilon_2 M$$
$$p_{01} = \frac{F\Gamma^3_{02}}{\Lambda}, \quad p_{02} = -\frac{E\Gamma^3_{02}}{\Lambda}, \quad \Lambda = EG - F^2 \tag{70}$$

Note that the M-LXIII equation (69) usually we use in the following form

$$Z_x = AZ \tag{71a}$$
$$Z_y = BZ \tag{71b}$$
$$Z_t = CZ \tag{71c}$$

where $Z = (\mathbf{r}_x, \mathbf{r}_y, \mathbf{n})^t$ and

$$A = \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} & L \\ \Gamma^1_{12} & \Gamma^2_{12} & M \\ p_{11} & p_{12} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \Gamma^1_{12} & \Gamma^2_{12} & M \\ \Gamma^1_{22} & \Gamma^2_{22} & N \\ p_{21} & p_{22} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \Gamma^1_{01} & \Gamma^2_{01} & \Gamma^3_{01} \\ \Gamma^1_{02} & \Gamma^2_{02} & \Gamma^3_{02} \\ \Gamma^1_{03} & \Gamma^2_{03} & 0 \end{pmatrix}. \tag{72}$$

3.3 The M-LXIV equation

In this subsection we derive the M-LXIV equation, which express some relations between coefficients of the M-LXIII equation (69) or (71). From (71) we have

$$A_y - B_x + [A, B] = 0 \tag{73a}$$

$$A_t - C_x + [A, C] = 0 \tag{73b}$$

$$B_t - C_y + [B, C] = 0 \tag{73c}$$

It is the M-LXIV equation. These equations are equivalent the relations

$$\mathbf{r}_{yxx} = \mathbf{r}_{xxy}, \quad \mathbf{r}_{yyx} = \mathbf{r}_{xyy} \tag{74a}$$

$$\mathbf{r}_{txx} = \mathbf{r}_{xxt}, \quad \mathbf{r}_{txy} = \mathbf{r}_{xyt}, \quad \mathbf{r}_{tyy} = \mathbf{r}_{yyt}. \tag{74b}$$

Note that (73a) is the well known Codazzi-Mainardi-Peterson equation (CMPE).

3.4 Orthogonal basis and LR of the M-LXIV equation

Let us introduce the orthogonal trihedral

$$\mathbf{e}_1 = \frac{\mathbf{r}_x}{\sqrt{E}}, \quad \mathbf{e}_2 = \mathbf{n}, \quad \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2. \tag{75}$$

Let $\mathbf{e}_1^2 = \beta = \pm 1, \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1$. Then these vectors satisfy the following equations

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = \frac{1}{\sqrt{E}} \begin{pmatrix} 0 & L & -\frac{\Lambda}{\sqrt{E}}\Gamma_{11}^2 \\ -\beta L & 0 & -\Lambda p_{12} \\ \frac{\beta\Lambda}{\sqrt{E}}\Gamma_{11}^2 & \Lambda p_{12} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{76a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = \frac{1}{\sqrt{E}} \begin{pmatrix} 0 & M & -\frac{\Lambda}{\sqrt{E}}\Gamma_{12}^2 \\ -\beta M & 0 & -\Lambda p_{22} \\ \frac{\beta\Lambda}{\sqrt{E}}\Gamma_{12}^2 & \Lambda p_{22} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{76b}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \frac{1}{\sqrt{E}} \begin{pmatrix} 0 & \Gamma_{01}^3 & -\frac{\Lambda}{\sqrt{E}}\Gamma_{01}^2 \\ -\beta\Gamma_{01}^3 & 0 & -\Lambda\Gamma_{03}^2 \\ \frac{\beta\Lambda}{\sqrt{E}}\Gamma_{01}^2 & \Lambda\Gamma^2 03 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \tag{76c}$$

The matrix form of this equation is

$$\hat{e}_{1x} = \frac{1}{\sqrt{E}}(L\hat{e}_2 - \frac{\Lambda}{\sqrt{E}}\Gamma_{11}^2\hat{e}_3) \tag{77a}$$

$$\hat{e}_{2x} = \frac{1}{\sqrt{E}}(-\beta L\hat{e}_1 - \Lambda p_{12}\hat{e}_3) \tag{77b}$$

$$\hat{e}_{3x} = \frac{1}{\sqrt{E}}(\frac{\beta\Lambda}{\sqrt{E}}\Gamma_{11}^2\hat{e}_1 + \Lambda p_{12}\hat{e}_2) \tag{77c}$$

$$\hat{e}_{1y} = \frac{1}{\sqrt{E}}(M\hat{e}_2 - \frac{\Lambda}{\sqrt{E}}\Gamma_{12}^2\hat{e}_3) \tag{78a}$$

$$\hat{e}_{2y}=\frac{1}{\sqrt{E}}(-\beta M\hat{e}_1-\Lambda p_{22}\hat{e}_3) \tag{78b}$$

$$\hat{e}_{3y}=\frac{1}{\sqrt{E}}(\frac{\beta\Lambda}{\sqrt{E}}\Gamma^2_{12}\hat{e}_1+\Lambda p_{22}\hat{e}_2) \tag{78c}$$

$$\hat{e}_{1t}=\frac{1}{\sqrt{E}}(\Gamma^3_{01}\hat{e}_2-\frac{\Lambda}{\sqrt{E}}\Gamma^2_{01}\hat{e}_3) \tag{79a}$$

$$\hat{e}_{2t}=\frac{1}{\sqrt{E}}(-\beta\Gamma^3_{01}\hat{e}_1-\Lambda\Gamma^2_{03}\hat{e}_3) \tag{79b}$$

$$\hat{e}_{3t}=\frac{1}{\sqrt{E}}(\frac{\beta\Lambda}{\sqrt{E}}\Gamma^2_{01}\hat{e}_1+\Lambda\Gamma^2 03\hat{E}_2 \tag{79c}$$

where

$$\hat{e}_1=g^{-1}\sigma_3 g, \quad \hat{e}_2=g^{-1}\sigma_2 g, \quad \hat{e}_3=g^{-1}\sigma_1 g. \tag{80}$$

Equations (77-79) we can rewrite in the form

$$[\sigma_3,U]=\frac{1}{\sqrt{E}}(L\sigma_2-\frac{\Lambda}{\sqrt{E}}\Gamma^2_{11}\sigma_1) \tag{81a}$$

$$[\sigma_2,U]=\frac{1}{\sqrt{E}}(-\beta L\sigma_3-\Lambda p_{12}\sigma_1) \tag{81b}$$

$$[\sigma_1,U]=\frac{1}{\sqrt{E}}(\frac{\beta\Lambda}{\sqrt{E}}\Gamma^2_{11}\sigma_3+\Lambda p_{12}\sigma_2) \tag{81c}$$

$$[\sigma_3,V]=\frac{1}{\sqrt{E}}(M\sigma_2-\frac{\Lambda}{\sqrt{E}}\Gamma^2_{12}\sigma_1) \tag{82a}$$

$$[\sigma_2,V]=\frac{1}{\sqrt{E}}(-\beta M\sigma_3-\Lambda p_{22}\sigma_1) \tag{82b}$$

$$[\sigma_1,V]=\frac{1}{\sqrt{E}}(\frac{\beta\Lambda}{\sqrt{E}}\Gamma^2_{12}\sigma_3+\Lambda p_{22}\sigma_2) \tag{82c}$$

$$[\sigma_3,W]=\frac{1}{\sqrt{E}}(\Gamma^3_{01}\sigma_2-\frac{\Lambda}{\sqrt{E}}\Gamma^2_{01}\sigma_1) \tag{83a}$$

$$[\sigma_2,W]=\frac{1}{\sqrt{E}}(-\beta\Gamma^3_{01}\sigma_3-\Lambda\Gamma^2_{03}\sigma_1) \tag{83b}$$

$$[\sigma_1,W]=\frac{1}{\sqrt{E}}(\frac{\beta\Lambda}{\sqrt{E}}\Gamma^2_{01}\sigma_3+\Lambda\Gamma^2_{03}\sigma_2 \tag{83c}$$

where

$$U=g_x g^{-1}, \quad V=g_y g^{-1}, \quad W=g_t g^{-1} \tag{84}$$

Hence we get

$$U=\frac{1}{2i\sqrt{E}}\left(\begin{array}{cc} -\sqrt{\Lambda}p_{12} & L+i\sqrt{\frac{\Lambda}{E}}\Gamma^2_{11} \\ L-i\sqrt{\frac{\Lambda}{E}}\Gamma^2_{11} & \sqrt{\Lambda}p_{12} \end{array}\right) \tag{85a}$$

$$V=\frac{1}{2i\sqrt{E}}\left(\begin{array}{cc} -\sqrt{\Lambda}p_{22} & M-i\sqrt{\frac{\Lambda}{E}}\Gamma^2_{12} \\ M+i\sqrt{\frac{\Lambda}{E}}\Gamma^2_{12} & \sqrt{\Lambda}p_{22} \end{array}\right) \tag{85b}$$

$$W=\frac{1}{2i\sqrt{E}}\left(\begin{array}{cc}-\sqrt{\Lambda}\Gamma_{03}^{2} & \Gamma_{01}^{3}-i\sqrt{\frac{\Lambda}{E}}\Gamma_{01}^{2}\\ \Gamma_{01}^{3}+i\sqrt{\frac{\Lambda}{E}}\Gamma_{01}^{2} & \sqrt{\Lambda}\Gamma_{03}^{2}\end{array}\right). \tag{85c}$$

Thus the matrix-function g satisfies the equations

$$g_x=Ug, \quad g_y=Vg, \quad g_t=Wg. \tag{86}$$

From these equations follow

$$U_y-V_x+[U,V]=0 \tag{87a}$$

$$U_t-W_x+[U,W]=0 \tag{87b}$$

$$V_t-W_y+[V,W]=0 \tag{87c}$$

This equation is the M-LXIV equation. Equation (87a) is the CMPE. Note that the M-LXIII equation in the form (76) have the same form with the mM-LXI equation (14) with the following identifications

$$k=\frac{L}{\sqrt{E}}, \quad \sigma=\frac{\Lambda}{E}\Gamma_{11}^{2}, \quad \tau=-\frac{\Lambda}{\sqrt{E}}p_{12} \tag{88a}$$

$$m_1=-\frac{\Lambda}{\sqrt{E}}p_{22}, \quad m_2=\frac{\Lambda}{E}\Gamma_{12}^{2}, \quad m_3=\frac{M}{\sqrt{E}} \tag{88b}$$

$$\omega_1=-\frac{1}{\sqrt{E}}\Lambda\Gamma_{03}^{2}, \quad \omega_2=\frac{\Lambda}{E}\Gamma_{03}^{2}, \quad \omega_3=\frac{1}{\sqrt{E}}\Gamma_{01}^{3} \tag{88c}$$

4 Self-cordination of the geometrical formalism and Hirota's bilinear method

The main goal of this section is the establishment self-coordination of the our geometrical formalism that presented above with the other powerful tool of soliton theory - the Hirota's bilinear method. We demonstrate our idea in some examples. Usually, for the spin vector $\mathbf{S}=(S_1,S_2,S_3)$ take the following transformation

$$S^{+}=S_1+iS_2=\frac{2\bar{f}g}{\bar{f}f+\bar{g}g}, \quad S_3=\frac{\bar{f}f-\bar{g}g}{\bar{f}f+\bar{g}g}. \tag{89}$$

Also n this section, we assume

$$\mathbf{S}=\mathbf{e}_1 \tag{90}$$

Now consider examples.

4.1 The Ishimori equation

It is well known that for the IE (26) the bilinear representation has the form

$$u_x = -2i\alpha^2 \frac{D_y(\bar{f} \circ f + \bar{g} \circ g)}{\bar{f} f + \bar{g} g}, \quad u_y = -2i \frac{D_x(\bar{f} \circ f + \bar{g} \circ g)}{\bar{f} f + \bar{g} g} \tag{91}$$

Then the IE (26) is transformed into the bilinear equations [7]

$$(iD_t - D_x^2 - \alpha^2 D_y^2)(\bar{f} \circ f - \bar{g} \circ g) = 0 \tag{92a}$$

$$(iD_t - D_x^2 - \alpha^2 D_y^2)\bar{f} \circ g = 0. \tag{92b}$$

Plus the additional condition, which follows from the condtion

$$u_{xy} = u_{yx} \tag{93}$$

Now we assume that

$$\tau = \frac{1}{2} u_y, \quad m_1 = \frac{1}{2\alpha^2} u_x \tag{94}$$

Then, the second equation of the IE (26b) has the same form with the third equation of then mM-LXII equation (17c). So, we get

$$e_1^+ = \frac{2\bar{f} g}{\Lambda}, \quad e_{13} = \frac{\bar{f} f - \bar{g} g}{\Lambda} \tag{95a}$$

$$\tau = -i \frac{D_x(\bar{f} \circ f + \bar{g} \circ g)}{\Lambda}, \quad m_1 = -i \frac{D_y(\bar{f} \circ f + \bar{g} \circ g)}{\Lambda} \tag{95b}$$

Similarly, after some algebra we obtain

$$e_2^+ = i \frac{\bar{f}^2 + \bar{g}^2}{\Lambda}, \quad e_{23} = i \frac{fg - \bar{f}\bar{g}}{\Lambda}, \quad e_3^+ = \frac{\bar{f}^2 - \bar{g}^2}{\Lambda}, \quad e_{33} = -\frac{fg + \bar{f}\bar{g}}{\Lambda} \tag{96}$$

and

$$k = -i \frac{D_x(g \circ f - \bar{g} \circ \bar{f})}{\Lambda}, \quad \sigma = -i \frac{D_x(g \circ f + \bar{g} \circ \bar{f})}{\Lambda}, \tag{97a}$$

$$m_2 = -i \frac{D_y(g \circ f + \bar{g} \circ \bar{f})}{\Lambda} \quad m_3 = -i \frac{D_y(g \circ f + \bar{g} \circ \bar{f})}{\Lambda} \tag{97b}$$

Here $\mathbf{e}_j = (e_{j1}, e_{j2}, e_{j3}), \quad e_j^{\pm} = e_{j1} \pm i e_{j2}$.

4.2 The M-I equation

Let us now consider the Myrzakulov I (M-I) equation, which looks like [1]

$$\mathbf{S}_t = (\mathbf{S} \wedge \mathbf{S}_y + u\mathbf{S})_x \tag{98a}$$

$$u_x = -\mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y). \tag{98b}$$

To this equation we take

$$\tau = 0, \quad m_1 = u \tag{99}$$

Then equations (17c) and (98b) have the same form. From (95) and (99) follow

$$D_x(\bar{f} \circ f + \bar{g} \circ g) = 0 \tag{100}$$

$$u = -i \frac{D_y(\bar{f} \circ f + \bar{g} \circ g)}{\Lambda} \tag{101}$$

4.3 The M-IX equation

In this case, we take

$$\tau = \frac{1}{2\alpha}[\alpha u_y - (2a+1)u_x], \quad m_1 = \frac{1}{2\alpha^2}[\alpha((2a+1)u_y - 4a(a+1)u_x] \quad (102)$$

So, for potential we have

$$u_x = 2i\alpha(2a+1)\frac{D_x(\bar{f}\circ f + \bar{g}\circ g)}{\Lambda} - 2i\alpha^2\frac{D_y(\bar{f}\circ f + \bar{g}\circ g)}{\Lambda} \quad (103a)$$

$$u_y = 8ia(a+1)\frac{D_x(\bar{f}\circ f + \bar{g}\circ g)}{\Lambda} - 2i\alpha(2a+1)\frac{D_y(\bar{f}\circ f + \bar{g}\circ g)}{\Lambda} \quad (103b)$$

5 Supersymmetry, geometry and soliton equations

In this section we establish a connection between geometry and supersymmetric (susy) soliton equations. As example we consider the susy generalizations of NLSE (1) and LLE (9). To this purpose, first we must construct a susy extensions of the SFE (9). Simple example of such extensions is the OSP(2|1) M-LXV equation [1]. It is convenient to work with the matrix form of the OSP(2|1) M-LXV equation, which we write in the form [1]

$$\hat{e}_{1x} = 2q\hat{e}_2 - 2p\hat{e}_3 + \beta\hat{e}_4 - \epsilon\hat{e}_5 \quad (104a)$$

$$\hat{e}_{2x} = pe_1 - 2i\lambda\hat{e}_2 + \epsilon\hat{e}_4 \quad (104b)$$

$$\hat{e}_{3x} = -qe_1 + 2i\lambda\hat{e}_3 + \beta\hat{e}_5 \quad (104c)$$

$$\hat{e}_{4x} = \epsilon e_1 - 2\beta\hat{e}_2 - i\lambda\hat{e}_4 - p\hat{e}_5 \quad (104d)$$

$$\hat{e}_{5x} = -\beta e_1 + 2\epsilon\hat{e}_2 - q\hat{e}_4 + i\lambda\hat{e}_5 \quad (104e)$$

Here, $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are bosonic matrices, $\hat{e}_4, \hat{e}_5$ are fermionic matrices, $p(q) = p(p) = 0, p(\beta) = p(\epsilon) = 1$ and

$$\hat{e}_1 = g^{-1}l_1g, \quad \hat{e}_2 = g^{-1}l_2g, \quad \hat{e}_3 = g^{-1}l_3g, \quad \hat{e}_4 = g^{-1}l_4g, \quad \hat{e}_5 = g^{-1}l_5g \quad (105)$$

Generators of the supergroup OSP(2|1) have the forms

$$l_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad l_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad l_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$l_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad l_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (106)$$

These generators satisfy the following commutation relations

$$[l_1, l_2] = 2l_2, \quad [l_1, l_3] = 2l_3, \quad [l_2, l_3] = l_1, \quad [l_1, l_4] = l_4, \quad [l_1, l_5] = -l_5$$

$$[l_2, l_4] = 0, \quad [l_2, l_5] = l_4, \quad [l_3, l_4] = l_5, \quad [l_3, l_5] = 0$$

$$\{l_4, l_4\} = -2l_2, \quad \{l_4, l_5\} = l_1, \{l_5, l_5\} = 2l_3 \tag{107}$$

From (9) follows

$$[l_1, U] = 2ql_2 - 2pl_3 + \beta l_4 - \epsilon l_5 \tag{108a}$$

$$[l_2, U] = pl_1 - 2i\lambda l_2 + \epsilon l_4 \tag{108b}$$

$$[l_3, U] = -ql_1 + 2i\lambda l_3 + \beta l_5 \tag{108c}$$

$$[l_4, U] = \epsilon l_1 - 2\beta l_2 - i\lambda l_4 - pl_5 \tag{108d}$$

$$[l_5, U] = -\beta l_1 + 2\epsilon l_2 - ql_4 + i\lambda l_5 \tag{108e}$$

where

$$g_x g^{-1} = U \tag{109}$$

Hence we get

$$U = i\lambda l_1 + ql_2 + pl_3 + \beta l_4 + \epsilon l_5 \tag{110}$$

Now we consider the (1+1)-dimensional M-V equation [1]

$$iR_t = \frac{1}{2}[R, R_{xx}] + \frac{3}{2}[R^2, (R^2)_{xx}] \tag{111}$$

Here $R \in osp(2|1)$, i.e it has the form

$$R = \begin{pmatrix} S_3 & S^- & \gamma_1 \\ S^+ & -S_3 & \gamma_2 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix} \tag{112}$$

and satisfies the condition

$$R^3 = R \tag{113a}$$

or in elements

$$S_3^2 + S^+S^- + 2\gamma_1\gamma_2 = 1. \tag{113b}$$

Here S_{ij} are bosonic functions and γ_j are fermionic functions, i.e. $p(S_{ij}) = 0, p(\gamma_j) = 1$. The M-V equation is the simplest supersymmetric generalization of the LLE (9) on group OSP(2|1). It admits two reductions: the UOSP(2|1) M-V equation and the UOSP(1,1|1) M-V equation [1]. As was established in [1], the gauge equivalent counterparts of the M-V equation (9) is the OSP(2|1) NLSE [8,9]. In [10] was studied the UOSP(1,1|1) M-V equation.

The LR of the M-V equation has the form [1]

$$\psi_x = U'\psi, \tag{114a}$$

$$\psi_t = V'\psi \tag{114b}$$

with

$$U' = i\lambda R, \tag{115a}$$

$$V' = 2i\lambda^2 R + \frac{3\lambda}{2}[R^2, (R^2)_x]. \tag{115b}$$

Now let us return to the our supercurves. To find the time evolution of this supercurves for the OSP(2|1) group case, we assume that

$$\hat{e}_1 \equiv R \tag{116}$$

Then $\hat{e}_1$ satisfies the the M-V equation, i.e

$$i\hat{e}_{1t} = \frac{1}{2}[\hat{e}_1, \hat{e}_{1xx}] + \frac{3}{2}[\hat{e}_1^2, (\hat{e}_1^2)_{xx}] \tag{117}$$

Now we are in position to write the time evolution of $\hat{e}_j$. We have

$$\hat{e}_{1t} = 2\lambda\hat{e}_{1x} - 2iq_x\hat{e}_2 - 2ip_x\hat{e}_3 - 2i\beta_x\hat{e}_4 - 2i\epsilon_x\hat{e}_5 \tag{118a}$$

$$\hat{e}_{2t} = 2\lambda\hat{e}_{2x} - 2i(pq + 2\beta\epsilon)\hat{e}_2 + ip_x\hat{e}_2 + 2i\epsilon_x\hat{e}_4 \tag{118b}$$

$$\hat{e}_{3t} = 2\lambda\hat{e}_{3x} + 2i(pq + 2\beta\epsilon)\hat{e}_3 + ir_x\hat{e}_1 - 2i\beta_x\hat{e}_5 \tag{118c}$$

$$\hat{e}_{4t} = 2\lambda\hat{e}_{4x} + 2i\epsilon_x\hat{e}_1 + 4i\beta_x\hat{e}_2 - i(pq + 2\beta\epsilon)\hat{e}_4 - ip_x\hat{e}_5 \tag{118d}$$

$$\hat{e}_{5t} = 2\lambda\hat{e}_{5x} - 2i\beta_x\hat{e}_1 + 4i\epsilon_x\hat{e}_3 + ir_x\hat{e}_4 + i(pq + 2\beta\epsilon)\hat{e}_5 \tag{118e}$$

Hence we obtain

$$[l_1, V - 2\lambda U] = -2iq_xl_2 - 2ip_xl_3 - 2i\beta_xl_4 - 2i\epsilon_xl_5 \tag{119a}$$

$$[l_2, V - 2\lambda U] = -2i(pq + 2\beta\epsilon)l_2 + ip_xl_2 + 2i\epsilon_xl_4 \tag{119b}$$

$$[l_3, V - 2\lambda U] = 2i(pq + 2\beta\epsilon)l_3 + ir_xl_1 - 2i\beta_xl_5 \tag{119c}$$

$$[l_4, V - 2\lambda U] = 2i\epsilon_xl_1 + 4i\beta_xl_2 - i(pq + 2\beta\epsilon)l_4 - ip_xl_5 \tag{119d}$$

$$[l_5, V - 2\lambda U] = -2i\beta_xl_1 + 4i\epsilon_xl_3 + ir_xl_4 + i(pq + 2\beta\epsilon)l_5 \tag{119e}$$

where

$$g_tg^{-1} = V \tag{120}$$

From (9) follows

$$V = 2\lambda U + i(pq + 2\beta\epsilon)l_1 - iq_xl_2 + ip_xl_3 - 2i\beta_xl_4 + 2i\epsilon_xl_5 \tag{121}$$

So for g we have the following set of the linear equations ce we obtain

$$g_x = Ug \tag{122a}$$

$$g_t = Vg \tag{122b}$$

The combatibility condition of these equations gives

$$iq_t + q_{xx} - 2rq^2 - 4q\beta\epsilon - 4\epsilon\epsilon_x = 0, \tag{123a}$$

$$ir_t - r_{xx} + 2qr^2 + 4r\beta\epsilon - 4\beta\beta_x = 0, \tag{123b}$$

$$i\epsilon_t + 2\epsilon_{xx} + 2q\beta_x + q_x\beta - \epsilon rq = 0, \tag{123c}$$

$$i\beta_t - 2\beta_{xx} - 2r\epsilon_x - r_x\epsilon + \beta rq = 0, \tag{123d}$$

It is the OSP(2|1) NLSE [8,9]. So we have proved that the M-V equation and the OSP(2|1) NLSE are equivalent to each other in geometrical sense.

6 Conclusion

To conclude, in this paper, starting from Lakshmanan's idea [2] we have discussed some aspects of the relation between differential geometry of curves/surfaces and soliton equations in 2+1 dimensions. Also we presented our point of view on the connection between geometry of curves and supersymmetric soliton equations. The self-cordination of geometry and Hirota's bilinear method is established.

Finally, we would like note that the above presented results are rather the formulation of problems than their solutions. The further studies of these problems seem to be very interesting. In this connection, I would like ask you, if you have or will have any results in these or close directions, dear colleaque, please inform me. Also any comments and questions are welcome.

7 Exercises

Finishing we also would like to pose the following particular questions as exercises:
Exercise N1: Write the vector form of the M-LIX equation.
Exercise N2: Write the vector form of the M-LXIV equation.
Exercise N3: Find a surface corresponding to the M-LIX equation.
Exercise N4: Find the integrable reductions of the M-LVIII.
Exercise N5: Find the integrable reductions of the M-LXIII.
Exercise N6: Find the integrable reductions of the M-LXV.
Exercise N7: As well known the M-XXXIV equation (9) is integrable. Find the other integrable equations among spin - phonon systems (3)-(9).
Exercise N8: Study the following version of the M-LIX equation

$$\alpha \mathbf{e}_{1y} = \frac{2a+1}{2}\mathbf{e}_{1x} + \frac{i}{2}\mathbf{e}_1 \wedge \mathbf{e}_{1x} + +c\mathbf{e}_2 - d\mathbf{e}_3 \tag{124a}$$

$$\alpha \mathbf{e}_{2y} = \frac{2a+1}{2}\mathbf{e}_{2x} + \frac{i}{2}\mathbf{e}_2 \wedge \mathbf{e}_{2x} + -c\mathbf{e}_1 + n\mathbf{e}_3 \tag{124b}$$

$$\alpha \mathbf{e}_{3y} = \frac{2a+1}{2}\mathbf{e}_{3x} + \frac{i}{2}\mathbf{e}_3 \wedge \mathbf{e}_{3x} + d\mathbf{e}_1 - n\mathbf{e}_2 \tag{124c}$$

Exercise N9: Find the physical applications of the above presented equations and spin-phonon systems from Appendix.

8 Appendix: Spin - phonon systems

Here we wish present some spin-phonon systems, which describe the nonlinear dynamics of compressible magnets [1]. May be some of these equation are integrable. For example, the M-XXXIV equation is integrable.

8.1 The 0-class

The M-LVII equation:

$$2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3] \tag{125}$$

The M-LVI equation:

$$2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3] \tag{126}$$

The M-LV equation:

$$2iS_t = \{(\mu \vec{S}_x^2 - u + m)[S, S_x]\}_x + h[S, \sigma_3] \tag{127}$$

The M-LIV equation:

$$2iS_t = n[S, S_{xxxx}] + 2\{(\mu \vec{S}_x^2 - u + m)[S, S_x]\}_x + h[S, \sigma_3] \tag{128}$$

The M-LIII equation:

$$2iS_t = [S, S_{xx}] + 2iuS_x \tag{129}$$

where $v_0, \mu, \lambda, n, m, a, b, \alpha, \beta, \rho, h$ are constants, u is scalar potential,

$$S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix}, \quad S^{\pm} = S_1 \pm iS_2, \quad r^2 = \pm 1 \quad S^2 = I.$$

8.2 The 1-class

The M-LII equation:

$$2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3] \tag{130a}$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda (S_3)_{xx} \tag{130b}$$

The M-LI equation:

$$2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3] \tag{131a}$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha (u^2)_{xx} + \beta u_{xxxx} + \lambda (S_3)_{xx} \tag{131b}$$

The M-L equation:

$$2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3] \tag{132a}$$

$$u_t + u_x + \lambda (S_3)_x = 0 \tag{132b}$$

The M-XLIX equation:

$$2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3] \tag{133a}$$

$$u_t + u_x + \alpha (u^2)_x + \beta u_{xxx} + \lambda (S_3)_x = 0 \tag{133b}$$

8.3 The 2-class

The M-XLVIII equation:

$$2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3] \tag{134a}$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(S_3^2)_{xx} \tag{134b}$$

The M-XLVII equation:

$$2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3] \tag{135a}$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S_3^2)_{xx} \tag{135b}$$

The M-XLVI equation:

$$2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3] \tag{136a}$$

$$u_t + u_x + \lambda(S_3^2)_x = 0 \tag{136b}$$

The M-XLV equation:

$$2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3] \tag{137a}$$

$$u_t + u_x + \alpha(u^2)_x + \beta u_{xxx} + \lambda(S_3^2)_x = 0 \tag{137b}$$

8.4 The 3-class

The M-XLIV equation:

$$2iS_t = \{(\mu\vec{S}_x^2 - u + m)[S, S_x]\}_x \tag{138a}$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(\vec{S}_x^2)_{xx} \tag{138b}$$

The M-XLIII equation:

$$2iS_t = \{(\mu\vec{S}_x^2 - u + m)[S, S_x]\}_x \tag{139a}$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(\vec{S}_x^2)_{xx} \tag{139b}$$

The M-XLII equation:

$$2iS_t = \{(\mu\vec{S}_x^2 - u + m)[S, S_x]\}_x \tag{140a}$$

$$u_t + u_x + \lambda(\vec{S}_x^2)_x = 0 \tag{140b}$$

The M-XLI equation:

$$2iS_t = \{(\mu\vec{S}_x^2 - u + m)[S, S_x]\}_x \tag{141a}$$

$$u_t + u_x + \alpha(u^2)_x + \beta u_{xxx} + \lambda(\vec{S}_x^2)_x = 0 \tag{141b}$$

8.5 The 4-class

The M-XL equation:

$$2iS_t = [S, S_{xxxx}] + 2\{((1+\mu)\vec{S}_x^2 - u + m)[S, S_x]\}_x \tag{142a}$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(\vec{S}_x^2)_{xx} \tag{142b}$$

The M-XXXIX equation:

$$2iS_t = [S, S_{xxxx}] + 2\{((1+\mu)\vec{S}_x^2 - u + m)[S, S_x]\}_x \tag{143a}$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(\vec{S}_x^2)_{xx} \tag{143b}$$

The M-XXXVIII equation:

$$2iS_t = [S, S_{xxxx}] + 2\{((1+\mu)\vec{S}_x^2 - u + m)[S, S_x]\}_x \tag{144a}$$

$$u_t + u_x + \lambda(\vec{S}_x^2)_x = 0 \tag{144b}$$

The M-XXXVII equation:

$$2iS_t = [S, S_{xxxx}] + 2\{((1+\mu)\vec{S}_x^2 - u + m)[S, S_x]\}_x \tag{145a}$$

$$u_t + u_x + \alpha(u^2)_x + \beta u_{xxx} + \lambda(\vec{S}_x^2)_x = 0 \tag{145b}$$

8.6 The 5-class

The M-XXXVI equation:

$$2iS_t = [S, S_{xx}] + 2iuS_x \tag{146a}$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(f)_{xx} \tag{146b}$$

The M-XXXV equation:

$$2iS_t = [S, S_{xx}] + 2iuS_x \tag{147a}$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(f)_{xx} \tag{147b}$$

The M-XXXIV equation:

$$2iS_t = [S, S_{xx}] + 2iuS_x \tag{148a}$$

$$u_t + u_x + \lambda(f)_x = 0 \tag{148b}$$

The M-XXXIII equation:

$$2iS_t = [S, S_{xx}] + 2iuS_x \tag{149a}$$

$$u_t + u_x + \alpha(u^2)_x + \beta u_{xxx} + \lambda(f)_x = 0 \tag{149b}$$

Here $f = \frac{1}{4}tr(S_x^2), \quad \lambda = 1.$

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