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4D STATIC SOLUTIONS WITH INTERACTING PHANTOM FIELDS

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Three static models with two interacting phantom and ghost scalar fields are considered: a model of a traversable wormhole, a branelike model and a spherically symmetric problem. It is shown numerically that regular solutions exist for all three cases.

Keywords: Scalar fields; static solutions.

1. Introduction

Scalar fields play a fundamental role in modern cosmology and astrophysics. Having come from the theory of elementary particles, they are widely used both in creation of models of compact objects and in research on the evolution of the universe as a whole. In particular, the models of quasi-star objects from scalar fields, the so-called boson stars, are well known.[1–3] In cosmology, scalar fields are the basis for creation of models of the early inflationary universe.[4]

Another field of application of scalar fields is connected with one of the most exciting events in astrophysics and cosmology of the last few decades — the discovery of the acceleration of the present universe.[5–10] This discovery has stimulated the appearance of a large number of various models trying to explain this phenomenon. A basis of all the models consists in violation of different energy conditions. For cosmology, the following two energy conditions are the most important (see e.g.

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Refs. 7–10). The first one, the so-called strong energy condition (SEC), states that $\rho + 3p \geq 0$, where ρ and p are the effective energy density and the pressure of matter, which determine the evolution of the universe. In hydrodynamical language this means that the parameter of the equation of state $w \geq -1/3$. This case corresponds to the decelerated expansion of the universe (the Friedmann models). Violation of the SEC leads to the accelerated expansion of the universe (the inflationary models). The second condition, the so-called weak energy condition (WEC), states that $\rho + p \geq 0$, from which it appears that $w \geq -1$. Failure to satisfy the WEC results in exponential or faster accelerated expansion of the universe. The substance providing such acceleration is called dark energy. This term usually means that the equation of state lies in the range $-1 \leq w \leq -1/3$ (the cosmological constant, quintessence,[11] Chaplygin gas,[12] the theories of gravitation with high derivatives,[13–18] etc.). All the models describe the present accelerated expansion of the universe within the accuracy of observations.

The above-mentioned violation of the SEC normally arises from some choice of the potential energy of the usual scalar fields (see examples of such potentials in Refs. 7–10). However, as was shown in Ref. 19 in a model-independent way on the basis of study of data sets containing 172 SNIa, it is possible in the present epoch that $w < -1$, i.e. the WEC can be violated. (For more recent papers with the latest observational data see Refs. 20–25.) Such an unusual state of matter is also known as phantom dark energy.[26] There are models of phantom energy within the framework of higher-order theories of gravity,[27] braneworld cosmology,[28–30] etc.

Another popular direction consists in consideration of ghost scalar fields with negative sign before the kinetic term.[31–33] In Ref. 34 we considered the cosmological model of the early universe with two interacting ghost scalar fields having a special form of the potential energy:

$$V(\phi, \chi) = \frac{\lambda_1}{4}(\phi^2 - m_1^2)^2 + \frac{\lambda_2}{4}(\chi^2 - m_2^2)^2 + \phi^2\chi^2 - V_0. \tag{1}$$

Here ϕ and χ are two scalar fields with masses m_1 and m_2, λ_1 and λ_2 are the self-coupling constants and V_0 is some constant which should be chosen on the assumption of a problem statement. The essential feature of this potential is the presence of two local minima at $\chi = 0$, $\phi = \pm\, m_1$. It allows existence of localized solutions with finite energy in problems with such potential. As is known,[35] for the case of one scalar field, localized solutions could exist only for a case with two or more minima of potential energy when solutions start in one minimum and tend to another. In the case of two scalar fields, a situation is possible when a solution starts and finishes in the same minimum. In Ref. 34 such a case was carried out: the solution started at $t \to -\infty$ and returned to the same local minimum at $t \to +\infty$. Such a soliton-like solutions, localized on a spacelike hypersurface, are known as spacelike branes (S-branes).[36]

The mentioned-in-Ref. 34 possibility of existence of the localized-in-*time* solutions for phantom fields allows us to hope for the presence of similar solutions and for a *static* case. In this paper we consider three models created by two interacting phantom and ghost scalar fields with the potential (1): (i) a traversable wormhole; (ii) a branelike solution — analog of a domain wall solution but with asymptotically nonflat space–time (anti-de Sitter space–time); (iii) a spherically symmetric particle-like solution.

The model of the so-called traversable wormhole was suggested in Ref. 37 (for a recent review, see Ref. 38). This wormhole is created by some special matter with the violated WEC. As phantom scalar fields also violate the WEC, they could be used in the modeling of traversable wormholes. Researches on such models have already been carried out (see e.g. Refs. 39 and 40). In these works some effective hydrodynamical energy–momentum tensor with the equation of state $w < -1$ was chosen as a source of matter. But a distribution of this matter was added by hand and, correspondingly, the *non-self-consistent* models of the traversable wormholes were considered. In this paper we consider a *self-consistent* model of a traversable wormhole created by two interacting phantom and ghost scalar fields with the potential (1). The above-specified features of this potential allow us to find regular solutions with localized energy density.

Domain walls are topological defects and they arise in different aspects in both particle physics and cosmology (see e.g. Refs. 4, 41 and 42 and references therein). They separate a space–time into several domains along a single coordinate. In the case of scalar fields, domain wall solutions exist when the scalar field potentials have isolated minima. The domain walls are surfaces separating minima of the potentials with different vacuum expectation values. The region of fast change of the scalar field corresponds to the domain wall. The domain wall is referred to as a *thin* wall if the energy density of the scalar field is localized at the domain wall surface and could be replaced by the delta function. The different variants of the domain wall solutions were found in Refs. 43–47 with asymptotically flat, de Sitter and Schwarzschild space–times. Also known are the so-called *thick* domain wall solutions,[48–51] which could exist at late-time phase transitions in the evolution of the universe. In this paper the consideration of the thick domain wall model with the potential (1) is presented.

The search for spherically symmetric solutions with various matter sources was always an important problem in special and general relativity. Such solutions are used both in investigation of different particle-like models of elementary particles and in creation of models of starlike objects and other large-scale configurations. The source of matter is fields with various spins, both interacting with each other and with the gravitational field (see e.g. Ref. 4). There are well-known regular solutions for the scalar fields for both noninteracting and self-interacting fields.[1–3] For the case of the usual (nonphantom) scalar fields, the model of a boson star with the potential (1) was considered in Ref. 52. It was shown there that there are regular solutions in the case under consideration. Recently the spherically symmetric model with self-gravitating matter with the hydrodynamic equation of state $w < -1$ was considered in Ref. 53. We show below that regular solutions for the spherically symmetric case exist for phantom and ghost scalar fields also.

This paper is organized as follows. In Sec. 2 the general gravitational and field equations for all the above-mentioned three models are presented. In Secs. 3–5 the models of the traversable wormhole, branelike and spherically symmetric solutions are considered. In Sec. 6 we present comments and conclusions.

2. General Equations

We choose the Lagrangian as follows:

$$L = -\frac{R}{16\pi G} + \epsilon\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V(\varphi,\chi)\right], \tag{2}$$

where R is the scalar curvature, G is the Newtonian gravity constant and the constant $\epsilon = \pm 1$. In the case $\epsilon = +1$ one has the theory of the usual scalar field plus gravitation. The case $\epsilon = -1$ corresponds to the theory of the ghost scalar field. The corresponding energy–momentum tensor will then be

$$T_i^k = \epsilon\left\{\partial_i\varphi\partial^k\varphi + \partial_i\chi\partial^k\chi - \delta_i^k\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V(\varphi,\chi)\right]\right\}, \tag{3}$$

and variation of the Lagrangian (2) gives the gravitational and field equations in the forms

$$G_i^k = 8\pi G T_i^k, \tag{4}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}g^{\mu\nu}\frac{\partial(\varphi,\chi)}{\partial x^\nu}\right] = -\frac{\partial V}{\partial(\varphi,\chi)}. \tag{5}$$

In our case, Eqs. (4) and (5) are the system of ordinary nonlinear differential equations with the potential energy from (1). As follows from the experience of previous researches on similar systems,[34] finding solutions to the system (4)–(5) is reduced to searching for eigenvalues of the parameters m_1, m_2. The procedure of the search for solutions and its application for investigation of the models mentioned in the Introduction will be considered in the next three sections.

3. Traversable Wormhole

We will search for static solutions to Eqs. (4) and (5) for the metric[54]

$$ds^2 = B(r)dt^2 - dr^2 - A(r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{6}$$

where $A(r), B(r)$ are the even functions depending only on the coordinate r, which covers the entire range, $-\infty < r < +\infty$. Using this metric, one can obtain from

(4) and (3) the equations (at $\epsilon = -1$)

$$\frac{A''}{A} - \frac{1}{2}\left(\frac{A'}{A}\right)^2 - \frac{1}{2}\frac{A'}{A}\frac{B'}{B} = \varphi'^2 + \chi'^2, \tag{7}$$

$$\frac{A''}{A} + \frac{1}{2}\frac{A'}{A}\frac{B'}{B} - \frac{1}{2}\left(\frac{A'}{A}\right)^2 - \frac{1}{2}\left(\frac{B'}{B}\right)^2 + \frac{B''}{B} = 2\left[\frac{1}{2}(\varphi'^2 + \chi'^2) + V\right], \tag{8}$$

$$\frac{1}{4}\left(\frac{A'}{A}\right)^2 - \frac{1}{A} + \frac{1}{2}\frac{A'}{A}\frac{B'}{B} = -\frac{1}{2}(\varphi'^2 + \chi'^2) + V, \tag{9}$$

where a prime denotes differentiation with respect to r. Equation (7) was obtained by subtracting the $\binom{r}{r}$ component from the $\binom{t}{t}$ component of Eq. (4), and the Eqs. (8) and (9) are the $\binom{\theta}{\theta}$ and $\binom{r}{r}$ components of Eq. (4). The corresponding field equations from (5) will be

$$\varphi'' + \left(\frac{A'}{A} + \frac{1}{2}\frac{B'}{B}\right)\varphi' = \varphi\left[2\chi^2 + \lambda_1(\varphi^2 - m_1^2)\right], \tag{10}$$

$$\chi'' + \left(\frac{A'}{A} + \frac{1}{2}\frac{B'}{B}\right)\chi' = \chi\left[2\varphi^2 + \lambda_2(\chi^2 - m_2^2)\right]. \tag{11}$$

In Eqs. (7)–(11) the following rescalings are used: $r \to \sqrt{8\pi G}r$, $\varphi \to \varphi/\sqrt{8\pi G}$, $\chi \to \chi/\sqrt{8\pi G}$, $m_{1,2} \to m_{1,2}/\sqrt{8\pi G}$.

As was shown in previous researches on problems with the potential (1) (see e.g. Ref. 34), regular solutions to the system of nonlinear differential Eqs. (7)–(11) could exist only for some values of the self-coupling constants λ_1, λ_2 and the masses of the scalar fields m_1, m_2, and also depend on boundary conditions which are set under the problem statement. In particular, specifying some values of the parameters λ_1, λ_2, one already has an effect on the shape of the potential (1), which in turn determines the possibility of existence of regular solutions to the system (7)–(11). The further task consists in a search for such parameters m_1, m_2 which give regular solutions. In this sense the problem reduces to a search *eigenvalues* of the parameters m_1, m_2 and the corresponding *eigenfunctions* A, B, φ, χ for the nonlinear system of differential Eqs. (7)–(11).

The technique of solution of systems similar to (7)–(11) is described in Ref. 55 in detail. The essence of this procedure is the following. In the first step one solves Eq. (10) with some arbitrary selected function χ, looking for a regular solution existing only at some value of the parameter m_1. In this step the influence of gravitation is not taken into account. Then this solution for the function φ is inserted into Eq. (11) and one searches for a value of the parameter m_2 yielding a regular solution. This procedure is reiterated several times (three is usually enough) so as to obtain to acceptable convergence of values of the parameters m_1, m_2. The obtained functions φ, χ are inserted into the gravitational equations (7) and (8). Equation (9), which is the constraint equation, is used for specifying the boundary conditions (see below). The obtained solutions for the metric functions A, B are inserted into the complete equations for the scalar fields (10) and (11), and they are solved again in order to search for eigenvalues of the parameters m_1, m_2 taking into account gravitation. This procedure is reiterated as many times as is necessary to obtain acceptable convergence of values of the parameters m_1, m_2.

The described procedure of the search for solutions to the system (7)–(11), also known as the shooting method, allows one to find rather quickly values of the parameters m_1, m_2 at which regular solutions exist. We have checked the obtained solutions using the NDSolve routine from *Mathematica*, substituting the eigenvalues m_1, m_2 and solving (7), (8), (10), (11) directly.

The boundary conditions are chosen taking into account $\mathbb{Z}_2$ symmetry in the forms

$$
\begin{aligned}
&\varphi(0) = \sqrt{3}, && \varphi'(0) = 0,\\
&\chi(0) = \sqrt{0.6}, && \chi'(0) = 0,\\
&A(0) = -\frac{1}{V(\phi(0), \chi(0))}, && A'(0) = 0,\\
&B(0) = 1.0, && B'(0) = 0,
\end{aligned}
\tag{12}
$$

where the condition for $A(0)$ is chosen to satisfy the constraint (9) at $r = 0$, $V(\phi(0), \chi(0))$ is the value of the potential at $r = 0$ and the self-coupling constants $\lambda_1 = 0.1$ and $\lambda_2 = 1$.

Then, using the above procedure for obtaining solutions to the system (7)–(11), we have the results presented in Figs. 1–4. These results are obtained for the masses $m_1 \approx 2.661776085$ and $m_2 \approx 2.928340304$. As one can see from Fig. 1, $\varphi \to m_1$ and $\chi \to 0$ as $r \to \pm\infty$. This corresponds to asymptotic transition of the solutions to the local minimum of the potential (1) (see Introduction). The arbitrary constant V_0 was chosen in such a way that the value of the potential in the local minimum was equal to zero, viz., $V_0 = (\lambda_2/4)m_2^4$. Such a choice of V_0 ensures a zero value of the energy density as $r \to \pm\infty$ (Fig. 2).

Let us estimate an asymptotic behavior of the solutions. For this purpose we will seek for solutions to Eqs. (10) and (11) in the forms

$$\varphi = m_1 - \delta\varphi, \quad \chi = \delta\chi, \tag{13}$$

where $\delta\varphi, \delta\chi \ll 1$ as $r \to \pm\infty$. Then the right-hand sides of Eqs. (7) and (8) go to zero and their particular solutions are

$$A \approx r^2 + r_0^2, \tag{14}$$

$$B \approx B_\infty \left(1 - \frac{r_0^2}{r^2}\right), \tag{15}$$

where r_0 and B_∞ are constants. Practically, r_0^2 defines the total mass of a wormhole and B_∞ the run of time at infinity. By corresponding redefinition of the time t, these solutions could be reduced to a flat form in spherical coordinates, i.e. one

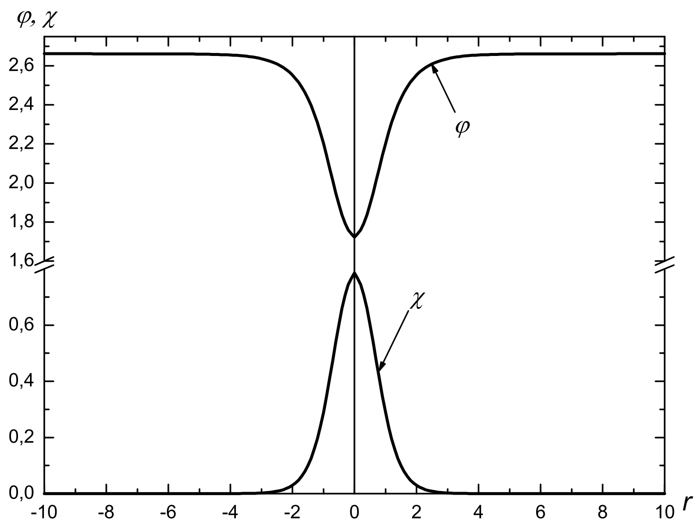

Fig. 1. The scalar fields φ, χ in the wormhole model for the boundary conditions given in (12).

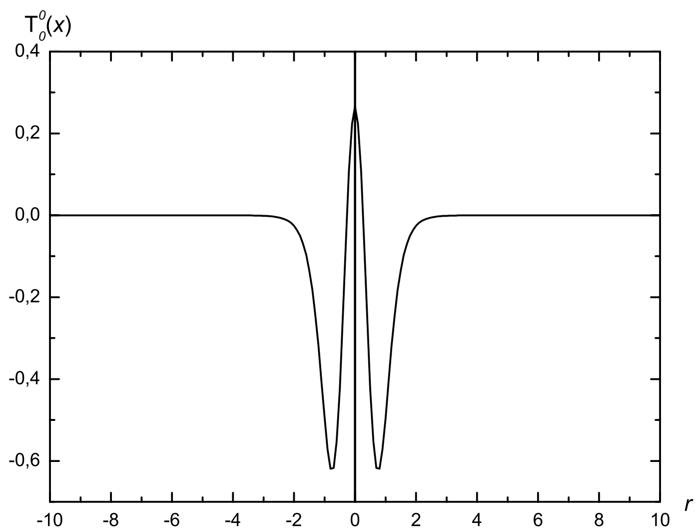

Fig. 2. The energy density $T_0^0(r)$ for the wormhole model.

has asymptotically flat Minkowski space–time (see Figs. 3 and 4). Then, taking into account (14)–(15), the corresponding asymptotic equations for the scalar fields (10) and (11) are rewritten as

$$\delta\varphi'' + \frac{2}{r}\delta\varphi' = 2\lambda_1 m_1^2 \delta\varphi, \tag{16}$$

$$\delta\chi'' + \frac{2}{r}\delta\chi' = (2m_1^2 - \lambda_2 m_2^2)\delta\chi, \tag{17}$$

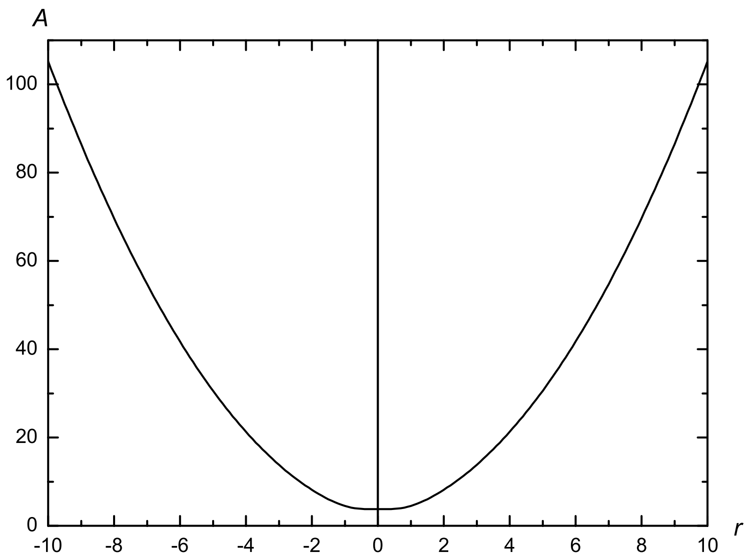

Fig. 3. The metric function A in the wormhole model.

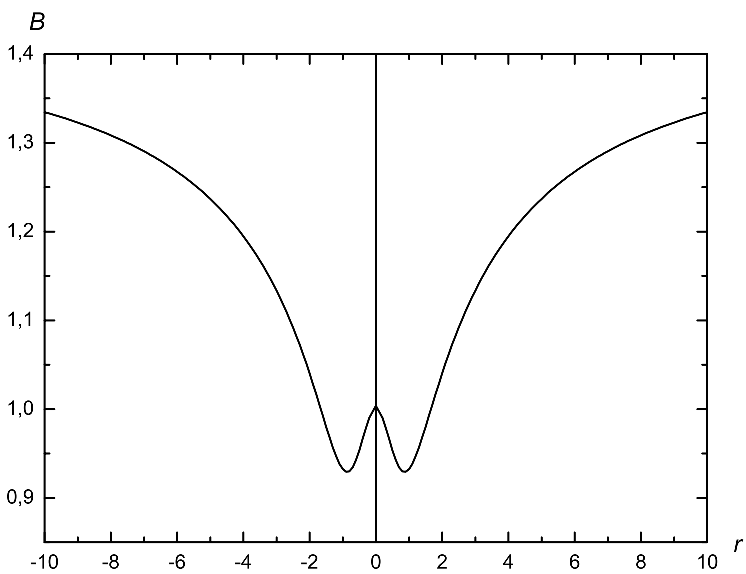

Fig. 4. The metric function B in the wormhole model.

with the exponentially fast damping solutions

$$\delta\varphi \approx C_\varphi \frac{\exp(-\sqrt{2\lambda_1 m_1^2} r)}{r}, \tag{18}$$

$$\delta\chi \approx C_\chi \frac{\exp(-\sqrt{(2m_1^2 - \lambda_2 m_2^2)} r)}{r}, \tag{19}$$

where C_φ, C_χ are integration constants. Thus the asymptotic solutions go to vacuum ones with the zero energy density (Fig. 2).

4. Branelike Solution

In this section we consider a branelike solution in 4D. Let us choose the metric in the form

$$ds^2 = a^2(x)(dt^2 - dy^2 - dz^2) - dx^2, \tag{20}$$

where the metric function $a(x)$ depends only on the coordinate x. This metric describes a (2+1)-dimensional space–time embedded in a (3+1)-dimensional space–time. Using (3)–(5), we can obtain the $\binom{x}{x}$ and $\binom{t}{t}$ components of the Einstein equations (4),

$$3\left(\frac{a'}{a}\right)^2 = -\frac{1}{2}\left(\varphi'^2 + \chi'^2\right) + V, \tag{21}$$

$$\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = \frac{1}{2}\left(\varphi'^2 + \chi'^2\right), \tag{22}$$

and the scalar field equations,

$$\varphi'' + 3\frac{a'}{a}\varphi' = \varphi\left[2\chi^2 + \lambda_1(\varphi^2 - m_1^2)\right], \tag{23}$$

$$\chi'' + 3\frac{a'}{a}\chi' = \chi\left[2\varphi^2 + \lambda_2(\chi^2 - m_2^2)\right], \tag{24}$$

where a prime denotes differentiation with respect to x and the arbitrary constant V_0 is chosen as follows:

$$V_0 = \frac{\lambda_1}{2}(\phi_0^2 - m_1^2)^2 + \frac{\lambda_2}{2}(\chi_0^2 - m_2^2)^2 + \phi_0^2\chi_0^2, \tag{25}$$

for the purpose of zeroing of a' at $x = 0$ [see Eqs. (21) and (26)]. (Here and further we use the same rescaling for all variables and parameters as in the previous section.)

We will solve the system of Eqs. (21)–(24) with the following boundary conditions at $x = 0$:

$$\begin{aligned} &\varphi(0) = \sqrt{3}, &&\varphi'(0) = 0,\\ &\chi(0) = \sqrt{1.8}, &&\chi'(0) = 0,\\ &a(0) = 1.0, &&a'(0) = 0. \end{aligned} \tag{26}$$

The procedure for finding solutions is the same as in the previous section. The obtained solutions with masses $m_1 \approx 2.59755, m_2 \approx 3.729$ and $\lambda_1 = 0.1, \lambda_2 = 1$ for the scalar fields are presented in Fig. 5, for the metric function $a(x)$ in Fig. 6 and for the energy density T_0^0 in Fig. 7.

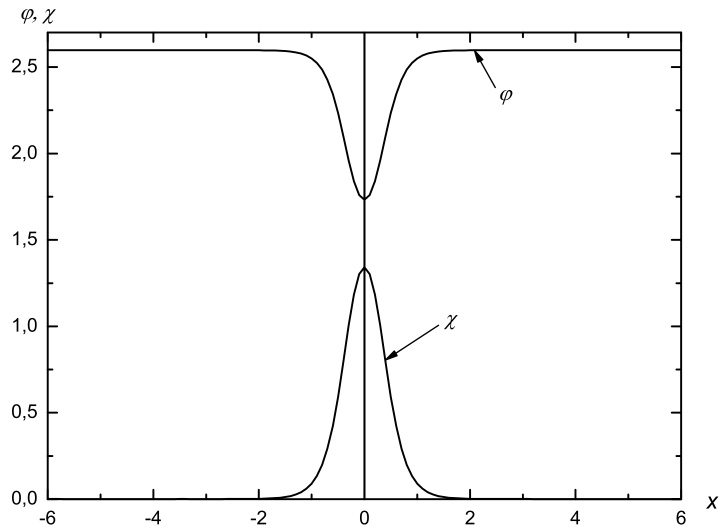

Fig. 5. The distribution of the scalar fields ϕ, χ near $x = 0$.

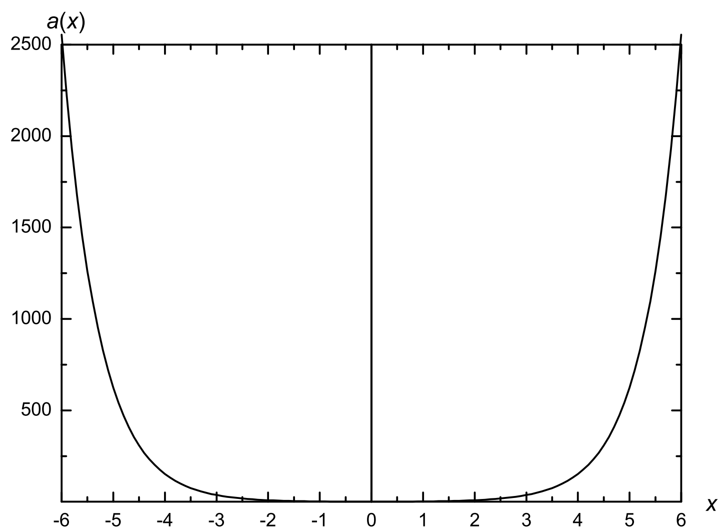

Fig. 6. The metric function $a(x)$.

We can easily estimate an asymptotic behavior of the solutions. We can see from Eq. (22) that asymptotically the right-hand side tends to zero and the solution to this equation is

$$a = a_0 e^{\alpha x}, \tag{27}$$

where a_0 and α are integration constants. This solution corresponds to the de Sitter-like solution for the space variable x. Then, using (27) and seeking for asymptotic

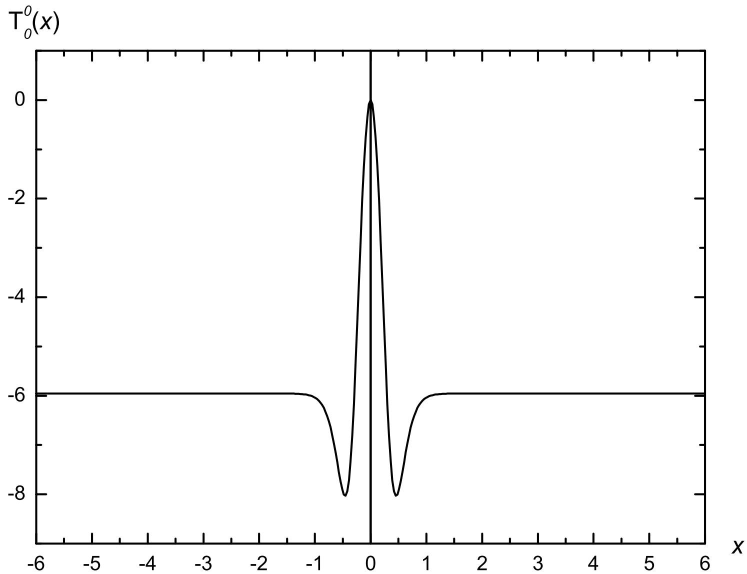

Fig. 7. The localized energy density $T_0^0(x)$ near $x = 0$ with asymptotic anti-de Sitter behavior.

solutions to Eqs. (23) and (24) in the forms

$$\varphi = m_1 - \delta\varphi, \quad \chi = \delta\chi, \tag{28}$$

where $\delta\varphi, \delta\chi \ll 1$ as $x \to \pm\infty$, we will have the following equations for $\delta\varphi$ and $\delta\chi$ from (23) and (24):

$$\delta\varphi'' + 3\alpha\delta\varphi' = 2\lambda_1 m_1^2 \delta\varphi, \tag{29}$$

$$\delta\chi'' + 3\alpha\delta\chi' = (2m_1^2 - \lambda_2 m_2^2)\delta\chi, \tag{30}$$

with the damping solutions

$$\delta\varphi \approx C_\varphi \exp\left[-\frac{x}{2}\left(3\alpha + \sqrt{9\alpha^2 + 8\lambda_1 m_1^2}\right)\right], \tag{31}$$

$$\delta\chi \approx C_\chi \exp\left[-\frac{x}{2}\left(3\alpha + \sqrt{9\alpha^2 + 4(2m_1^2 - \lambda_2 m_2^2)}\right)\right], \tag{32}$$

where C_φ, C_χ are integration constants. So we have the solutions that tend asymptotically to the local minimum of the potential (1) at $\varphi = m_1$ and $\chi = 0$.

5. Spherically Symmetric Solution

For consideration of the spherically symmetric problem we take the metric in Schwarzschild coordinates:

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{33}$$

Then the $\binom{t}{t}$ and $\binom{r}{r}$ components of the Einstein equations (4) are

$$\frac{1}{r}\frac{A'}{A^2} + \frac{1}{r^2}\left(1 - \frac{1}{A}\right) = -\frac{1}{2A}\left(\varphi'^2 + \chi'^2\right) - V(\varphi, \chi), \qquad (34)$$

$$\frac{1}{r}\frac{B'}{AB} - \frac{1}{r^2}\left(1 - \frac{1}{A}\right) = -\frac{1}{2A}\left(\varphi'^2 + \chi'^2\right) + V(\varphi, \chi), \qquad (35)$$

$$\frac{B''}{B} - \frac{1}{2}\left(\frac{B'}{B}\right)^2 - \frac{1}{2}\frac{A'}{A}\frac{B'}{B} - \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right) = 2A\left[\frac{1}{2A}\left(\varphi'^2 + \chi'^2\right) + V(\varphi, \chi)\right], \qquad (36)$$

and the scalar field equations (5) are

$$\varphi'' + \left(\frac{2}{r} + \frac{B'}{2B} - \frac{A'}{2A}\right)\varphi' = A\varphi\left[2\chi^2 + \lambda_1(\varphi^2 - m_1^2)\right], \qquad (37)$$

$$\chi'' + \left(\frac{2}{r} + \frac{B'}{2B} - \frac{A'}{2A}\right)\chi' = A\chi\left[2\varphi^2 + \lambda_2(\chi^2 - m_2^2)\right], \qquad (38)$$

where a prime denotes differentiation with respect to r and in the potential (1) $V_0 = (\lambda_2/4)m_2^4$ for the purpose of zeroing of the energy density as $r \to \infty$. Equation (36) is a consequence of the preceding equations (34) and (35).

Choosing the boundary conditions at $r = 0$ in the forms

$$\begin{aligned} &\varphi(0) = \sqrt{3}, && \varphi'(0) = 0, \\ &\chi(0) = \sqrt{0.6}, && \chi'(0) = 0, \\ &A(0) = 1.0, && B(0) = 1.0, \end{aligned} \qquad (39)$$

and following the above procedure for obtaining solutions, we find the masses $m_1 \approx 2.329305, m_2 \approx 3.0758999$ at $\lambda_1 = 0.1, \lambda_2 = 1$. The results of numerical calculations for the scalar fields are presented in Fig. 8, for the metric functions $A(r), B(r)$ in Fig. 9 and for the energy density T_0^0 in Fig. 10.

One can see from (34) and (35) that the asymptotic behavior of the metric functions $A(r)$ and $B(r)$ is

$$A \approx \frac{1}{1 + \frac{r_0}{r}}, \quad B \approx B_\infty\left(1 + \frac{r_0}{r}\right), \qquad (40)$$

where r_0 and B_∞ are constants. Practically, r_0 defines the total mass and B_∞ the run of time at infinity. Redefining the time variable t, we can choose $B = 1$ as $r \to \infty$, i.e. we have asymptotically flat Minkowski space–time. The corresponding asymptotic scalar field Eqs. (37) and (38) taking into account

$$\varphi = m_1 - \delta\varphi, \quad \chi = \delta\chi \qquad (41)$$

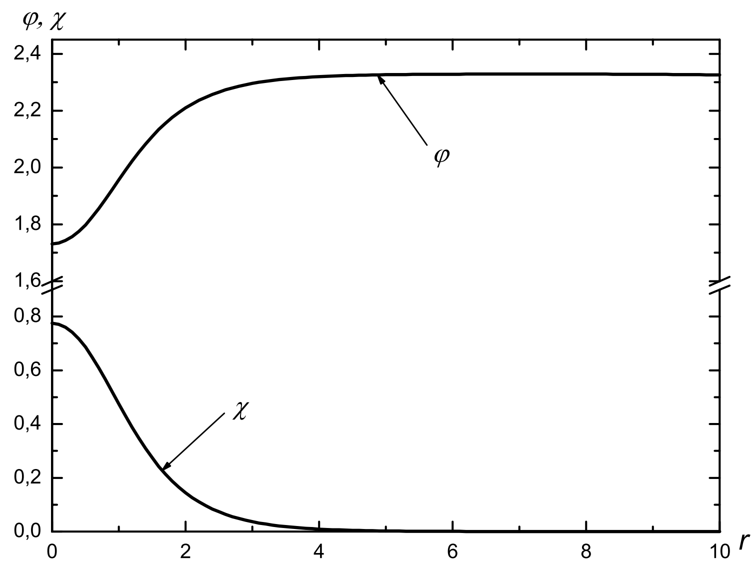

Fig. 8. The scalar fields ϕ, χ for the spherically symmetric case.

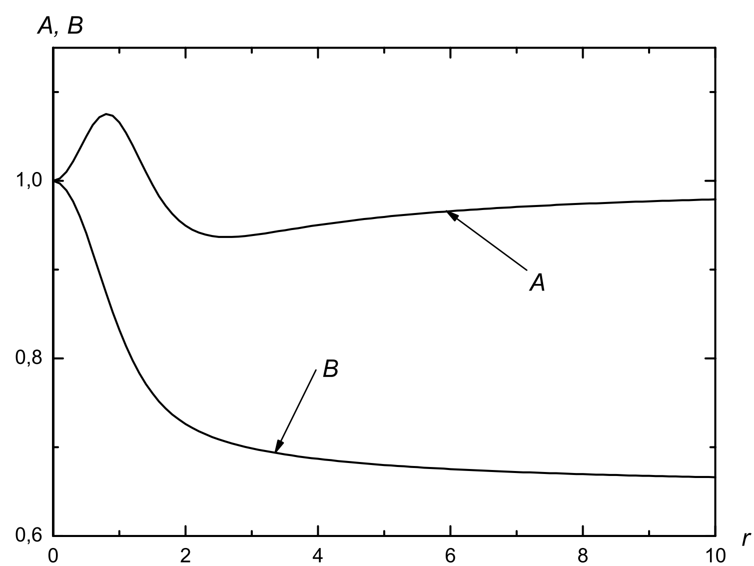

Fig. 9. The metric function $A(r), B(r)$ for the spherically symmetric case.

will be

$$\delta\varphi'' + \frac{2}{r}\delta\varphi' = 2\lambda_1 m_1^2 \delta\varphi, \tag{42}$$

$$\delta\chi'' + \frac{2}{r}\delta\chi' = (2m_1^2 - \lambda_2 m_2^2)\delta\chi, \tag{43}$$

with the exponentially fast damping solutions

$$\delta\varphi \approx C_\varphi \frac{\exp(-\sqrt{2\lambda_1 m_1^2} r)}{r}, \tag{44}$$

$$\delta\chi \approx C_\chi \frac{\exp(-\sqrt{(2m_1^2 - \lambda_2 m_2^2)} r)}{r}, \tag{45}$$

where C_φ, C_χ are integration constants. Thus the asymptotic solutions go to vacuum ones with the zero energy density (Fig. 10).

Finally, let us show the evolution of the effective equation of state $w(r) = p(r)/\varepsilon(r)$, where $\varepsilon(r)$ and p are the effective energy density and pressure of the scalar fields. In the case under consideration we have from (3) and (33)

$$T_0^0 = \varepsilon(r) = -\left[\frac{1}{2A}\left(\varphi'^2 + \chi'^2\right) + V(\varphi,\chi)\right], \tag{46}$$

$$T_1^1 = -p(r) = -\left[-\frac{1}{2A}\left(\varphi'^2 + \chi'^2\right) + V(\varphi,\chi)\right]. \tag{47}$$

Then the corresponding effective equation of state will be

$$w(r) = \frac{p(r)}{\varepsilon(r)} = -\frac{-\frac{1}{2A}\left(\varphi'^2 + \chi'^2\right) + V(\varphi,\chi)}{\frac{1}{2A}\left(\varphi'^2 + \chi'^2\right) + V(\varphi,\chi)}. \tag{48}$$

Using the numerical solution obtained earlier, we have the following graph for the equation of state (see Fig. 11). As we can see from the figure, there is some point $r = r_*$ at which the denominator in (48) tends to zero (in the case under consideration

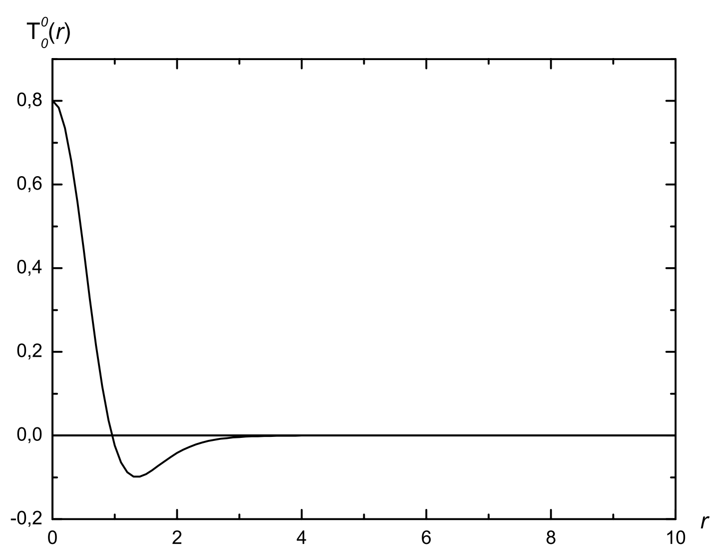

Fig. 10. The energy density $T_0^0(r)$ for the spherically symmetric case.

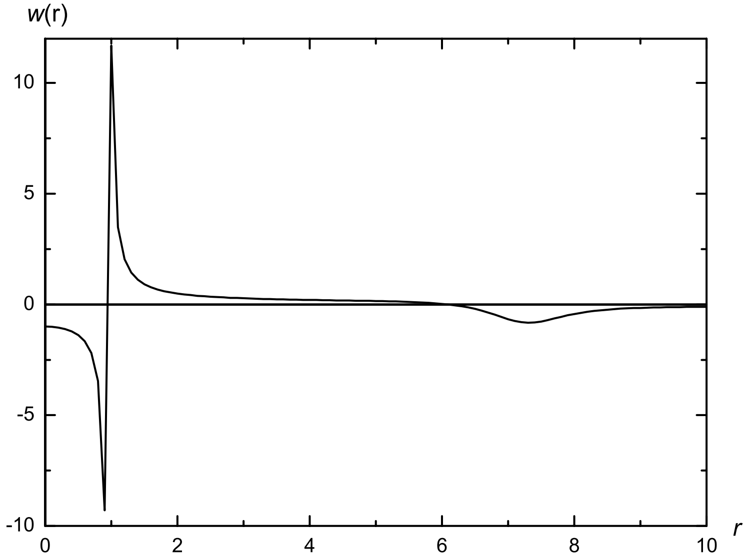

Fig. 11. The equation of state $w(r)$ for the spherically symmetric case.

$r_* \approx 0.956$). In the range $0 < r < r_*$ we have $w \leq -1$ ($w = -1$ at $r = 0$), and in the range $r_* < r < 10$ we have $w > -1$. At $r \to r_*$ from the left $w \to -\infty$, and at $r \to r_*$ from the right $w \to +\infty$. Asymptotically $w(r)$ tends to zero, i.e. we have the dustlike equation of state.

It follows from this that in the entire range $0 < r < r_*$ the WEC is violated, i.e. the scalar fields efficiently describe phantom matter. On the other hand, in the range $r > r_*$ the WEC is not violated and the scalar fields are nonphantom. The scalar fields with the equation of state varying in time are used in the theories describing the present acceleration of the universe (see e.g. Ref. 56). In such theories there is some point of time at which the violation of the WEC and the transition of the usual scalar fields to phantom ones take place. In our case we deal with inhomogeneous distribution of the scalar fields with the varying-in-space equation of state and the transition of the usual scalar fields to phantom ones at some point $r = r_*$.

6. Conclusion

We have considered three static solutions for the model of two interacting phantom and ghost scalar fields: a model of a traversable wormhole, a branelike model and a spherically symmetric problem. The self-consistent problems taking into account the back reaction of the scalar fields on gravitation were investigated. The choice of a potential in the form (1) ensures the existence of two local minima that allow one to find regular solutions which start and finish in one of these minima. In this case the nonlinear problems on evaluation of eigenvalues of the parameters m_1, m_2 ensuring the mentioned regular solutions were solved. Note that the existence of such solutions depends on values of the self-coupling constants λ_1, λ_2 and the boundary conditions. There is some range of these parameters appropriate for the existence of regular solutions. Particularly, this range is defined by conditions on the existence of the local and global minima of the potential (1): $\lambda_1 > 0, m_1^2 > \lambda_2 m_2^2/2$; $\lambda_2 > 0, m_2^2 > \lambda_1 m_1^2/2$; $\lambda_2 m_2^4 > \lambda_1 m_1^4$.

For the wormhole model, it was shown that there exist regular solutions in the entire range, $-\infty < r < +\infty$, with asymptotically flat Minkowski space–time [see Eqs. (14) and (15)], i.e. the obtained wormhole solution connects two flat space–times. The radius of the wormhole throat is defined by a minimal value of the function $A(r)$, i.e. $R_0 = \min_{r\in(-\infty,+\infty)}\{A(r)\}$ at $r = 0$. As one can see in Fig. 2, for a remote observer the phantom energy looks like a compact object localized near the throat of the wormhole with some negative energy density.

For the branelike case, the obtained solutions describe a (2+1)-dimensional space–time embedded in a (3+1)-dimensional space–time. The asymptotic value of the potential (1), $V(x = \pm\infty) < 0$, plays the role of a negative cosmological constant, and so the metric (20) is asymptotically anti-de Sitter, with the corresponding anti-de Sitter horizon. In this sense such a solution differs from the domain wall solution, which has flat asymptotics.

For the spherically symmetric case we have found particle-like solutions with asymptotically flat Minkowski space–time. It was shown that the effective equation of state $w(r)$ for the scalar fields changes essentially along the radius r (see Fig. 11). There exists some point $r = r_*$ which divides the whole space into two regions: in the entire range, $0 < r < r_*$, $w < -1$, which corresponds to phantom-like behavior of the scalar fields. On the other hand, in the range $r_* < r < \infty$ the equation of state $w > -1$, which corresponds to nonphantom (ghost) scalar fields. Similar behavior of the equation of state could be obtained for the wormhole and branelike models as well.

Let us note that for all three models, the obtained solutions correspond to soliton-like solutions starting and finishing in the same minimum [in the cases under consideration, in the local minimum $\phi = m_1, \chi = 0$ of the potential (1)]. In models with one scalar field, soliton-like solutions exist only in a case of the presence of two or more minima and they start from one minimum of a potential and tend asymptotically to another. In the terminology of Ref. 35, such solutions refer to topological solutions, and our solutions to nontopological ones.

Note here that the known constraint on the possibility of existence of regular static solutions in spaces with dimensionality $D \geq 3$ does not work in our case. As was shown in Ref. 57 (see also Ref. 35), the mentioned solutions for the usual (nonphantom) scalar fields do not exist if the potential $V \geq 0$ in the whole space. On the other hand, for phantom fields the condition $V \leq 0$ should always be satisfied (see, in this connection, Ref. 58). In our case the potential (1) changes its sign, which allows one to avoid the constraint of the theorem from Ref. 57 and obtain static regular solutions.

Our attempts to find regular solutions with $\epsilon = +1$ did not succeed. However, this does not exclude the possibility of their existence at some parameters λ_1, λ_2 in the potential (1) and some special boundary conditions.

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