APPROACH TO DESIGN OF ROBUST CONTROL SYSTEMS IN THE CLASS OF STRUCTURALLY STABLE MAPS WITH EXAMPLE OF SIMPLEST WALKING ROBOT

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1. Introduction

There are a lot of methods of design of robust control that are developed with increasing interest and some of them become classical. Commonly all of them are dedicated to defining the ranges of parameters (if uncertainty of parameters takes place) within which the system will function with desirable properties, first of all, will be stable [1, 2]. Thus there are many research efforts which successfully attenuate the uncertain changes of parameters in small (regarding to magnitudes of their own nominal values) ranges. But no one existing method can guarantee the stability of designed control system at arbitrarily large ranges of uncertainly changing parameters of plant. The approach that is offered in the present work relies on the results of catastrophe theory [3, 4, 5, 6, 7], uses nonlinear structurally stable functions, and due to bifurcations of equilibrium points in designed nonlinear systems allows to stabilize a dynamic plant with ultimately wide ranges of changing of parameters.

It is known that the catastrophe theory deals with several functions which are characterized by their stable structure. Today there are many classifications of these functions but originally they are discovered as seven basic nonlinearities named as 'catastrophes':

$x^3 + k_1 x$ (fold);

$x^4 + k_2 x^2 + k_1 x$ (cusp);

$x^5 + k_3 x^3 + k_2 x^2 + k_1 x$ (swallowtail);

$x^6 + k_4 x^4 + k_3 x^3 + k_2 x^2 + k_1 x$ (butterfly);

$x_2^3 + x_1^3 + k_1 x_2 x_1 - k_2 x_2 + k_3 x_1$ (hyperbolic umbilic);

$x_2^3 - 3x_2 x_1^2 + k_1\left(x_1^2 + x_2^2\right) - k_2 x_2 - k_3 x_1$ (elliptic umbilic);

$x_2^2 x_1 + x_1^4 + k_1 x_2^2 + k_2 x_1^2 - k_3 x_2 - k_4 x_1$ (parabolic umbilic).

A part of the catastrophe which does not contain parameters k_i is called as 'germ' of catastrophe. Adding any of them to dynamic system as a controller will give effect shown below. On the example of the catastrophe 'elliptic umbilic' added to dynamical systems we shall see that:

1) new (one or several) equilibrium point appears so there are at least two

equilibrium point in new designed system,

2) these equilibrium points are stable but not simultaneous, i.e. if one exists (is stable) then another does not exist (is unstable),

3) stability of the equilibrium points are determined by values or relations of values of parameters of the system,

4) what value(s) or what relation(s) of values of parameters would not be, every time there will be one and only one stable equilibrium point to which the system will attend and thus be stable.

Let us consider the cases of second-order systems (2) and examples (possible applications) of design of control systems (3): double pendulum's stable oscillations.

2. Second order systems

A. Integrators in series. Let us consider a control plant presented by two integrators connected in series, as shown in Fig. 1:

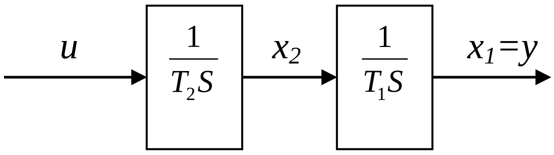


Figure 1. "Integrators in series" structure.

where T_1 and T_2 are the parameters of integration. The structure of several integrators (more than 2 integrators) is famous of its instability, i.e. no one linear controller can provide the stability to such system and more over with uncertainly changeable parameters [8, 9]. The example of two integrators in series allows us to see the advantages of using non-linear catastrophe as controller.

Let us choose a feedback control law as following form:

$$u = -x_2^3 + 3x_2 x_1^2 - k_1\left(x_1^2 + x_2^2\right) + k_2 x_2 + k_3 x_1, \qquad (1)$$

and in order to study stability of the system let us suppose that there is no input signal in the system (equal to zero). Hence, the system with proposed controller can be presented as:

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{1}{T_1} x_2, \\ \dfrac{dx_2}{dt} = \dfrac{1}{T_2}\left(-x_2^3 + 3x_2 x_1^2 - k_1\left(x_1^2 + x_2^2\right) + k_2 x_2 + k_3 x_1\right). \end{cases} \qquad (2)$$

$y = x_1$.

The system (2) has following equilibrium points

$$x_{1s}^1 = 0,\ x_{2s}^1 = 0; \tag{3}$$

$$x_{1s}^2 = \frac{k_3}{k_1},\ x_{2s}^2 = 0. \tag{4}$$

Stability conditions for equilibrium point (3) obtained via linearization are

$$\begin{cases} -\dfrac{k_2}{T_2} > 0, \\ \dfrac{k_3}{T_1 T_2} < 0. \end{cases} \tag{5}$$

Stability conditions of the equilibrium point (4) are

$$\begin{cases} -\dfrac{3k_3^2 + k_2 k_1^2}{k_1^2 T_2} > 0, \\ \dfrac{k_3}{T_1 T_2} > 0. \end{cases} \tag{6}$$

By comparing the stability conditions given by (5) and (6) we find that the signs of the expressions in the second inequalities are opposite. Also we can see that the signs of expressions in the first inequalities can be opposite due to squares of the parameters k_1 and k_3 if we properly set their values.

Let us suppose that parameter T_1 can be perturbed but remains positive. If we set k_2 and k_3 both negative and $|k_2| < 3\frac{k_3^2}{k_1^2}$ then the value of parameter T_2 is irrelevant. It can assume any values both positive and negative (except zero), and the system given by (2) remains stable. If T_2 is positive then the system converges to the equilibrium point (3) (becomes stable). Likewise, if T_2 is negative then the system converges to the equilibrium point (4) which appears (becomes stable). At this moment the equilibrium point (3) becomes unstable (disappears).

Let us suppose that T_2 is positive, or can be perturbed staying positive. So if we can set the k_2 and k_3 both negative and $|k_2| > 3\frac{k_3^2}{k_1^2}$ then it does not matter what value (negative or positive) the parameter T_1 would be (except zero), in any case the system (2) will be stable. If T_1 is positive then equilibrium point (3) appears (becomes stable) and equilibrium point (4) becomes unstable (disappears) and vice versa, if T_1 is negative then equilibrium point (4) appears (become stable) and equilibrium point (3) becomes unstable (disappears).

Results of MatLab simulation for the first and second cases are presented in Fig. 2 and 3 respectively. In both cases we see how phase trajectories converge to equilibrium points $(0{,}0)$ and $\left(\frac{k_3}{k_1};0\right)$.

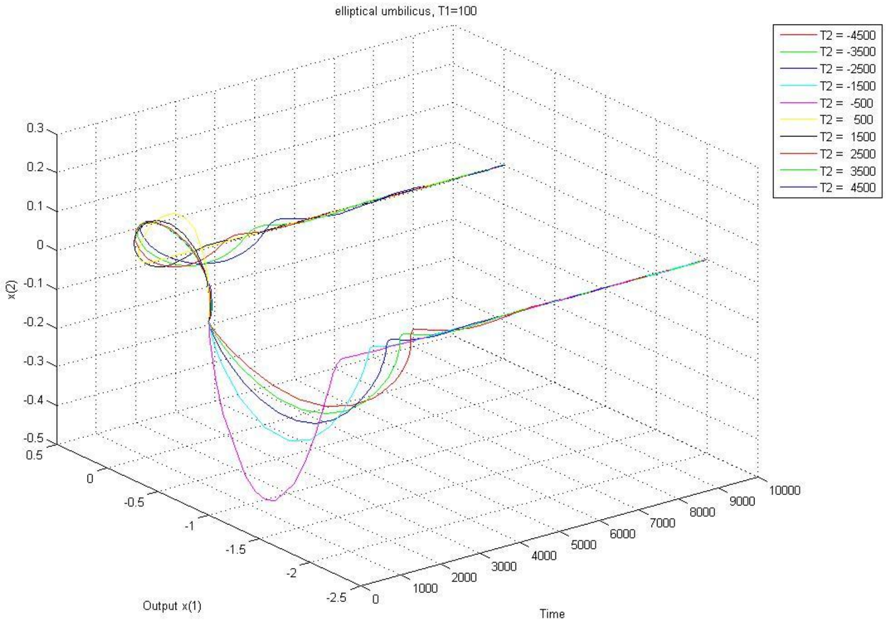


Figure 2. Behavior of output of designed control system in the case of integrators in series at various T_2.

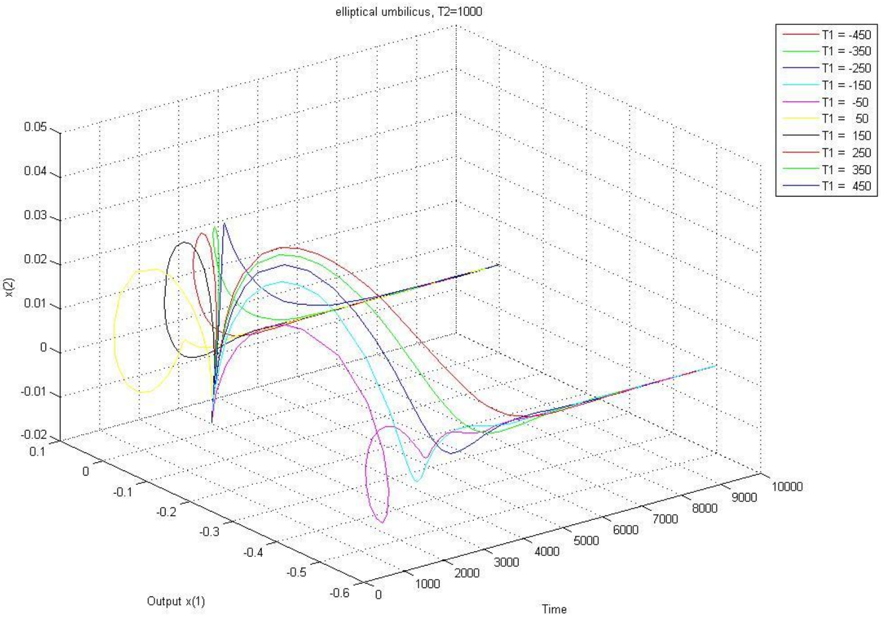


Figure 3. Behavior of output of designed control system in the case of integrators in series at various T_1.

B. Canonical controllable form (CCF). This form is important if we would like to affect to the last term of characteristic polynomial a_n which corresponds to general gain of the system.

Let us consider the second order system which is identical to CCF:

$$\begin{cases} \dfrac{dx_1}{dt} = x_2, \\ \dfrac{dx_2}{dt} = -a_2 x_1 - a_1 x_2. \end{cases}$$

$y = x_1$.

It is known that the system will be stable if and only if the parameters a_1 and a_2 are positive. If for example the small perturbation will make the a_2 negative then system will become unstable.

Let us set the control law in the form (1). Hence we will obtain the following equations of designed control system.

$$\begin{cases} \dfrac{dx_1}{dt} = x_2, \\ \dfrac{dx_2}{dt} = -a_2 x_1 - a_1 x_2 - x_2^3 + 3x_2 x_1^2 - k_1\left(x_1^2 + x_2^2\right) + k_2 x_2 + k_3 x_1. \end{cases} \tag{7}$$

$y = x_1$.

The system (7) has following equilibrium points:

$$x_{1s}^1 = 0,\ x_{2s}^1 = 0; \tag{8}$$

$$x_{1s}^2 = \frac{k_3 - a_2}{k_1},\ x_{2s}^2 = 0; \tag{9}$$

Stability conditions for equilibrium points (8) and (9) respectively are

$$\begin{cases} a_1 - k_2 > 0, \\ a_2 - k_3 > 0. \end{cases} \tag{10}$$

$$\begin{cases} a_1 - k_2 + 3\dfrac{(k_3 - a_2)^2}{k_1^2} > 0, \\ k_3 - a_2 > 0. \end{cases} \tag{11}$$

From inequalities (10) and (11) it is easy to see that here it does not matter what value except zero parameter a_2 will be. Similar to above we can resume that system (7) will be stable.

Example

Double pendulum's stable oscillations. As another example let us consider the dynamics of inverted double pendulum oscillations. For more simplicity let us take a model of simplest walker from [10] but assuming that one of the legs, namely stance leg, is fixed by its heel. The model is shown in the Fig. 9.

It has two rigid legs, connected by a frictionless hinge at a hip. Here θ is the angle of stance leg with respect to the slope normal, ϕ is the angle between the stance leg and the swing leg, γ is the ramp slope (which is used here only for the orientation of the pendulum in the space), *g* is the acceleration due to gravity.

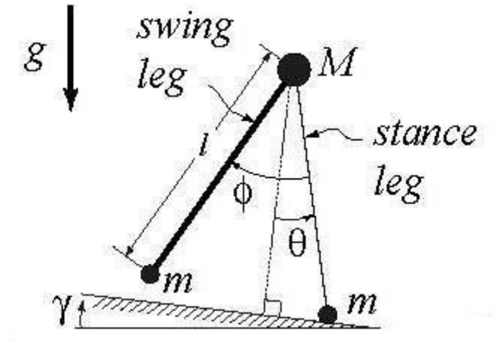


Figure 9. Double pendulum with slope

We assume that hip mass *M* is much larger than the foot mass *m* so that the motion of a swinging foot does not affect the motion of the hip. Finally, assuming that leg length *l* is equal to 9.8 m the dynamics of this construction is described by following equations

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \sin(x_1 - \gamma), \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = \sin(x_1 - \gamma) + x_2^2 \sin x_3 - \cos(x_1 - \gamma)\sin x_3. \end{cases} \quad (30)$$

where θ, $\dot{\theta}$, ϕ, and $\dot{\phi}$ are denoted as x_1, x_2, x_3, and x_4 correspondingly.

If our task is to obtain a stable position of the stance leg and stable periodic oscillation of the swinging leg at possible uncertain perturbations of initial conditions and parameter γ of the system (30) in some ranges we can use the offered approach of using class of structurally stable functions from catastrophe theory.

Let us choose the ‘cusp’ catastrophe as a controller and take it without the germ x^4

$$u = \begin{bmatrix} k_{21}x_1^2 + k_{11}x_1 \\ k_{22}x_1^2 + k_{12}x_1 \\ k_{23}x_1^2 + k_{13}x_1 \\ k_{24}x_1^2 + k_{14}x_1 \end{bmatrix}$$

and set parameters of the controller as

$$\begin{aligned} &k_{21}=2, \quad k_{11}=-1,\\ &k_{22}=0, \quad k_{12}=-4,\\ &k_{23}=0, \quad k_{13}=0,\\ &k_{24}=0, \quad k_{14}=0. \end{aligned}$$

Thus the designed control system is

$$\begin{cases} \dot{x}_1 = x_2 + 2x_1^2 - x_1, \\ \dot{x}_2 = \sin(x_1-\gamma) - 4x_1, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = \sin(x_1-\gamma) + x_2^2 \sin x_3 - \cos(x_1-\gamma)\sin x_3. \end{cases} \tag{32}$$

By simulation experiments it is found out that offered controller (31) can provide stable oscillation of swinging leg and stable position of stance leg at various values of parameter γ and various values of initial conditions θ_0, $\dot{\theta}_0$, ϕ_0, $\dot{\phi}_0$ which can be changed (uncertainly), for example, in the ranges [-0.4, +0.3] and [-0.4,+0.4], [-0.05,+0.05], [-0.8,+0.8], [-0.1,+0.1] respectively.

The Fig. 10 shows different behaviors of dynamics of θ in the control system (32) at corresponding different values of parameter γ varied in the ranges [-0.4, +0.3] radians, and at initial conditions $\theta_0 = 0.5, \quad \dot{\theta}_0 = 0.05, \quad \phi_0 = 1, \quad \dot{\phi}_0 = -0.1$.

The Fig. 11 shows different behaviors of dynamics of ϕ in the control system (32) at corresponding different values of parameter γ varied in the ranges [-0.4, +0.3] radians, and at initial conditions $\theta_0 = 0.5, \quad \dot{\theta}_0 = 0.05, \quad \phi_0 = 1, \quad \dot{\phi}_0 = -0.1$.

The Fig. 12 shows different behaviors of dynamics of θ in the control system (32) at constant value of parameter γ, equal to 0.3 and different values of initial conditions varied in the ranges $\theta \in [-0.4,+0.4]$, $\dot{\theta} \in [-0.05,+0.05]$, $\phi \in [-0.8,+0.8]$, $\dot{\phi} \in [-0.1,+0.1]$.

The Fig. 13 shows different behaviors of dynamics of ϕ in the control system (32) at constant value of parameter γ, equal to 0.3 and different values of initial conditions varied in the ranges $\theta \in [-0.4,+0.4]$, $\dot{\theta} \in [-0.05,+0.05]$, $\phi \in [-0.8,+0.8]$, $\dot{\phi} \in [-0.1,+0.1]$.

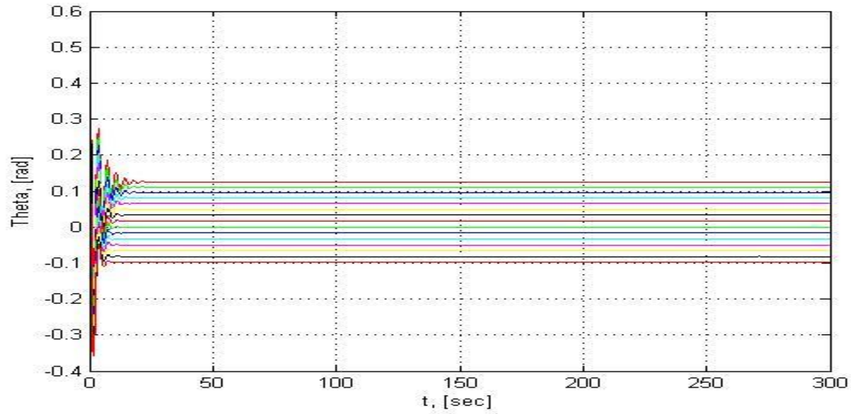


Figure 10. Behavior θ at varied initial conditions of the system (32)

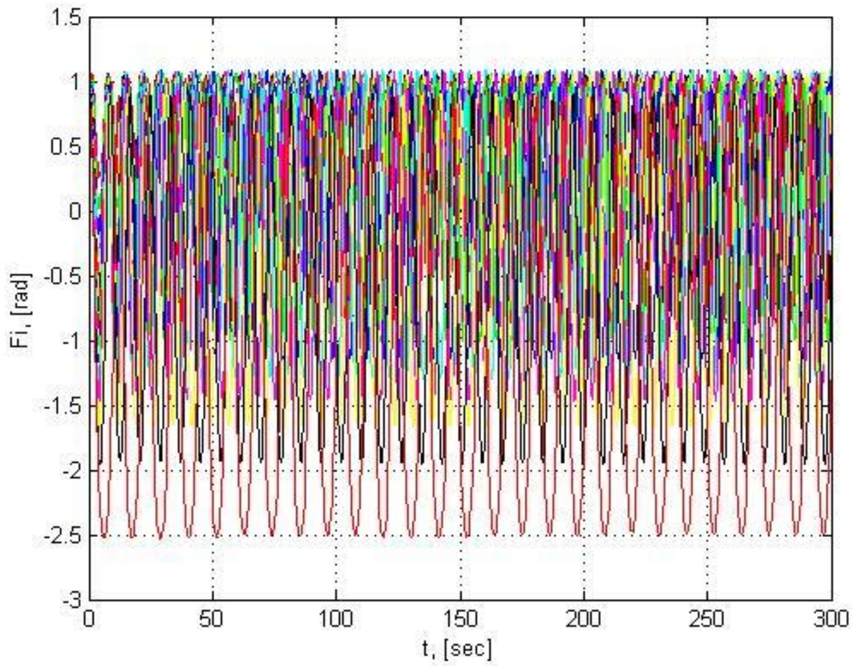


Figure 11. Behavior φ at varied initial conditions of the system (32)

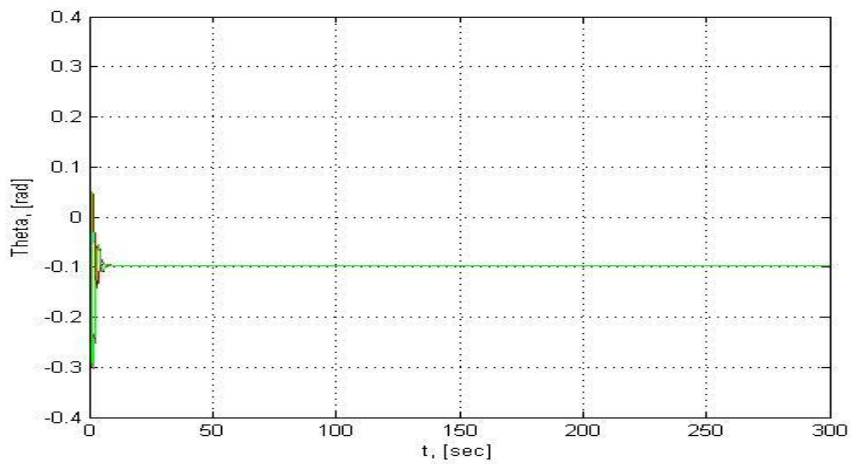


Figure 12. Behavior θ at varied values of the parameter γ of the system (32)

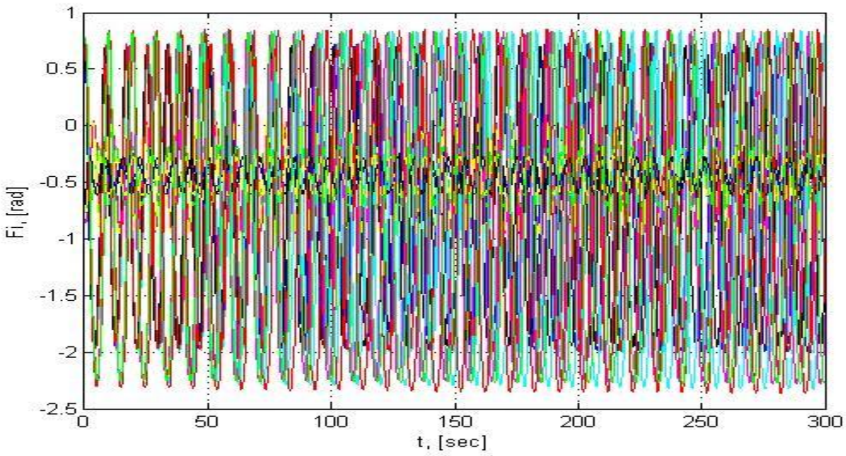
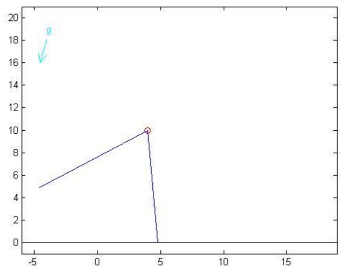


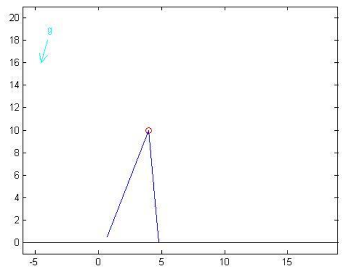
Figure13. Behavior φ at varied values of the parameter γ of the system (32)

All graphs confirm that finally stance leg takes stable almost vertical position and swinging leg oscillates with constant amplitude and at stable position with respect to stance leg.

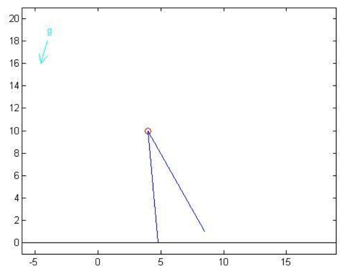
The dynamics of the system (32) corresponds to the motion of the mechanism presented in the Fig. 14.



Position 1



Position 2



Position 3

Figure 14. Motion of the designed controlled pendulum

Conclusion

Structurally stable functions from catastrophe theory are very useful as controllers and give many advantages. The main of them is that in linear case the safe ranges of parameters are widened significantly because the designed system stay stable within unbounded ranges of perturbation of parameters even the sign of them changes. The behaviors of designed control systems obtained by MATLAB simulation of double pendulum's oscillations confirm the efficiency of the offered method. The offered approach of design can be applied not only for linear but also for some set or class of nonlinear dynamic plants. For further research and investigation many perspective tasks can occur such that synthesis of control systems with special requirements, design of optimal control, control of chaos, etc.

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