Violation of First Law of Thermodynamics in $f(R,T)$ Gravity

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Abstract: In this Letter, we derived the first law of thermodynamics using the method proposed by Wald. Treating the entropy as Noether charge and comparing with the usual first law of thermodynamics, we obtained the expression of entropy explicitly which contains infinitely many non-local terms (i.e. the integral terms). We have proved, in general, that the first law of black bole thermodynamics is violated for $f(R,T)$ gravity. But there might exist some special cases in which the first law for $f(R,T)$ gravity is recovered.

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From the cosmological observational data [1–4], we know that the present observable Universe is undergoing an accelerated expansion. While the source driving this cosmic acceleration is known as 'dark energy' its origin has not been fully understood due to absence of a consistent theory of quantum gravity. The 'cosmological constant' is the most simple and natural candidate for explaining cosmic acceleration but it faces serious problems of fine-tuning and large mismatch between theory and observations [5–7]. Hence there has been significant development in the construction of dark energy models by modifying the geometrical part of the Einstein-Hilbert action. This phenomenological approach is called as the Modified Gravity which is compatible with the observational data [8–14] (also see a recent review [15] on $f(R)$ gravity and its cosmological implications).

In a recent paper [16], the authors considered a generalized gravity model $f(R,T)$, with R and T being the trace of Riemann curvature tensor and stress-energy tensor, respectively, manifesting a coupling between matter and geometry. By choosing different functional forms of f, they solved the dynamical equations relevant to astrophysical and cosmological interest. Different aspects of this model discussed in the literatures [17–21]. In this paper we discuss the first law of thermodynamics and the Wald's entropy expression for $f(R,T)$ gravity without imposing any restrictions on the action. We show that the entropy can not be obtained by a simple replacement of the Newtonian gravitational constant $8\pi G$ by an effective value $8\pi G_{\text{eff}} = 8\pi G + f_T$ (we will set $G = 1$ in later calculations) as some authors discussed [22]. Further, we will show that the first law of the thermodynamics is violated in $f(R,T)$ gravity. Only when $f_{TT} << 1$, the entropy can be obtained using G_{eff}, but even in this limit the first law of the thermodynamics is violated in contradiction to previous statements [22].

The action of $f(R,T)$ gravity is given by [16, 23]

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\sqrt{-g}f(R,T) + \int d^4x\mathcal{L}_m\sqrt{-g}, \tag{1}$$

where $f(R,T)$ is an arbitrary function of the scalar curvature $R = R^\mu_\mu$ and the trace $T = T^\mu_\mu$ of the energy-momentum tensor $T_{\mu\nu}$. This model is an extension of a former model $f(R)$. In this extension, there is a non-minimally coupling between the Ricci scalar R and the trace of the energy-momentum tensor. We define the Lagrangian density for matter field $\mathcal{L}_m$ by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \tag{2}$$

The equation of motion (EOM) is obtained by varying the action (1) with respect to $g^{\mu\nu}$ [16]

$$f_R R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R = 8\pi T_{\mu\nu} - f_T T_{\mu\nu} - f_T\Theta_{\mu\nu}. \tag{3}$$

where $f_R \equiv \frac{\partial f}{\partial R}$ and $f_T \equiv \frac{\partial f}{\partial T}$. Here ∇_μ is the operator for covariant derivative and box operator (or d' Alembert operator) $\Box$ is defined via

$$\Box \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu), \quad \Theta_{\mu\nu} \equiv g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}.$$

If $f(R,T)$ is replaced with $f(R)$, this equation is just the $f(R)$ gravity equation. It is easy to show that the left hand side of this equation is the same as the $f(R)$ model. The main difference backs to the right hand side. The contributions from f_T comes in the energy-momentum tensor part in the right hand side of the gravitational field equation. Performing a contraction of indices in (3), we obtain

$$Rf_R + 3\Box f_R - 2f = 8\pi T - Tf_T - \Theta f_T. \tag{4}$$

Here $\Theta \equiv g^{\mu\nu}\Theta_{\mu\nu}$. For a perfect fluid with the following energy-momentum tensor

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - pg_{\mu\nu}, \tag{5}$$

we have

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu}. \tag{6}$$

We can write the field equation for dust $p = 0$ in the following form

$$f_R R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R = (8\pi + f_T)T_{\mu\nu}. \tag{7}$$

The left hand side of this equation is the same as the equation of the motion in the $f(R)$ gravity, if $f(R,T) \longrightarrow f(R)$. So, it is comparable with the $f(R)$ gravity. But if $p \neq 0$, then in the right hand side an extra term $pf_T g_{\mu\nu}$ remains. This term can be combined in the left hand side as

$$-(pf_T + \frac{1}{2}f)g_{\mu\nu}$$

This term causes some problems in the process of obtaining the entropy, since now $p(R,T)$ is arbitrary. So it is not clear how we can treat this term in our calculations. As we know that when the pressure is included, a *work term* is required to modify the first law of the thermodynamics. It seems that the physical interpretation of such term is not easy. Therefore we focus only on the dust case with $p = 0$.

We adapt Wald's approach to construct entropy and first law of thermodynamics ($\delta Q = T\delta S$) in any gravity theory [24]. Consider a heat flux passing through an open patch $dH = dAd\lambda$, on a null surface of black hole horizon,

$$\delta Q = \int_H T_{\mu\nu}\xi^\mu k^\nu dAd\lambda, \tag{8}$$

where $T_{\mu\nu}$ is the stress energy tensor for energy-matter, ξ^μ is the killing vector, H represents the null surface, λ is the affine parameter, $k^\mu = \frac{dx^\mu}{d\lambda}$ is the tangent vector to H. Using (7) in (8), we obtain

$$\begin{aligned}\delta Q &= \int\limits_H \frac{1}{8\pi + f_T}[f_R R_{\mu\nu} - \nabla_\mu\nabla_\nu f_R]\xi^\mu k^\nu dA d\lambda, \\ &= \int\limits_H [f_R R_{\mu\nu} - \nabla_\mu\nabla_\nu f_R]\Big(\frac{\xi^\mu}{8\pi + f_T}\Big)k^\nu dA d\lambda, \\ &= \int\limits_H \Big[f_R\nabla_\mu\nabla_\nu\Big(\frac{\xi^\mu}{8\pi + f_T}\Big) - \Big(\frac{\xi^\mu}{8\pi + f_T}\Big)\nabla_\mu\nabla_\nu f_R\Big]k^\nu dA d\lambda,\end{aligned}$$

where we used $R_{\mu\nu}\xi^\mu = \nabla_\mu\nabla_\nu\xi^\mu$ in the above steps. We can further simplify

$$\begin{aligned}\delta Q &= \int\limits_H k^\nu\nabla^\mu\Big[f_R\nabla_\nu\Big(\frac{\xi_\mu}{8\pi + f_T}\Big)\Big]dA d\lambda, \\ &= \int\limits_H k^\nu l^\mu f_R\nabla_\nu\Big(\frac{\xi_\mu}{8\pi + f_T}\Big)dA d\lambda, \\ &= \int\limits_H k^\nu l^\mu f_R\Big[\frac{\nabla_\nu\xi_\mu}{8\pi + f_T} - \frac{\xi_\mu\nabla_\nu f_T}{8\pi + f_T}\Big]dA d\lambda, \\ &= \frac{1}{8\pi}\int\limits_H k^\nu l^\mu f_R\Big[\nabla_\nu\xi_\mu\Big(1 - \frac{f_T}{8\pi} + \Big(\frac{f_T}{8\pi}\Big)^2 - \dots\Big) - \xi_\mu\nabla_\nu f_T\Big(1 - \frac{f_T}{8\pi} + \Big(\frac{f_T}{8\pi}\Big)^2 - \dots\Big)\Big]dA d\lambda,\end{aligned}$$

where we assumed $f_T < 1$. Simplifying further

$$\begin{aligned}\delta Q &= \frac{1}{8\pi}\int\limits_H k^\nu l^\mu f_R\nabla_\nu\xi_\mu dA d\lambda - \frac{1}{(8\pi)^2}\int\limits_H k^\nu l^\mu f_R f_T\nabla_\nu\xi_\mu dA d\lambda + \frac{1}{(8\pi)^3}\int\limits_H k^\nu l^\mu f_R f_T^2\nabla_\nu\xi_\mu dA d\lambda - \dots \\ &\quad - \frac{1}{8\pi}\int\limits_H k^\nu l^\mu f_R\xi_\mu\nabla_\nu f_T dA d\lambda + \frac{1}{(8\pi)^2}\int\limits_H k^\nu l^\mu f_R f_T\xi_\mu\nabla_\nu f_T dA d\lambda - \frac{1}{(8\pi)^3}\int\limits_H k^\nu l^\mu f_R f_T^2\xi_\mu\nabla_\nu f_T dA d\lambda \\ &\quad + \dots\end{aligned}$$

$$\begin{aligned}\delta Q &= \frac{\kappa}{2\pi}\Big(\frac{f_R dA}{4}\Big)\Big|_0^{d\lambda} - \frac{1}{8\pi}\frac{\kappa}{2\pi}\Big(\frac{f_R f_T dA}{4}\Big)\Big|_0^{d\lambda} + \frac{1}{64\pi^2}\frac{\kappa}{2\pi}\Big(\frac{f_R f_T^2 dA}{4}\Big)\Big|_0^{d\lambda} - \dots \\ &\quad - \frac{1}{8\pi}\int\limits_H k^\nu l^\mu f_R\xi_\mu\nabla_\nu f_T dA d\lambda + \frac{1}{(8\pi)^2}\int\limits_H k^\nu l^\mu f_R f_T\xi_\mu\nabla_\nu f_T dA d\lambda - \frac{1}{(8\pi)^3}\int\limits_H k^\nu l^\mu f_R f_T^2\xi_\mu\nabla_\nu f_T dA d\lambda \\ &\quad + \dots\end{aligned} \tag{9}$$

It should be stressed that in deriving the first law in Eq. (8), we have used the formula

$$R_{\mu\nu}\xi^\mu = \nabla_\mu\nabla_\nu\xi^\mu$$

which is valid only for an exact Killing vector ξ^{μ}. However in general for $f(R,T)$, there does not exist any exact Killing vector in a dynamic spacetime.

It should be mentioned that we have proved that, in general, the first law of black bole thermodynamics is violated for $f(R,T)$ gravity. But there might exist some special cases in which the first law for $f(R,T)$ recovers. Note that for black holes with the same metric $g_{\mu\nu}$, we have many different choices of energy-momentum tensor $T_{\mu\nu}$ by which they can be related with each other by some diffeomorphism invariance transformations. Those black holes have the same $k^{\nu}\nabla_{\nu}\xi_{\mu}$ but different other terms. So, for some special cases of the $f(R,T)$ models, the two terms might cancel each other and the second term of the last line of Eq. (8) vanishes.

Now using (9), we find

$$\delta Q \ = \ T\delta S - T\delta S_1 + T\delta S_2 - \ldots - \frac{1}{8\pi}\int\limits_{H} k^{\nu}l^{\mu}f_R\xi_{\mu}\nabla_{\nu}f_T dA d\lambda + \ldots \tag{10}$$

where

$$\delta S_1 = \frac{1}{8\pi}\left(\frac{f_R dA}{4}\right)\Big|_0^{d\lambda}, \quad \delta S_2 = \frac{1}{(8\pi)^2}\left(\frac{f_R f_T dA}{4}\right)\Big|_0^{d\lambda}, \ldots$$

Thus the final expression of entropy in $f(R,T)$ model becomes

$$S = \frac{f_R A}{4} - \frac{1}{8\pi}\frac{f_R f_T A}{4} + \frac{1}{64\pi^2}\frac{f_R f_T^2 A}{4} - \frac{1}{4\kappa}\int\limits_{H} k^{\nu}l^{\mu}f_R\xi_{\mu}\nabla_{\nu}f_T dA d\lambda - \ldots \tag{11}$$

The last term indicates the entropy production in $f(R,T)$ theory, even in the static cases, and this result is the same as the previous result in $f(T)$ gravity [25].

Since $\nabla_{\nu}f_T = f_{TT}T_{\nu}$, and if f_{TT} is small, than (11) reduces to

$$S = \frac{f_R A}{4} - \frac{1}{8\pi}\frac{f_R f_T A}{4} + \frac{1}{(8\pi)^2}\frac{f_R f_T^2 A}{4} - \ldots \tag{12}$$

Alternatively (if $S_{f(R)} \equiv \frac{f_R A}{4}$)

$$S = S_{f(R)} - \frac{f_T}{8\pi}S_{f(R)} + \frac{f_T^2}{(8\pi)^2}S_{f(R)} - \ldots \tag{13}$$

The first term on right hand side of (13) is the Wald entropy for $f(R)$ gravity which can be obtained by several methods. But the remaining terms make our results new compared with $f(R)$ gravity. These new terms in entropy arise due to non-minimal coupling of curvature with matter in the action. It shows that the thermodynamical aspects $f(R)$ and $f(R,T)$ gravities are completely different. Further these extra terms modify the Friedmann equations of $f(R,T)$ gravity.

In this paper, we discussed the thermodynamical properties of $f(R,T)$ gravity. We derived the first law of thermodynamics using the method proposed by Wald. Treating the entropy as Noether charge and comparing with the usual first law of thermodynamics, we obtained the expression of entropy explicitly which contains infinitely many non-local terms (i.e. the integral terms) using the method proposed method by Wald. Further we corrected some errors about entropy in a previous work [22]. The extra terms in (10) and (11) can be understood as 'entropy production' and thus refer to non-equilibrium thermodynamics [26]. The first law of thermodynamics does not hold, as a consequence of entropy production terms in $f(R,T)$ gravity.

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