

Some Models of Cyclic and Knot Universes

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Abstract

In this paper, we obtained a class of oscillatory, cyclic and knot type solutions from the non-linear Friedmann equations. This is performed by choosing specific forms of energy density and pressure of matter. All the expressions written here are in dimensionless form. We show that evolutionary path taken by the spatial coordinates in the model follow various knots, specifically trefoil and eight-knots. We provide several examples and plot relevant cosmological parameters in figures. Our cyclic models can be interpreted as a periodic cosmological model, such that early and late time acceleration are unified under the same mechanism. Finally we have presented some examples of knot universes for the Bianchi - I spacetime.

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1 Introduction

The understanding of formation and evolution of the observable universe is the main theme of modern cosmology. About the former problem, there are several candidates including the standard cosmological model which predicts a singularity out of which our universe was born, while it predicts nothing about what happened before that. Other models like braneworlds and cyclic models predict a cyclic nature of cosmic evolution, according to which universe evolves from one singular state (big bang) to another one [1]. Another interesting theoretical problem is the phenomenon of cosmic acceleration occurring at present time, usually attributed to ‘dark energy’ (DE). There are numerous approaches to study this phenomena [2], however, it is fair to say that none of them is particularly well motivated, and many of them appear like acts of desperation on the part of theorists. From this situation we sometimes come to the conclusion that perhaps we must return to the origin and study in detail the mathematical nature of the basic gravitational equations. Following this idea we have studied some integrable and non-integrable reductions of the Einstein equation (Friedmann equation) in [3]-[5]. In [6], we have established the relationship between solutions of the Einstein equation and the Ramanujan and Chazy equations.

Returning to a cyclic scenario which in fact was proposed long time ago [7] we note that the idea to consider the whole universe as one cyclic system has already attracted many authors. In particular, it has achieved a lot of successes in recent decades (see e.g. Refs. [8]-[19] and references therein). One of interesting aspects of cyclic scenario is its possible connection with the knot scenario proposed in [20] (see also Ref. [6]). One of important branches of mathematics (algebraic geometry), the knot theory describe a depth properties of the 3-dimensional space [21]-[26]. So that these arguments in our opinion give us physical and mathematical motivations in order to further investigation the relationship between cyclic universe models and knot universe models. In this paper, we study in more detail the relationship between some models of the cyclic universe and the knot theory. As examples we consider the trefoil and figure-eight knot universe models.

The paper is organized as follows. In Sec. II we briefly give some basic equations of the Einstein gravity. Sec. III is devoted to study the relation between the Friedmann equation with some Equation of state and the trefoil knot and Sec. IV with the figure-eight knot. Some similar models of a cyclic universe are studied in Sec.V. A new realizations of knot universes for the Bianchi - I models are considered in Sec. VI. The last section is devoted to the conclusion.

2 The model and basic equations

In this section we briefly review some basic facts about the Einstein’s field equation. We start from the standard gravitational action (chosen units are $c = 8\pi G = 1$)

$$S = \frac{1}{4} \int d^4x \sqrt{-g} (R - 2\Lambda + L_m), \quad (2.1)$$

where R is the Ricci scalar, Λ is the cosmological constant and L_m is the matter Lagrangian. For a general metric $g_{\mu\nu}$, the line element is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad \mu, \nu = 0, 1, 2, 3 \quad (2.2)$$

the corresponding Einstein field equations are given by

$$R_{\mu\nu} + \left(\Lambda - \frac{1}{2} R \right) g_{\mu\nu} = -T_{\mu\nu}, \quad (2.3)$$

Here $R_{\mu\nu}$ is the Ricci tensor. This equation forms the mathematical basis of the theory of general relativity. In (3), $T_{\mu\nu}$ is the energy-momentum tensor of the matter field defined as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta g^{\mu\nu}}, \quad (2.4)$$

and satisfies the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0, \quad (2.5)$$

where ∇_μ is the covariant derivative which is the relevant operator to smooth a tensor on a differentiable manifold. Eq. (5) yields the conservations of energy and momentums, corresponding to the independent variables involved. The general Einstein equation (3) is a set of non-linear partial differential equations. We consider the Friedmann-Robertson-Walker (FRW) metric which represents a spatial and homogeneous spacetime relevant to cosmological perspective. This metric has the form (in suitable dimensionless coordinates)

$$ds^2 = d\tau^2 - a(\tau)^2[(dx'_1)^2 + (dx'_2)^2 + (dx'_3)^2]. \quad (2.6)$$

Here we employed the following notations $x'_1 = x_1/x_{10}$, $x'_2 = x_2/x_{20}$, $x'_3 = x_3/x_{30}$ and $\tau = t/t_0$ for making the above equation dimensionless. Quantities with subscript 0 refer to their values at present epoch, that is $x_{10}, x_{20}, x_{30}, t_0$ are some constants. We write the corresponding Einstein equations (Friedmann equation and the conservation equation) in the following dimensionless form

$$3H^2 - \rho = 0, \quad (2.7)$$

$$2\dot{H} + 3H^2 + p = 0, \quad (2.8)$$

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (2.9)$$

where $H = (\ln a)_{,\tau} = a_{,\tau}/a$, is the dimensionless Hubble parameter and $\dot{H} = dH/d\tau$, ρ is the density of energy and p is the pressure. With the use of dimensionless variables, all the equations in this paper become dimensionless. Out of the three equations (7)-(9), only two are independent i.e. (7) and (8) or (7) and (9).

3 Knot universe: the trefoil knot case

The aim of this and next sections is to establish a connection between the cyclic universe models with the knot theory. In this section we want to construct the simplest examples of the knot universe, namely, the trefoil knot universe. Let us consider some explicit examples.

3.1 Example 1

Let the equation of state (EoS) that is the pressure p and energy density ρ have the following forms [6]

$$\rho(\tau) = 3r^2 \cos^2(2\tau), \quad (3.1)$$

$$p(\tau) = 6 \sin(3\tau) \cos(2\tau) + 4r \sin(2\tau) - 3r^2 \cos^2(2\tau), \quad (3.2)$$

where

$$r \equiv 2 + \cos(3\tau),$$

is another dimensionless parameter. Now substituting these expressions for the energy density and the pressure into the equations (7)-(8), we get

$$H(\tau) = r \cos(2\tau). \quad (3.3)$$

Such periodic Hubble's parameters are familiar from previous works on occurrence of an oscillating universe, it has been shown using an inhomogeneous equation of state for dark energy fluid, the Hubble parameter presents a periodic behavior such that early and late time acceleration are unified under the same mechanism [27].

Upon integration, we obtain the scale factor for this model

$$a(\tau) = a_0 e^{[0.5 \sin(\tau) + \sin(2\tau) + 0.1 \sin(5\tau)]}, \quad (3.4)$$

where $a_0 = a(\tau = 0)$ is the value of scale factor taken at present time. Having obtained relevant parameters, we can discuss some cosmological implications of this model. The first quantity is deceleration parameter q defined as

$$q = -1 - \frac{\dot{H}}{H^2}. \quad (3.5)$$

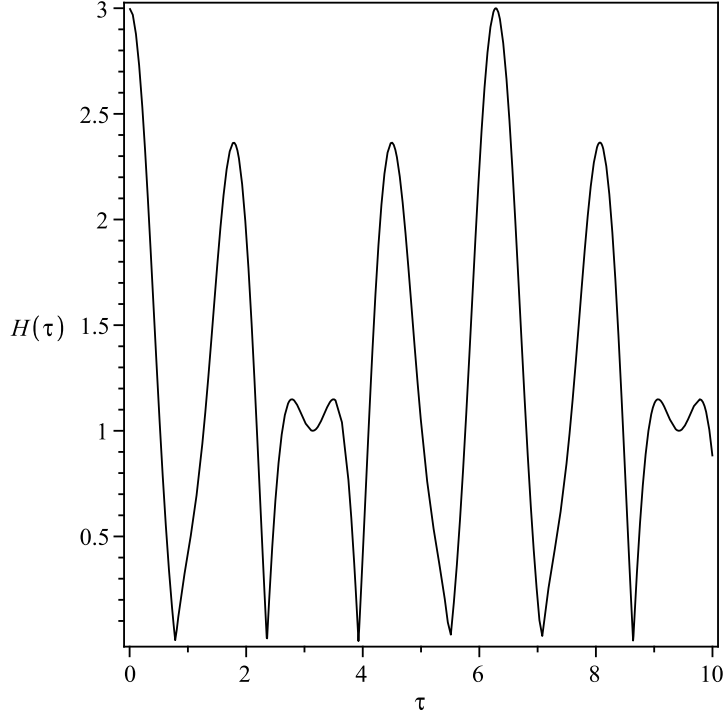


Figure 1: The Hubble parameter is plotted as given in (12).

A universe expanding with acceleration (deceleration) has $q < 0$ ($q > 0$). Since the observable universe is currently in accelerated expansion, we demand $q > -1$. Thus using (12) and (14) we have

$$q = -1 + \frac{3 \sin(3\tau) \cos(2\tau) + 2(2 + \cos(3\tau)) \sin(2\tau)}{(2 + \cos(3\tau))^2 \cos^2(2\tau)}, \quad (3.6)$$

The EoS parameter for this example is given by

$$\omega \equiv \frac{p}{\rho} = \frac{6 \sin(3\tau) \cos(2\tau) + 4r \sin(2\tau) - 3r^2 \cos^2(2\tau)}{3r^2 \cos^2(3\tau)}. \quad (3.7)$$

Figure (1) shows time evolution of the H given by (12). It may be non differentiable [27]. It can be used for a periodic behavior such that early and late time acceleration are unified under the same mechanism.

Let us introduce three new variables as

$$x = H, \quad (3.8)$$

$$y = \sqrt{r^2 - H^2}, \quad (3.9)$$

$$z = \sqrt{1 - (r - 2)^2}. \quad (3.10)$$

Using the above expressions of H and r , we can rewrite equations (17)-(19) as [6]

$$x = r \cos(2\tau), \quad (3.11)$$

$$y = r \sin(2\tau), \quad (3.12)$$

$$z = \sin(3\tau). \quad (3.13)$$

Eqs. (20-22) are nothing but the parametric equations for the trefoil knot (see Figure 2). This curve lies entirely on the torus

$$(r - 2)^2 + z^2 = 1, \quad (3.14)$$

making trefoil the simplest example of a torus knot. In fact, the trefoil is a (2,3)-torus knot because the curve winds around the torus thrice in one direction and twice in the other direction.

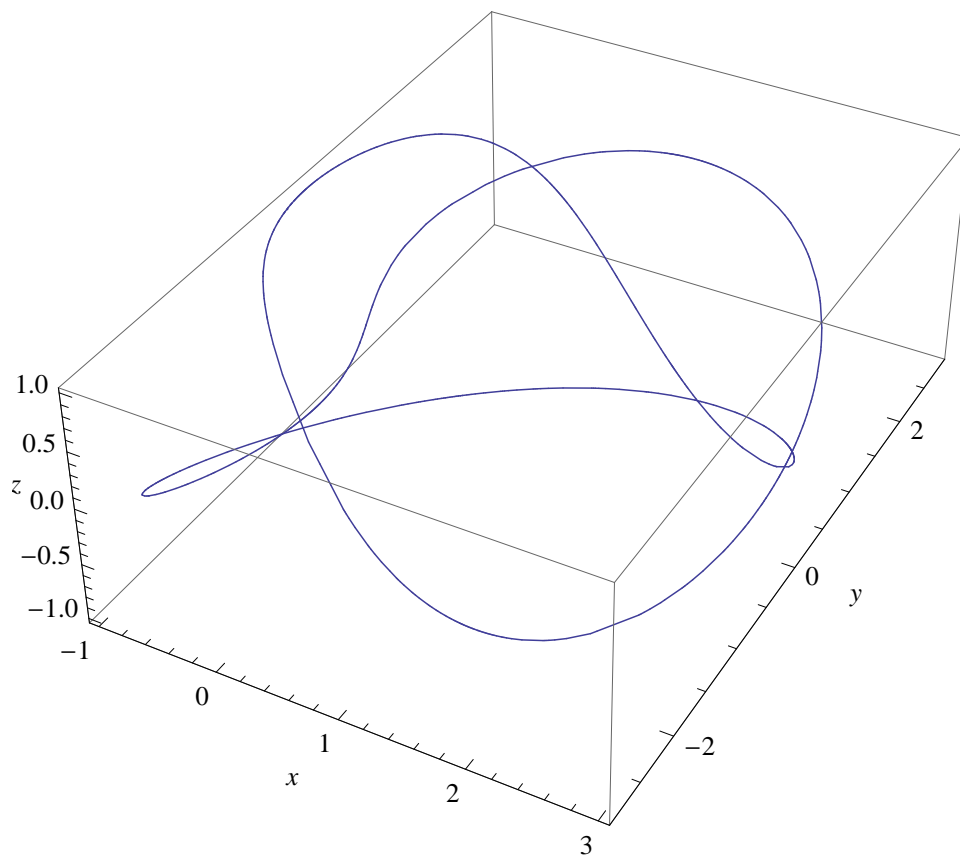


Figure 2: The trefoil knot for the equations (21)-(23).

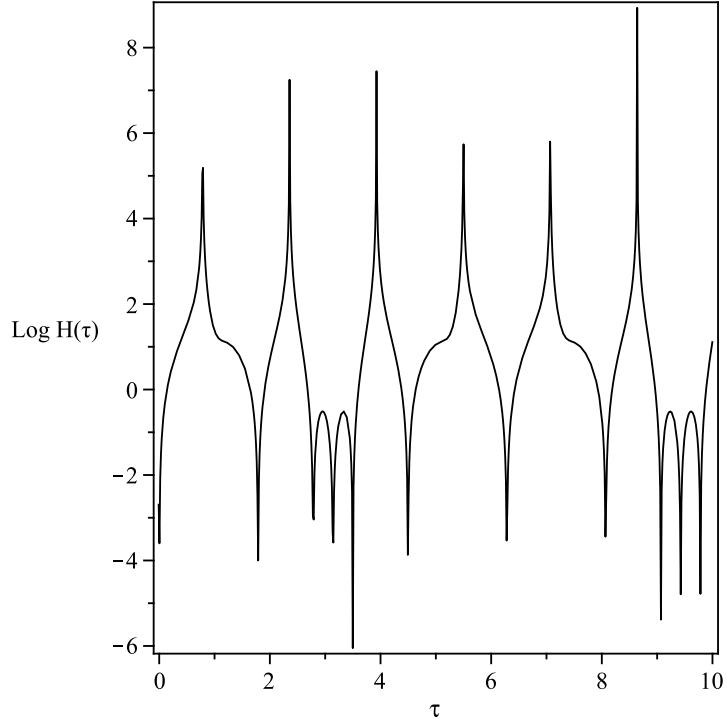


Figure 3: The Log-Hubble parameter given by (25).

3.2 Example 2

We consider the scale factor of the form

$$a' = \frac{a}{a_0} = r \cos(2\tau), \quad (3.15)$$

where $a_0 = a(t = 0)$ and assume $a_0 = 1$. Thus the Hubble parameter becomes

$$H = [\ln(r \cos(2\tau))]_{,\tau} = -\frac{3 \sin(3\tau) \cos(2\tau) + 2(2 + \cos(3\tau)) \sin(2\tau)}{(2 + \cos(3\tau)) \cos(2\tau)}. \quad (3.16)$$

In figure 3, we show the time evolution of $\log H$ vs τ . It describes a periodic behavior such that early and late time acceleration are unified under the same mechanism.

Making use of (7), (8) and (25), the energy density and pressure become

$$\rho = 3\{[\ln(r \cos(2\tau))]_{,\tau}\}^2, \quad (3.17)$$

$$p = -2[\ln(r \cos(2\tau))]_{,\tau\tau} - 3\{[\ln(r \cos(2\tau))]_{,\tau}\}^2. \quad (3.18)$$

The corresponding parameter of the EoS is

$$\omega = -1 - \frac{2}{3} \frac{[\ln(r \cos(2\tau))]_{,\tau\tau}}{\{[\ln(r \cos(2\tau))]_{,\tau}\}^2}. \quad (3.19)$$

To describe the trefoil knot, we introduce three new variables x, y, z as:

$$x = a', \quad (3.20)$$

$$y = \sqrt{r^2 - a'^2}, \quad (3.21)$$

$$z = \sqrt{1 - (r - 2)^2}, \quad (3.22)$$

where $a' = a/a_0$, $a_0 = \text{constant}$. In terms of τ these functions take the same form as in (20)-(22) so that they describe the trefoil knot curve. This means that our model with the EoS (28) corresponds to a knot universe if be exactly to the trefoil knot universe.

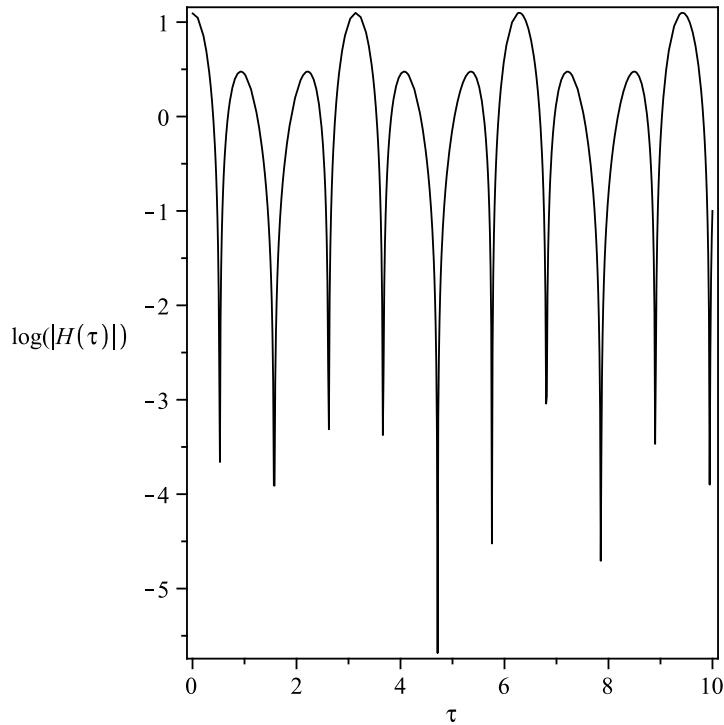


Figure 4: The Log-Hubble parameter given by (34).

4 Knot universe: the figure-eight knot case

Now consider a more complex case than the previous trefoil knot universe, namely, the figure-eight knot. Here some examples of such figure-eight knot universes.

4.1 Example 1

Let the EoS is given in the parametric form [6]

$$\rho = 3h^2 \cos^2(3\tau), \quad (4.1)$$

$$p = 4 \sin(2\tau) \cos(3\tau) + 6h \sin(3\tau) - 3h^2 \cos^2(3\tau), \quad (4.2)$$

where

$$h = 2 + \cos(2\tau).$$

Equations (7)-(8) then give

$$H = h \cos(3\tau). \quad (4.3)$$

Figure 4 shows the time evolution of the H for this example. It may be interpreted as a periodic cosmological model, such that early and late time acceleration are unified under the same mechanism.

This result (34) tells us that the scale factor has the form

$$a(\tau) = a_0 e^{[0.5 \sin(\tau) + \frac{2}{3} \sin(3\tau) + 0.1 \sin(5\tau)]}. \quad (4.4)$$

The corresponding EoS parameter for this model is

$$\omega = \frac{p}{\rho} = \frac{4 \sin(2\tau) \cos(3\tau) + 6h \sin(3\tau) - 3h^2 \cos^2(3\tau)}{3h^2 \cos^2(3\tau)}. \quad (4.5)$$

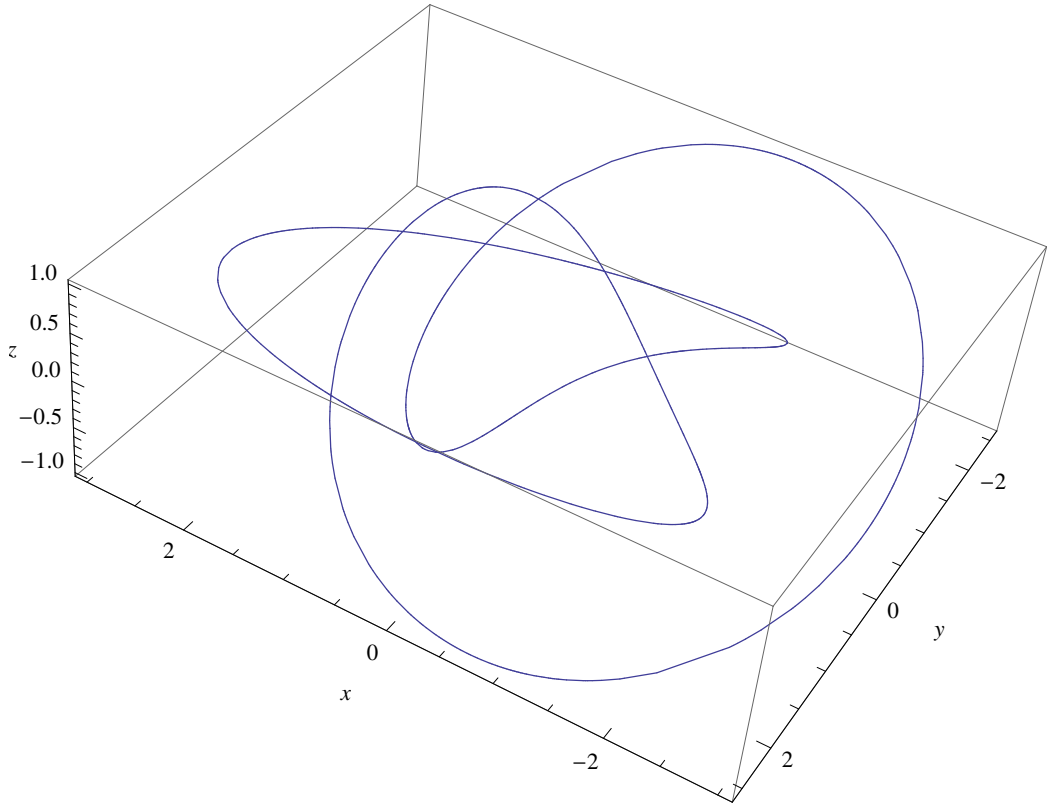


Figure 5: The figure-eight knot for the equations (40)-(42).

As above, we introduce three new functions as:

$$x = H, \quad (4.6)$$

$$y = \sqrt{h^2 - H^2}, \quad (4.7)$$

$$z = 2(h-2)\sqrt{1-(h-2)^2}. \quad (4.8)$$

This system x, y, z can be rewritten in parametric form as [6]

$$x = h \cos(3\tau), \quad (4.9)$$

$$y = h \sin(3\tau), \quad (4.10)$$

$$z = \sin(4\tau), \quad (4.11)$$

which is nothing but the parametric equations of the figure-eight knot (see Figure 5). Note that this figure-eight knot satisfies the equation

$$4(h-2)^4 - 4(h-2)^2 + z^2 = 0. \quad (4.12)$$

4.2 Example 2

Consider the scale factor of the form

$$a' = h \cos(3\tau) = (2 + \cos(2\tau)) \cos(3\tau), \quad (4.13)$$

so that the Hubble parameter takes the form

$$H = [\ln(h \cos(3\tau))]_{,\tau} = -\frac{2 \sin(2\tau) \cos(3\tau) - 3(2 + \cos(2\tau)) \sin(3\tau)}{(2 + \cos(2\tau))^2 \cos^2(3\tau)}. \quad (4.14)$$

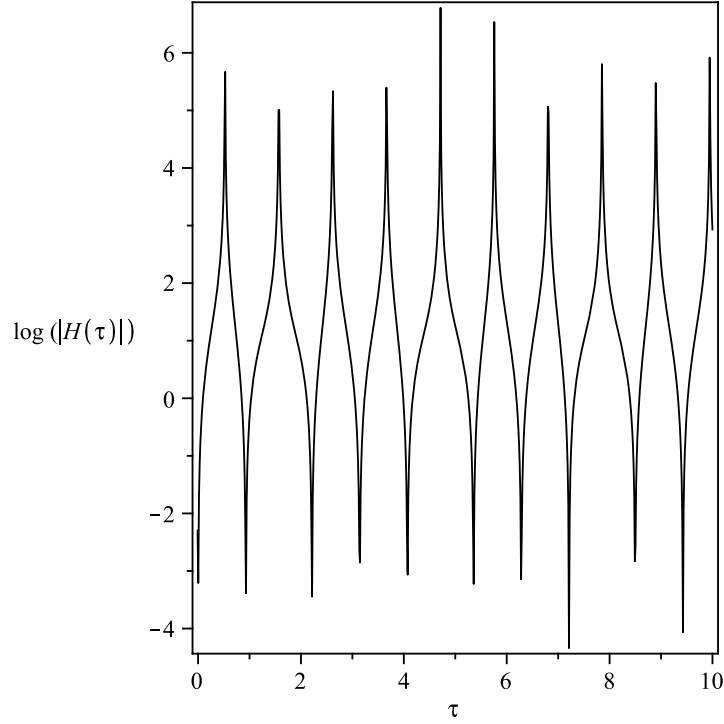


Figure 6: The Hubble parameter given by (46).

Figure 6 shows the time evolution of the H for this example. It is a periodic cosmological model, such that the early and late time accelerations are unified under the same mechanism.

Then the equations (7)-(8) give the following parametric EoS

$$\rho = 3\{[\ln(h \cos(3\tau))]_{,\tau}\}^2, \quad (4.15)$$

$$p = -2[\ln(h \cos(3\tau))]_{,\tau\tau} - 3\{[\ln(h \cos(3\tau))]_{,\tau}\}^2 \quad (4.16)$$

which gives the parameter of the EoS in the form

$$\omega = -1 - \frac{2}{3} \frac{[\ln(r \cos(2\tau))]_{,\tau\tau}}{\{[\ln(r \cos(2\tau))]_{,\tau}\}^2}. \quad (4.17)$$

Again we introduce three new variables x, y, z as:

$$x = a', \quad (4.18)$$

$$y = \sqrt{h^2 - a'^2}, \quad (4.19)$$

$$z = 2(h-2)\sqrt{1 - (h-2)^2}, \quad (4.20)$$

where $a' = a/a_0$, $a_0 = \text{const}$. If we rewrite these equations in terms of τ then we get again the system (40)-(42) which are the parametric equations for the figure-eight knot. So this model with the EoS (46)-(47) describes the figure-eight knot universe.

5 Similar models of a cyclic universe

5.1 Example 1

Let us consider an oscillating universe described by the Hubble parameter

$$H(\tau) = \alpha + \beta \cos(\nu\tau). \quad (5.1)$$

Then, by substituting this expression into the equations (7), (8), we obtain

$$\rho = 3(\alpha + \beta \cos(\nu\tau))^2, \quad (5.2)$$

$$p = 2\beta\nu \sin(\nu\tau) - 3(\alpha + \beta \cos(\nu\tau))^2, \quad (5.3)$$

so that the EoS parameter becomes

$$\omega = -1 + \frac{2\beta\nu \sin(\nu\tau)}{3(\alpha + \beta \cos(\nu\tau))^2}. \quad (5.4)$$

The corresponding scale factor is

$$a(\tau) = a_0 e^{[\alpha\tau + \frac{\beta}{\nu} \sin(\nu\tau)]}. \quad (5.5)$$

5.2 Example 2

One can consider now an oscillating universe described by the Hubble parameter

$$H(\tau) = \beta \sin(\nu\tau), \quad (5.6)$$

where β and ν are constants. Then, by substituting this expression into the equations (7),(8), we get

$$\rho = 3\beta^2 \sin^2(\nu\tau), \quad (5.7)$$

$$p = -2\beta\nu \cos(\nu\tau) - 3\beta^2 \sin^2(\nu\tau). \quad (5.8)$$

so that the EoS parameter is given by

$$\omega = -1 - \frac{2\beta\nu \cos(\nu\tau)}{3\beta^2 \sin^2(\nu\tau)}. \quad (5.9)$$

For the scale factor we get

$$a(\tau) = a_0 e^{[-\frac{\beta}{\nu} \cos(\nu\tau)]}. \quad (5.10)$$

5.3 Example 3

Let's take

$$a(\tau) = \alpha + \beta \cos^2(\nu\tau). \quad (5.11)$$

Then, the Hubble parameter becomes

$$H(\tau) = -\frac{2\beta\nu \cos(\nu\tau) \sin(\nu\tau)}{\alpha + \beta \cos^2(\nu\tau)}. \quad (5.12)$$

By substituting this expression into the equations (7)-(8), it yields,

$$\rho = 12 \frac{\beta^2 \nu^2 \cos^2(\nu\tau) \sin^2(\nu\tau)}{(\alpha + \beta \cos^2(\nu\tau))^2}, \quad (5.13)$$

$$p = -12 \frac{\beta^2 \nu^2 \cos^2(\nu\tau) \sin^2(\nu\tau)}{(\alpha + \beta \cos^2(\nu\tau))^2} + \frac{4\beta\nu^2 [\beta \cos^4(\nu\tau) + \beta \cos^2(\nu\tau) \sin^2(\nu\tau) + \alpha(\cos^2(\nu\tau) - \sin^2(\nu\tau))]}{(\alpha + \beta \cos^2(\nu\tau))^2}. \quad (5.14)$$

So that the EoS parameter is given by

$$\omega = -1 + \frac{\beta \cos^4(\nu\tau) + \beta \cos^2(\nu\tau) \sin^2(\nu\tau) + \alpha(\cos^2(\nu\tau) - \sin^2(\nu\tau))}{3\beta \cos^2(\nu\tau) \sin^2(\nu\tau)}. \quad (5.15)$$

5.4 Example 4

Our next example is:

$$a(\tau) = \alpha + \beta \cos(\nu\tau). \quad (5.16)$$

Then, the Hubble parameter becomes

$$H(\tau) = -\frac{\beta\nu \sin(\nu\tau)}{\alpha + \beta \cos(\nu\tau)}. \quad (5.17)$$

By substituting this expression into the equations (7)-(8), we obtain

$$\rho = \frac{3\beta^2\nu^2 \sin^2(\nu\tau)}{(\alpha + \beta \cos(\nu\tau))^2}, \quad (5.18)$$

$$p = -\frac{3\beta^2\nu^2 \sin^2(\nu\tau)}{(\alpha + \beta \cos(\nu\tau))^2} + \frac{2\beta\nu^2(\beta + \alpha \cos^2(\nu\tau))}{(\alpha + \beta \cos(\nu\tau))^2}. \quad (5.19)$$

So that the EoS parameter simplifies to

$$\omega = -1 + \frac{2}{3\beta} \frac{\beta + \alpha \cos(\nu\tau)}{\sin^2(\nu\tau)}. \quad (5.20)$$

5.5 Example 5

General model of a cyclic universe: We introduce

$$\rho = \varrho_0 \cos^2(2\pi\nu\tau + \theta) \quad (5.21)$$

$$p = -\frac{\sqrt{\varrho_0}}{3} \left(3 (\cos(2\pi\nu\tau + \theta))^2 \sqrt{\varrho_0} - 4\pi\nu\sqrt{3} \sin(2\pi\nu\tau + \theta) \right). \quad (5.22)$$

We get

$$H = \sqrt{\frac{\varrho_0 \cos(2\pi\nu\tau + \theta)}{3}}. \quad (5.23)$$

Hence we have

$$a(\tau) = a_0 e^{\left[\sqrt{\frac{\varrho_0}{3}} \frac{1}{\pi\nu} \text{EllipticE}(\pi\nu\tau + \frac{\theta}{2}, 2) \right]}, \quad (5.24)$$

where EllipticE is an elliptic integral of second kind.

The EoS of the model is

$$\omega = -\frac{3 (\cos(2\pi\nu\tau + \theta))^2 \sqrt{\varrho_0} - 4\sqrt{3} \sin(2\pi\nu\tau + \theta) \pi\nu}{3 (\cos(2\pi\nu\tau + \theta))^2 \sqrt{\varrho_0}}. \quad (5.25)$$

One can find a new set of coordinates x, y, z such as (40)-(42), and show that there are some knot like figures in a configuration space of these coordinates.

6 Conclusion

Going to some mathematical structures, the knot theory, we studied the relations between some oscillatory solutions of the FRW equations and the geometrical picture of some types of the knot. We show that, by assuming the periodic forms for pressure and energy density as a functions of time, there exists a coordinate set, in which the time evolutions of the space is knot like. This formal similarity repeated when we examined other types of the matter density and the pressure. Our work, exhibited some interesting features of the rich, hidden, mathematical structures of the non-linear FRW equations. Also, we discussed some models, described the existence of a cyclic universe. We obtained the exact solutions for the scale factor, the EoS parameter ω is all models. Also we have considered some examples knot universes for the Bianchi - I spacetime. Finally it is interesting to extend the results of this paper to the F(R) and F(G) gravity theories (see e.g. the ref. [38]-[39]) as well as F(T) gravity [?]-[?].

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