Harmonic Analysis

Norm inequalities for convolution operators

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Abstract

We study norm convolution inequalities in Lebesgue and Lorentz spaces. First, we improve the well-known O’Neil’s inequality for the convolution operators and prove corresponding estimate from below. Second, we obtain Young–O’Neil-type estimate in the Lorentz spaces for the limit value parameters, i.e., $\|K \ast f\|_{L(p,h_1) \to L(p,h_2)}$. Finally, similar estimates in the weighted Lorentz spaces are presented. To cite this article: E. Nursultanov et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009). © 2009 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Let $(\Omega, \mu)$ be a measurable space and $L_p(\Omega, \mu)$ be the collection of all those measurable functions $f$ satisfying $\|f\|_{L_p(\Omega, \mu)} = (\int_\Omega |f(x)|^p \, d\mu)^{1/p} < \infty$. The distribution of a measurable function $f$ on $\Omega$ is defined by $m(\sigma, f) = \mu(\{x \in \Omega : |f(x)| > \sigma\})$. Then $f^*(t) = \inf[\sigma : m(\sigma, f) \leq t]$ is the decreasing rearrangement of $f$.

Let $0 < p < \infty$ and $0 < q < \infty$. The Lorentz space $L_p,q(\Omega, \mu)$ is defined [3, Ch. 4] by those measurable functions $f$ such that

$$\|f\|_{L_p,q} = \left( \int_0^\infty (t^{1/p} f^*(t))^{q} \frac{dt}{t} \right)^{1/q} < \infty,$$

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In this paper we study norm estimates for the convolution operator
\[ (Af)(y) = (K * f)(y) = \int_D K(x-y)f(x) \, dx, \quad y \in \Omega \tag{1} \]
in the Lebesgue and Lorentz spaces.

In Section 2 we consider the upper and lower estimates of \( \|A\|_{L_p \to L_q} \). The upper estimate sharpens the known O’Neil and Stepanov inequalities. Section 3 is devoted to the O’Neil type inequalities in the Lorentz spaces. We study the case of limit value parameters, that is, \( \|A\|_{L_{p_1} \to L_{p_2}} \) for \( 1 \leq p \leq \infty \). Finally, in Section 4 we give norm convolution estimates in the more general Lorentz spaces. Detailed proof of these results can be found in [9–11].

2. Norm convolution inequalities in the Lebesgue spaces

The Young convolution inequality of the form \( \|K*f\|_{L_q} \leq \|f\|_{L_p} \|K\|_{L_r}, \quad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} \) plays a very important role both in Harmonic Analysis and PDE.

O’Neil [12] extended Young’s inequality as follows. Let \( \mu \) be the linear Lebesgue measure and \( L_p(\mathbb{R}) = L_{p,\mu}(\mathbb{R}, dx) \). Then (1 < \( p, q, r < \infty \))
\[ \|A\|_{L_p(\mathbb{R}, dx) \to L_q(\mathbb{R}, dx)} \leq C \|K\|_{L_{r,\mu}(\mathbb{R}, dx)}, \tag{2} \]
where \( A \) is given by (1) with \( \Omega = D = \mathbb{R} \).

Another extension of Young’s inequality was proved by Stepanov [13] using the Wiener amalgam space \( W(L_{r,\mathbb{R}}([0, 1], L_{r,\mathbb{Z}})) \) (see e.g. [6]): for \( 1 < p < q < +\infty \) and \( 1/r = 1 - 1/p + 1/q \) one has
\[ \|A\|_{L_p \to L_q} \leq C \|K\|_{W(L_{r,\mathbb{R}}([0, 1], L_{r,\mathbb{Z}}))}, \tag{3} \]
where \( \|K\|_{W(L_{r,\mathbb{R}}([0, 1], L_{r,\mathbb{Z}}))} := \|\tilde{K}\|_{L_{r,\mathbb{R}}([0, 1], L_{r,\mathbb{Z}})} := \sup_{n \in \mathbb{N}} n^{1/r} (\sup_{0 \leq t \leq 1} t^{1/r} \tilde{K}^*(t, \cdot))_n \), and \( \tilde{K}(m, x) := K(m + x), \quad m \in \mathbb{Z}, \quad x \in [0, 1] \). In [13] it was also shown that inequalities (2) and (3) are not comparable.

In this section we sharpen O’Neil and Stepanov inequalities (2) and (3) and give an estimate of \( \|A\| \) from below. We will need the following definitions. Let \( I \) be an interval with \( |I| = d \). Then \( T_I = \{I + kd\}_{k \in \mathbb{Z}} \) is a partition of \( \mathbb{R} \), i.e., \( \mathbb{R} = \bigcup_{k \in \mathbb{Z}} (I + kd) \). We define two collections of sets \( \mathcal{L}(I) \subset \mathcal{U}(I) \):
\[ \mathcal{L}(I) = \left\{ e: \ e = \bigcup_{k=1}^m ([a, b] + kd), \ [a, b] \subseteq I, \ m \in \mathbb{N} \right\} \]
and
\[ \mathcal{U}(I) = \left\{ e: \ e = \bigcup_{k=1}^m \omega_k, \ m \in \mathbb{N} \right\}, \]
where \( \{\omega_k\}_{k=1}^m \) is any collection of compact sets of equal measure \( |\omega_k| = d \) and such that each \( \omega_k \) belongs to a different element of \( T_I \).

**Theorem 1.** Let \( 1 < p < q < \infty \). Then for \( Af = K*f \) we have
\[ C_1 \sup_{I \in \mathcal{L}(I)} \sup_{e \in \mathcal{L}(I)} \frac{1}{|e|^{1/p-1/q}} \left| \int_E K(x) \, dx \right| \leq \|A\|_{L_p \to L_q} \leq C_2 \inf_{I \in \mathcal{U}(I)} \sup_{e \in \mathcal{U}(I)} \frac{1}{|e|^{1/p-1/q}} \left| \int_E K(x) \, dx \right|, \tag{4} \]
where the constants \( C_1 \) and \( C_2 \) depend only on \( p \) and \( q \).

For the certain regular kernels \( K \), for instance, monotone or quasi-monotone, the upper and lower bounds in (4) coincide, that is, we get the equivalent relation for \( \|A\|_{L_p \to L_q} \).
Corollary 1. Let \( 1 < p < q < \infty \) and \( K \) satisfy the following condition:
\[
|K(x)| \leq C \left| \frac{1}{x} \int_0^x K(t) \, dt \right|, \quad x \in \mathbb{R} \setminus \{0\}.
\]

Then a necessary and sufficient condition for \( Af = K \ast f \) to be bounded from \( L_p(\mathbb{R}) \) to \( L_q(\mathbb{R}) \) is
\[
S := \sup_{|x| > 0} \frac{1}{|x|^{1/p - 1/q}} \left( \int_0^x K(y) \, dy \right) < \infty.
\]

Moreover, \( \| A \|_{L_p \to L_q} \approx S \).

Compared with O’Neil and Stepanov’s estimates, we prove in [11] that the right-hand side estimate of (4) implies both (2) and (3). It can also be shown that for the function
\[
K(x) = \begin{cases} 
2^{k/r}, & \text{for } x \in [-k, -k + 2^{-k}], \ k \in \mathbb{N}; \\
1, & \text{for } x \in [k, k + 1/k), \ k \in \mathbb{N}; \\
0, & \text{otherwise}
\end{cases}
\]
we have \( \inf_{f} \sup_{x \in \mathbb{R}} \frac{1}{|x|^{1/p - 1/q}} \int_x^0 K(x) \, dx | < \infty, \ K \|_{L_{r,\infty}(\mathbb{R},dx)} = \infty, \ K \|_{W(L_{r,\infty}(0,1), h, L(\mathbb{Z}))} = \infty. \)

3. Young–O’Neil type inequalities in the Lorentz spaces

The Young–O’Neil inequality for the convolution \( Af = K \ast f \) in the Lorentz spaces is given by
\[
\| Af \|_{L_{q,h}} \leq C \| f \|_{L_{p,h}} \| K \|_{L_{r,h}}, \quad \text{where } 1 < p, q, r < \infty, 0 < h_1, h_2, h \leq \infty, 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}, \text{ and } \frac{1}{h} = \frac{1}{h_1} - \frac{1}{h_2}. \]

In this section we study the boundedness of the operator \( A \) from \( L_{p,h_1} \) into \( L_{p,h_1} \), i.e., the limiting case of the Young–O’Neil inequality \( (p = q \text{ and } r = 1) \). First, we note (see [5, Theorem 2]) that if \( h_1 < h_2 \leq \infty \) and \( K \geq 0 \), then \( A : L_{p,h_1} \to L_{p,h_1}, \) \( dx \) implies \( A \equiv 0 \), i.e., \( K \leq 0 \). We however show that in the case when \( \Omega \) is of finite measure, the same problem has a nontrivial solution.

Next two theorems provide the boundedness of the convolution \( A \), given by (1) with the 1-periodic functions \( K \) and \( f \) and \( \Omega = D = [0, 1] \) from \( L_{p,h_1}([0, 1], dx) \) in \( L_{p,h_2}([0, 1], dx) \).

Theorem 2. Let \( 1 \leq h_1, h_2 \leq \infty \), and \( \frac{1}{h} = \frac{1}{h_1} - \frac{1}{h_2} \). We have
\[
\| Af \|_{L_{q,h_1}} \leq C \| f \|_{L_{p,h_2}} |L_{q,h_2}| \| K \|_{L_{r,h}}.
\]

for \( 1 < p \leq \infty \) and
\[
\| K \|_{L_{r,h}} \leq C \| f \|_{L_{p,h_2}} \| K \|_{L_{r,h}}.
\]

Remark. In both inequalities (5) and (6), the factor \( \| K \|_{L_{r,h}} \) cannot be changed to \( \| K \|_{L_{r,h}} \).

Our second goal is to give an analogue of Young–O’Neil’s inequality in the \( L_{\infty,q}([0,1]) \) spaces. Following Bennett et al. [2] (see also [1]), we define \( L_{\infty,q}([0,1]) \) as follows
\[
L_{\infty,q}([0,1]) = \left\{ f \in L_1([0,1]): \| f \|_{L_{\infty,q}([0,1])} := \| f \|_{L_1([0,1])} + \left( \int_0^1 \frac{(f^{**} - f^{*})^q}{t} \, dt \right)^{1/q} < \infty \right\}.
\]

Note that \( L_{\infty,0}([0,1]) \mapsto L_{\infty,q}([0,1]) \mapsto L_{\infty,q_1}([0,1]) \mapsto L_p([0,1]), \) for \( 1 \leq p < \infty \) and \( 1 \leq q < q_1 < \infty \).

Theorem 3. Let \( 1 \leq h_1, h_2, h < \infty \) and \( \frac{1}{h} = \frac{1}{h_1} - \frac{1}{h_2} \). We have \( \| Af \|_{L_{\infty,h_1}} \leq 2 \| f \|_{L_{\infty,h_2}} \| K^{**} \|_{L_{1,h}} \).
4. Convolution operator in the Lorentz spaces with weights

In the case of non-homogeneous measures, the convolution operator does not satisfy all requirements from [12] and needs thorough investigation. The following theorem provides sufficient conditions for the convolution operator to be bounded in the weighted Lorentz space.

Theorem 4. Let \( 1 < p, q < \infty \) and let the measures \( \mu \) and \( \nu \) be defined on measurable subsets \( \Omega \) and \( D \) of \( \mathbb{R}^n \), respectively. Assume that a function \( K(z) \) defined on \( D - \Omega = \{ z = x - y: \ x \in D, \ y \in \Omega \} \) satisfies the following condition: there exists \( \gamma > 0 \) such that

\[
\sup_{e \in M_1} \frac{1}{(\mu(e))^{1/q - 1/yp'}} \left| \int_e K(x - y) \, d\mu_y \right| \leq B \quad \text{for a.e.}\ x \in D,
\]

\[
\sup_{w \in M_2} \frac{1}{(\nu(w))^{1/p - \gamma/q}} \left| \int_w K(x - y) \, d\nu_x \right| \leq B \quad \text{for a.e.}\ y \in \Omega,
\]

where \( M_1 = \{ e \subset \Omega: 0 < \mu(e) < \infty \} \) and \( M_2 = \{ w \subset D: 0 < \nu(w) < \infty \} \). Then

\[
Af(y) = \int_D K(x - y) f(x) \, d\nu_x
\]

is bounded from \( L_{p,h_1}(D, \nu) \) to \( L_{q,h_2}(\Omega, \mu) \) with \( 1 \leq h_1 \leq h_2 \leq \infty \) and, moreover, \( \| A \|_{L_{p,h_1}(D, \nu) \to L_{q,h_2}(\Omega, \mu)} \leq CB \), where \( C = C(p, q, h_1, h_2) \).

For the power weights the Young–O’Neil inequality was generalized by Kerman [8]. We continue this investigation by presenting the following result.

Theorem 5. Let \( \alpha, \beta \in [0, 1), \ 1 < p, q < \infty \), and \( 0 < \frac{1}{r} = 1 - \frac{1 - \alpha}{p} - \frac{1 - \beta}{q} \). Suppose that measures \( \mu \) and \( \nu \) are define as follows \( \mu(e) = \int e \frac{dy}{|y|^p} \) and \( \nu(\omega) = \int_\omega \frac{dx}{|x|^q} \). Then the convolution operator (7) with \( D = \mathbb{R} \) satisfies

\[
\| A \|_{L_p(R, \nu) \to L_q(R, \mu)} \leq C \sup_{0 \leq |e| < \infty} \frac{1}{|e|^r} \int e |K(t)| \, dt,
\]

where \( |e| \) is the linear measure of \( e \). Moreover, if the kernel \( K(t) \) is non-negative, then \( C \sup_{d > 0} \frac{1}{d^r} \int_{-d}^d K(t) \, dt \leq \| A \|_{L_p(R, \nu) \to L_q(R, \mu)} \).

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