Gravitational model of the string

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Received 8 November 2005, revised 2 March 2006, accepted 3 March 2006 by F.W. Hehl
Published online 5 April 2006

Key words 5D Kaluza-Klein gravity, flux tube solution, isometrical embedding.

PACS 04.50.+h

It is shown that an infinite gravitational flux tube solution in 5D Kaluza-Klein gravity with the cross section in the Planck region after $5D \to 4D$ reduction and isometrical embedding in a Minkowski spacetime can be considered as a moving infinite string-like object. Such an object carries an electric and a magnetic flux. The 4D gravitational waves on the tube are considered.

1 Introduction

The string in string theory is an 1-dimensional structureless object. Historically the concept of string in the modern physical context has arisen in quantum chromodynamics (QCD). In QCD the confinement problem is connected with a hypothesized flux tube filled with the longitudinal color electric field and stretched between quark and antiquark. If we let the cross section of the flux tube to zero, we obtain a string which can give us some information about quark confinement. The problems with such string approximation for confinement are well known. These problems have resulted in the fact that the string is an 1-dimensional fundamental structureless object in the modern string theory. Nevertheless there is an open question: has any field theory string-like solutions? If yes, then it is necessary to compare the classical and quantum properties such string-like solutions with real strings in string theory. Evidently the field theories like gauge theories will have the same problems with the quantization of string-like solutions as the quantization of string in QCD.

In this note we will show that in the 5D Kaluza-Klein gravity there is a solution which can be considered as a string-like object in some external embedding spacetime. The idea is that any pseudo-Riemannian spacetime can be locally and isometrically embedded in a Minkowski spacetime $\mathcal{M}$ [1]. The local topology of the discussed solution is $\mathbb{R}_1 \times \mathbb{R}_2 \times S^1 \times S^2$ where the time is spanned on $\mathbb{R}_1$, the $5^{th}$ coordinate on the 1D sphere $S^1$, the longitudinal coordinate on $\mathbb{R}_2$ and the polar angles $\theta, \varphi$ on the 2D sphere $S^2$. The factor $\mathbb{R}_2 \times S^2$ describes a tube. If the cross section of the tube and $S^1$ can be chosen arbitrary small, then, after embedding in the above mentioned Minkowski spacetime $\mathcal{M}$, we will have a string-like object moving in $\mathcal{M}$. The most significant difference with the ordinary string is that $\mathcal{M}$ can have more than one time.

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2 5D flux tube

At first we will present the regular 5D wormhole-like flux tube metric which is the solution of the 5D vacuum Einstein’s equations [2]

\[ ds^2 = \cosh^2 \left( \frac{r}{r_0} \right) dt^2 - dr^2 - r_0^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) - [d\chi + \omega(r) dt + Q \cos \theta d\varphi]^2, \]

\[ \omega(r) = \sqrt{2} \sinh \left( \frac{r}{r_0} \right), \quad Q = \sqrt{2}r_0, \]

where \( 0 \leq \chi \leq 2\pi r_0 \) is the 5th extra coordinate; \( \theta, \varphi \) are polar angles in the spherical coordinate system; \( r \in (-\infty, +\infty) \) is the longitudinal coordinate; \( Q \) is the magnetic charge, \( r_0 \) is the cross sectional size of the flux tube. This solution is a tube with the cross section \( 4\pi r_0^2 \), the coordinate \( r \) is directed along the tube and the metric (1) is the 4D generalization of well known 4D Levi-Civita–Bertotti–Robinson solution [3].

On the 4D language we have the following components of the 4D electromagnetic potential \( A_\mu \),

\[ A_t = \sqrt{2} \sinh \left( \frac{r}{r_0} \right) \quad \text{and} \quad A_\varphi = Q \cos \theta = \sqrt{2}r_0 \cos \theta, \]

and the Maxwell tensor is

\[ F_{rt} = \frac{\sqrt{2}}{r_0} \cosh \left( \frac{r}{r_0} \right) \omega' \quad \text{and} \quad F_{\theta\varphi} = -Q \sin \theta = -\sqrt{2}r_0 \sin \theta. \]

The \((\chi t)\)-Einstein equation can be written in the following way:

\[ \left( 4\pi r_0^2 \frac{1}{\cosh(r/r_0)} \omega' \right)' = 0. \]

The 5D Kaluza-Klein gravity after the dimensional reduction indicates that the Maxwell tensor is

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

It allows us to write the electric field as \( E_r = \omega' \). Eq.(4) with the electric field defined by (5) can be compared with the Maxwell equation in a continuous medium

\[ \text{div} \bar{D} = 0, \]

where \( \bar{D} = \epsilon \bar{E} \) is the electric displacement and \( \epsilon \) the permittivity. Comparing Eq. (6) with Eq. (4), we recognize that the quantity \( \frac{1}{\cosh(r/r_0)} \omega' \) can be identify with the electric displacement. Thus, the permittivity is

\[ \epsilon = \frac{1}{\cosh(r/r_0)}. \]

This means that \( q = \sqrt{2}r_0 \) can be taken as the Kaluza-Klein electric charge because the flux of the electric field is \( \Phi = 4\pi r_0^2 \bar{D} = 4\pi q = \text{const.} \)

We see that this solution is a gravitational flux tube with the following properties:

- the electric and magnetic charges are equal
  \[ q = Q; \]

- the radial electric displacement is equal to the radial magnetic field
  \[ D_r = H_r = \frac{\sqrt{2}}{r_0} = \text{const}; \]
the tube is filled with the constant parallel electric and magnetic fields;
• the linear size of the 5th dimension is \( r_0 \);
• the cross sectional size of the flux tube is \( r_0 \) and can be arbitrary.

The last item allows us to consider the case \( r_0 \to 0 \) but with one essential remark. If we believe that we live in Nature with a minimal (Planck) length then the minimal value of \( r_0 \) is \((r_0)_{\text{min}} = l_{P} \) and for a macroscopical observer this minimal length is a point. It means that the presented solution in the limit \((r_0)_{\text{min}} = l_{P} \) is the flux tube with almost zero cross section, i.e. a string-like object with a flux of electric and magnetic fields. Though the tube has some thickness, it is the minimal one and for a macroscopical observer it is the string-like object.

In the next section we will show that the solution presented here after 5D \(ightarrow 4D\) reduction can be isometrically embedded in a Minkowski spacetime and by \((r_0)_{\text{min}} = l_{P} \) we will have a string-like object moving in an embedding spacetime (in a bulk).

3 Isometrical embedding of the flux tube

Let us consider the case

\[
ds^2 = (dt^1)^2 + (dt^2)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - (dx^4)^2.\tag{9}
\]

According to Kaluza-Klein point of view on the 5D gravity the 5D metric (1) can be considered as 4D metric

\[
ds^2 = \cosh^2 \left( \frac{r}{r_0} \right) dt^2 - dr^2 - r_0^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right)\tag{10}
\]

and 4D electromagnetic field with the potential (2) and electric and magnetic fields (3). From this point of view we will consider embedding of 4D metric (10) in 6D Minkowski spacetime (9). One can see that this embedding can be realized by the following way

\[
t^1 = r_0 \cosh \left( \frac{r}{r_0} \right) \cos \left( \frac{t}{r_0} \right), \quad t^2 = r_0 \cosh \left( \frac{r}{r_0} \right) \sin \left( \frac{t}{r_0} \right), \quad x^1 = r_0 \sinh \left( \frac{r}{r_0} \right), \quad x^2 = r_0 \sin \theta \cos \phi, \quad x^3 = r_0 \sin \theta \sin \phi, \quad x^4 = r_0 \cos \phi.\tag{11}
\]

Now we would like to show that this flux tube in the limit \( r_0 \to 0 \) can be considered as a string-like object (gravitational flux tube with almost zero cross section). We see that the map of the 4D spacetime with the metric (10) into a 6D Minkowski spacetime \((t^1, t^2, x^1, x^2, x^3, x^4)\) can be factorized on 2D sphere in \((x^2, x^3, x^4)\) space with the equation

\[
(x^2)^2 + (x^3)^2 + (x^4)^2 = r_0^2\tag{13}
\]

and the hypersurface in the Minkowski spacetime \((t^1, t^2, x^1)\) with the equation

\[
(t^1)^2 + (t^2)^2 = r_0^2 + (x^1)^2.\tag{14}
\]

Let us consider the case \( r_0 \to 0 \). We can not put \( r_0 \) simply equal to zero as in Nature exists a minimal length \( l_{P} \). We can not say anything about the spacetime structure inside Planck region consequently the Planck volume is a physical point for the a macroscopical observer. After this remark we can put \((r_0)_{\text{min}} = l_{P} \). In this case the 2D sphere in \((x^2, x^3, x^4)\) space becomes a point and Eq. (14) in the Minkowski spacetime describes the world sheet of a moving string-like object. It is easy to see that any spacelike section of the worldsheet (14) is the moving string-like object.
4 The properties of the derived string-like object

The most significant difference between derived string-like object and the ordinary string in string theory is that the derived here string moves in the spacetime with two times. It results in that each point of the string moves on a closed path. It is very interesting that an observer living on a brane will see a moving particle but not a string! For example, if we consider the brane \( t^1 = \text{const} \) then on the spacetime spanned on \( (t^1, x^1) \) coordinates the string-like object is the world line of a tachyon. Of curse one can choose another section of the worldsheet in such a way that it will be a particle moving with the speed \( v < c \).

The equations of motion of the isometrically embedding string-like object in the Minkowski spacetime are 5D Einstein’s vacuum equations + the embedding conditions

\[
R_{(A)(B)} - \frac{1}{2} \eta_{(A)(B)} R = 0, \quad (15)
\]

\[
(\partial_A X^\mu) (\partial_B X^\nu) \eta_{\mu\nu} = G_{AB} = e^{(A)}_A e^{(B)}_B \eta_{(A)(B)}, \quad (16)
\]

where \( A, B = 0, 1, 2, 3, 5 \) are 5 world indexes; \( (C), (D) \) are 5-bein indexes; \( X^\mu \) are the coordinates of the embedded spacetime in the Minkowski spacetime \( (9) \); \( \eta_{(A)(B)} \) is the 5-bein metric; \( \eta_{\mu\nu} \) is the metric of the 6D embedding Minkowski spacetime.

The equations of the moving bosonic string in string theory are

\[
\partial_A \partial^A X^\mu = 0. \quad (17)
\]

We see that equations of motions (15), (16) and (17) are different. The reason is that the isometrically embedded gravitational flux tube has a finite thickness. The situation is similar to QCD string where the finite thickness of QCD flux tube leads to a rigidity and consequently the string action (and string equations) changes a little [5, 6].

There is an interesting difference between isometrical embedded gravitational string and QCD string. In both cases the string has a finite thickness but in the first case the electric and magnetic fields are concentrated inside flux tube only. In the second case the color electric field is distributed in the whole space (though the most part of force lines are concentrated inside the string).

5 Gravitational waves on the string-like object

In this section we would like to show that on the gravitational flux tube (1) may exist some perturbations which after \( r_0 \to 0 \) or more exactly \( (r_0)_{\text{min}} = l_\text{Pl} \) will give us some perturbation of the worldsheet.

Let us introduce the 5-bein for the perturbed metric

\[
e^{(A)}_B e^{(A)}_C = G_{BC}, \quad (18)
\]

\[
e^{(A)}_C e^{C}_{(B)} = \eta_{(A)(B)} = \text{diag} \{+1, -1, -1, -1, -1\}, \quad (19)
\]

where \( G_{AB} \) is the perturbed metric (1). We will consider the next perturbation

\[
e^{(A)}_B = e^{(A)}_B + \tilde{e}^{(A)}_B, \quad (20)
\]

\[
e^{(0)}_B = \cosh \left( \frac{r}{r_0} \right) + \tilde{e}^{(0)}_B, \quad e^{(1)}_B = 1 + \tilde{e}^{(1)}_B, \quad (21)
\]
where $\tilde{e}^{(A)}_B$ is the 5-bein for the metric (1); $\tilde{e}^{(A)}_B$ are the perturbations. The perturbed metric is

$$ds^2 = \left[ \cosh \left( \frac{r}{r_0} \right) + \tilde{e}^{(0)}_0 \right]^2 dt^2 - \left[ 1 + \tilde{e}^{(1)}_1 \right]^2 dr^2 - r_0^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) - [d\chi + \omega(r) dt + Q \cos \theta d\varphi]^2,$$

$$\omega(r) = \sqrt{2} \sinh \left( \frac{r}{r_0} \right), \quad Q = \sqrt{2} r_0. \tag{22}$$

Let us note that we consider the perturbation of the 4D metric only, the perturbations of $G_{55}$ and the electromagnetic field $A_\mu = \tilde{e}^{(5)}_\mu$ are frozen. Additionally we take

$$\tilde{e}^{(0)}_0 = -\tilde{e}^{(1)}_1 \cosh \left( \frac{r}{r_0} \right). \tag{23}$$

Then the perturbed 5D Einstein’s equations give us

$$\frac{\partial^2 \tilde{e}^{(1)}_1}{\partial \tau^2} + \cosh^2 x \frac{\partial^2 \tilde{e}^{(1)}_1}{\partial x^2} + 3 \sinh x \cosh x \frac{\partial \tilde{e}^{(1)}_1}{\partial x} + 2 \cosh^2 x \tilde{e}^{(1)}_1 = 0, \tag{24}$$

where $\tau = t/r_0$ and $x = r/r_0$ are correspondingly the dimensionless time and longitudinal coordinates.

One can search the solution in the form

$$\tilde{e}^{(1)}_1 = T(t) X(x) \tag{25}$$

with the following equations for $T(t)$ and $X(x)$

$$\ddot{T}(t) = -\omega^2 T(t), \tag{26}$$

$$X'' + 3 \tanh x X' + \left( 2 - \frac{\omega^2}{\cosh^2 x} \right) X = 0. \tag{27}$$

Eqs. (26) (27) have the following special solutions

$$T(t) = A \sin (\omega \tau + \alpha), \quad X(x) = \frac{C}{\cosh x} \quad \text{for } \omega^2 = +1, \tag{28}$$

$$T(t) = D e^{[\omega] \tau} + E e^{-[\omega] \tau}, \quad X(x) = F \frac{\sinh x}{\cosh^2 x} \quad \text{for } \omega^2 = -3, \tag{29}$$

i.e. oscillating gravitational waves

$$\tilde{e}^{(1)}_1 = G \frac{\sin(\tau + \alpha)}{\cosh x}, \quad \tilde{e}^{(0)}_0 = -G \sin(\tau + \alpha) \tag{30}$$

and damped gravitational waves

$$\tilde{e}^{(1)}_1 = H \frac{\sinh x}{\cosh^3 x} e^{-[\sqrt{3}] \tau}, \quad \tilde{e}^{(0)}_0 = -H \frac{\sinh x}{\cosh^2 x} e^{-[\sqrt{3}] \tau}, \tag{31}$$

where $A, C, D, F, G, H, \alpha$ are some constants. The exponential increasing gravitational waves with $T(\tau) = D e^{[\sqrt{3}] \tau}$ have to be investigated more carefully as for the big time $\tau$ such perturbation analysis is incorrect.

Evidently the perturbed 4D metric

$$ds^2 = \left[ \cosh \left( \frac{r}{r_0} \right) + \tilde{e}^{(0)}_0 \right]^2 dt^2 - \left[ 1 + \tilde{e}^{(1)}_1 \right]^2 dr^2 - r_0^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right)$$

can be embedded in some multidimensional Minkowski spacetime with more than two times.
6 Discussion and conclusions

In this note we have shown that in 5D Kaluza-Klein gravity there exists a solution which after $5D \rightarrow 4D$ reduction can be isometrically embedded in the 6D Minkowski spacetime with two times. We have considered here the case with the infinite gravitational flux tube ($q = Q$). In [4] it is shown that there is the case with a finite gravitational flux tube solution ($q \approx Q, q > Q$). Such finite flux tube with the cross section in the Planck region can be isometrically embedded in a Minkowski spacetime as well. The difference with the infinite flux tube solution ($q = Q$) is that in the case ($q \approx Q, q > Q$) the whole spacetime looks as two spacetimes connected by a very long gravitational flux tube ($0 \leq 1 - \frac{q}{Q} \ll 1$) filled with almost equal longitudinal electric and magnetic fields. After isometrical embedding into a Minkowski spacetime with several times, this solution will look as two branes connected with a string-like object.

The investigation presented here shows that if the cross section of such flux tube is in Planck region then such tube can be considered as a string-like object but with some differences with the ordinary string: the string-like object has finite thickness, moves in 6D Minkowski spacetime with two times and equations of motions for string and string-like object are different.

Acknowledgements I am very grateful to D. Singleton for the fruitful comments.

References