PHYSICAL SINGULARITY IN THE REGULAR SPACE–TIME 
AND FUNDAMENTAL LENGTH

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It is shown that formally regular solutions in 5D Kaluza–Klein gravity have singularities. 
This phenomenon is connected with the existence of a minimal length in nature. The 
calculation of the derivative of the $G_{55}$ metric component leads to the appearance of the 
Dirac $\delta$-function. In this case, the Ricci scalar becomes singular since there is a square 
of this derivative.

Keywords: 5D gravity; regular solution; singularity.

1. Introduction

In practice, any fundamental physical theory has singularities and one of the main 
aims of modern physics is the struggle against these singularities. For example, in 
classical electrodynamics there is the divergence of the electric field near a point-like 
electric charge; in quantum electrodynamics there are divergences connected 
with loops in Feynman diagrams. In general relativity, there are cosmological, 
Schwarzschild, and Reissner–Nordström singularities and so on (for review see, for 
example, Ref. 1). Most of these singularities are connected with a punctual origin 
of a field (electric, gravitational, etc.). Probably the most general receipe to tackle 
singularities is to extend the point-like particle in some directions. String theory 
does it by extending the particle in one dimension, which leads to the replacement 
of a zero-dimensional particle to with a one-dimensional string.

Intuitively, we understand what the singularity is: a place where some invariants 
are divergent, for example, the field strength and scalar curvature. These singular-
ities are mathematical in the sense that they are present in the solution of the
corresponding field equations. In this paper, we show that there exist situations when the mathematical solution is regular but from the physical point of view there is a physical (soft) singularity. In the case presented here, it is connected with the fact that in nature there is a minimal length (Planck length). Some physical quantities vary quickly during the Planck length, for example, from $+1$ to $-1$. In this case, this quantity is similar to the step (Heaviside) function. The derivative of this function is the Dirac $\delta$-function. It is not a problem but if some invariant has such a derivative in a square then it is: we will have a physical singularity. In other words, such a physical singularity is present in a mathematically regular solution only if some physical quantity varies too quickly during the minimal length.

2. Regular Wormhole-Like Flux Tube Solution

In this paper, we show that in 5D Kaluza–Klein gravity there are regular solutions but at some points there are the conditions for the appearance of a physical singularity: the $G_{55}$ metric component changes too quickly. First, we will present the regular 5D wormhole-like flux tube metric:

$$ds^2 = \frac{dt^2}{\Delta(r)} - \Delta(r)e^{2\psi(r)} [d\chi + \omega(r)dt + Q\cos\theta d\varphi]^2$$
$$- dr^2 - a(r)(d\theta^2 + \sin^2 \theta d\varphi^2),$$

where $\chi$ is the fifth extra coordinate; $r, \theta, \varphi$ are 3D spherical-polar coordinates; $r \in \{-\infty, +\infty\}$ is the longitudinal coordinate; $Q$ is the magnetic charge.

In 4D language, we have the following components of the 4D electromagnetic potential $A_\mu$:

$$A_t = \omega(r) \quad \text{and} \quad A_\varphi = Q\cos\theta,$$

and the Maxwell tensor is

$$F_{rt} = \omega'(r) \quad \text{and} \quad F_{\theta\varphi} = -Q\sin\theta.$$  

Substituting this ansatz into the five-dimensional Einstein vacuum equations

$$R_{AB} - \frac{1}{2}\eta_{AB}R = 0,$$

where $A, B = 0, 1, 2, 3, 5$ and $\eta_{AB}$ is the metric signature, gives us

$$\frac{\Delta''}{\Delta} - \frac{\Delta'^2}{\Delta^2} + \frac{\Delta' a'}{\Delta a} + \frac{\Delta' \psi'}{\Delta} + \frac{q^2}{a^2\Delta^2}e^{-4\psi} = 0,$$

$$\frac{a''}{a} + \frac{a' \psi'}{a} - \frac{2}{a} + \frac{Q^2}{a^2 \Delta^2} e^{2\psi} = 0.$$
where $q$ is the electric charge. As the electric $q$ and magnetic $Q$ charges are varied, it has been found that the solutions to the metric in Eqs. (5)–(8) evolve in the following way:

(i) $0 \leq Q < q$. In this case, the solution is a regular gravitational flux tube. The solution is filled with both electric and magnetic fields. The longitudinal distance between the $\pm r_H$ surfaces increases, and the cross-sectional size does not increase as rapidly as $r \to r_H$ with $q \to Q$. The values $r = \pm r_H$ are defined by $\Delta (\pm r_H) = 0$.

(ii) $Q = q$. In this case, the solution is an infinite flux tube filled with constant electric and magnetic fields. The cross-sectional size of this solution is constant ($a = \text{const.}$).

(iii) $0 \leq q < Q$. In this case, we have a singular gravitational flux tube located between two $(\pm)$ and $(\mp)$ electric and magnetic charges at $\pm r_{\text{sing}}$. At $r = \pm r_{\text{sing}}$, this solution has real singularities where the charges are placed.

We will consider the case with $q \approx Q$ but $q > Q$. In this case, there is a region (throat) $|r| < r_H$ where the solution is likened to a tube filled with almost equal electric and magnetic fields. The length $L = 2r_H$ of the throat depends on relation $\delta = 1 - Q/q$ as $L \approx 2a(0)$.

In Fig. 1, the profile of the $G_{55} = \Delta e^{2\psi}$ metric component is presented. We see that near the value $r = r_H$ this function changes drastically from the value $G_{55} \approx +1$ by $r \lesssim r_H - l_0$ to $G_{55} \approx -1$ by $r \gtrsim r_H + l_0$. The same occurs near $r = -r_H$.

At the throat for $|r| < r_H - l_0$, the solution is approximately (since $q \approx Q$)

$$a(r) \approx \frac{Q_0^2}{2} = \text{const.},$$

$$e^{2\psi(r)} \approx \frac{1}{\Delta} = \cosh^2 \left( r \frac{r}{\sqrt{a(0)}} \right),$$

$$\omega(r) \approx \sqrt{2} \sinh \left( r \frac{r}{\sqrt{a(0)}} \right).$$

$$G_{55}(r) = \Delta e^{2\psi(r)} \approx 1.$$
This approximation is valid only for $|r| \lesssim r_H - l_0$, where $l_0$ is some small quantity $l_0 \ll r_H$.

Now, we would like to estimate the length $l_0$ of the region where the change of the metric component $G_{55} = \Delta e^{2\psi}$ is

$$\Delta e^{2\psi} |_{r \approx r_H + l_0} - \Delta e^{2\psi} |_{r \approx r_H - l_0} \approx 2.$$  \hspace{1cm} (14)

For this estimation, Eq. (6) will be used. At the throat for $|r| < r_H - l_0$, this equation is approximately

$$-\frac{2}{a} + \frac{Q^2}{a^2} \Delta e^{2\psi} \approx 0.$$  \hspace{1cm} (15)

We can estimate $l_0$ by solving Einstein's equations (5)–(8) near $r = +r_H$ (for $r = -r_H$, the analysis is the same) and define $r = r_H - l_0$, where the last two terms in Eq. (6) have the same order:

$$\left. \left( \frac{2}{a} \right) \right|_{r = r_H - l_0} \approx \left. \left( \frac{Q^2}{a^2} \Delta e^{2\psi} \right) \right|_{r = r_H - l_0}.$$  \hspace{1cm} (16)

For the solution close to $r = +r_H$, we search in the following form:

$$\Delta(r) = \Delta_1 (r_H - r) + \Delta_2 (r_H - r)^2 + \cdots.$$  \hspace{1cm} (17)

Substitution in Eq. (5) gives us the following solution:

$$\Delta_1 = \frac{q e^{-2\psi_H}}{a_H}.$$  \hspace{1cm} (18)

After the substitution into Eq. (16), we have

$$l_0 \approx \sqrt{a(0)} = l_p.$$  \hspace{1cm} (19)
where we took into account that numerical analysis shows that \( a_H = a(\pm r_H) \approx 2a(0) \). This means that the change of macroscopic dimensionless function \( G_{55} = \Delta e^{2\psi} \) as in Eq. (14) occurs during the Planck length. The metric (1) for \( |r| \approx r_H - l_{Pl} \) is approximately

\[
ds^2 \approx e^{2\psi_H} dt^2 - dr^2 - a(r_H)(d\theta^2 + \sin \theta d\varphi^2) - (d\chi + \omega dt + Q \cos \theta d\phi)^2.
\]

For \( |r| \approx r_H + l_{Pl} \), the metric is approximately

\[
ds^2 \approx -e^{2\psi_H} dt^2 - dr^2 - a(r_H)(d\theta^2 + \sin \theta d\varphi^2) + (d\chi + \omega dt + Q \cos \theta d\phi)^2.
\]

Here, we took into account that numerical calculations show that \( \psi \approx \psi_H = \text{const.} \) for \( |r| \lesssim r_H \). We see that during the Planck length, the metric signature changes from \( \{+, -, -, -, -\} \) to \( \{-, -, -, +, +\} \). Simultaneously, it is important to mention that the metric (1) is non-singular for \( |r| = r_H \) and is approximately

\[
ds^2 \approx e^{2\psi_H} dt^2 - e^{\psi_H} dt(d\chi + Q \cos \theta d\varphi) - dr^2 - a(r_H)(d\theta^2 + \sin^2 \theta d\varphi^2)
\]

\[
= \left[ e^{\psi_H} dt - \frac{1}{2} (d\chi + Q \cos \theta d\varphi) \right]^2 - dr^2 - a(r_H)(d\theta^2 + \sin^2 \theta d\varphi^2)
\]

\[
- \frac{1}{4} (d\chi + Q \cos \theta d\varphi)^2,
\]

where \( \psi_H \) is some constant.

If we write the metric (1) in the five-bein formalism

\[
ds^2 = \omega^A \omega^B \eta_{AB}, \quad \omega^A = \epsilon^A_\mu dx^\mu, \quad x^\mu = t, r, \theta, \varphi, \chi,
\]

then we see that

\[
\eta_{AB} = \{+1, -1, -1, -1, -1\} \quad \text{for} \quad |r| \lesssim r_H - l_{Pl},
\]

\[
\eta_{AB} = \{-1, -1, -1, -1, +1\} \quad \text{for} \quad |r| \gtrsim r_H + l_{Pl}.
\]

### 3. Physical Singularity

Let us consider more carefully the situation at \( r = \pm r_H \). Here, it is more convenient to present the metric (1) in the form

\[
ds^2 = \frac{e^{2\psi(r)}}{\Delta(r)} dt^2 - \Delta(r)[d\chi + \omega(r)dt + Q \cos \theta d\varphi]^2 - dr^2 - a(r)(d\theta^2 + \sin^2 \theta d\varphi^2),
\]

where

\[
G_{55} = \hat{\Delta}(r) = \Delta(r)e^{2\psi(r)}.
\]

By the definition, we have

\[
\hat{\Delta}'(r)_{|r=r_H} = \lim_{\Delta r \to 0} \frac{\hat{\Delta}(r_H + \Delta r) - \hat{\Delta}(r_H - \Delta r)}{2\Delta r}.
\]

Now, we have to take into account the assumption that in nature there is a minimal length (Planck length). In this case, one can see that here there is one subtlety. The
point is that \( \Delta r \) cannot converge to zero since there exists a minimal length. Strictly speaking, the introduction of the minimal length can be made in quantum gravity only. Now, we do not have such a theory but there are some algebraic considerations which allow us to understand how the Planck length can appear. Ordinarily, the existence of the minimal length does not result in problems with the calculation of a derivative but not in this case. Equations (14) and (28) give us existence of the minimal length does not result in problems with the calculation of which allow us to understand how the Planck length can appear. Ordinarily, the existence of the minimal length does not result in problems with the calculation of a derivative but not in this case. Equations (14) and (28) give us

\[
\hat{\Delta}'(r)|_{r=r_H} \approx \frac{\hat{\Delta}(r_H + l_{Pl}) - \hat{\Delta}(r_H - l_{Pl})}{2l_{Pl}} = \frac{\Theta(r_H + l_{Pl}) - \Theta(r_H - l_{Pl})}{2l_{Pl}} \\
\approx \Theta'(r)|_{r=r_H} \approx \delta(r - r_H)|_{r=r_H} = \delta(0) = \infty,
\]

where

\[
\Theta(x) = \begin{cases} 
+1, & \text{if } x > 0, \\
0, & \text{if } x = 0, \\
-1, & \text{if } x < 0, 
\end{cases}
\]

is the step function, \( \delta(r) \) is the Dirac \( \delta \)-function. More exactly

\[
\hat{\Delta}'(\pm r_H) \approx \frac{1}{l_{Pl}}.
\]

This means that near \( r = \pm r_H \)

\[
\hat{\Delta}'(r) \approx \delta(r).
\]

It is not a problem if we only have \( \hat{\Delta}'(r) \) rather than \( \hat{\Delta}^2(r) \). But in our case the situation is a problem since, for example, the Ricci scalar is

\[
\frac{1}{2} R(r) = \frac{1}{2} \left( \frac{\hat{\Delta}^2(r)}{\Delta^2(r)} - \frac{q^2 e^{-4\psi(r)}}{a^2(r)\Delta^2(r)} \right) + \psi^2(r) - \frac{\psi'(r)\hat{\Delta}'(r)}{\Delta(r)} + 2\psi''(r) \\
+ 2\frac{a'(r)\psi'(r)}{a(r)} - \frac{1}{2} a^2(r) + 2\frac{a''(r)}{a(r)} + \frac{1}{2} Q^2 \frac{\hat{\Delta}(r)}{a^2(r)}.
\]

At the first glance, we do not have any problems since

\[
\frac{\hat{\Delta}^2(r)}{\Delta^2(r)} - \frac{q^2 e^{-4\psi(r)}}{a^2(r)\Delta^2(r)}|_{r=\pm r_H} = 0,
\]

as a consequence of Einstein’s equation (5). But this is not the whole story since the above-mentioned analysis of the derivative \( \hat{\Delta}'(r) \) in Eq. (29) shows us that the Ricci scalar has \( \hat{\Delta}^2(r) = \delta^2(0) = \infty \). Approximately, we have

\[
R(\pm r_H) = \left( \frac{\hat{\Delta}^2}{\Delta^2} - \frac{q^2 e^{-4\psi}}{a^2\Delta^2} \right)|_{r=\pm r_H} \approx \frac{1}{l_{Pl}^2} \frac{1}{\Delta(\pm r_H)} = \frac{1}{l_{Pl}^2} \frac{1}{0}
\]

This means that for \( r = \pm r_H \) we have a singularity — a soft singularity. The word “soft” signifies that in this case the singularity differs from the hard singularity of
the Schwarzschild black hole. One can call such a singularity a quantum singularity since it is connected with a quantum gravity phenomenon such as the minimal length.

4. Conclusions

In this paper, we have shown that a quantum gravity phenomenon such as the minimal length leads to very interesting physical consequences: the appearance of a physical singularity. The reason for such a singularity is that there is some physical quantity which changes too quickly during the Planck length. Formal mathematical calculations of the derivative of this physical quantity do not show any problems but the presence of the minimal length leads to the fact that the corresponding derivative at this point is the Dirac $\delta$-function. Finally, in the Ricci scalar the corresponding term is in a square, which leads to a singularity. This analysis is correct only in the case when the cross-section of the gravitational flux tube is $\approx l_{Pl}^2$; if this quantity $\gg l_{Pl}^2$ then we will not have any problems.

It is also interesting to note that in Refs. 5–7 the 4D analog (Levi–Civita–Robinson–Bertotti metric\textsuperscript{8–10}) of the 5D gravitational flux tube solutions is considered as a model of the electric charge.

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