ON SCORES IN MULTIPARTITE HYPERTOURNAMENTS

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Abstract. In this paper, we discuss two types of hypertournaments, one \( [\alpha_i]^k \)-multipartite hypertournament \(( [\alpha_i]^k]-\text{MHT} \) and the second \(( \alpha_i)^k \)-multipartite hypertournament \(( (\alpha_i)^k]-\text{MHT} \). We obtain necessary and sufficient conditions for the \( k \) lists of non-negative integers in non-decreasing order to be the losing score lists (score lists) of \([\alpha_i]^k]-\text{MHT} \) and that of \((\alpha_i)^k]-\text{MHT} \). We extend this concept to more general class of \([\alpha_i]^k]-\text{multipartite multihypertournament} \(( [\alpha_i]^k]-\text{MMHT} \) and \((\alpha_i)^k]-\text{multipartite multihypertournament} \(( (\alpha_i)^k]-\text{MMHT} \).

1 Introduction

A hypergraph is a generalization of a graph [1]. While an edge of a graph is a pair of vertices, an edge of a hypergraph is a subset of the vertex set. An edge consisting of \( k \) vertices is called a \( k \)-edge and a hypergraph whose all edges are \( k \)-edges is called a \( k \)-hypergraph. A \( k \)-hypertournament is a complete \( k \)-hypergraph with each \( k \)-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyperedge. In a \( k \)-hypertournament, the score \( s(v_i) \) or \( s_i \) of a vertex \( v_i \) is the number of arcs containing \( v_i \) and in which \( v_i \) is not the last element, the losing score \( r(v_i) \) or \( r_i \) of a vertex \( v_i \) is the number of arcs containing \( v_i \) and in which \( v_i \) is the last element. The score sequence (losing score sequence) is formed by listing the scores (losing scores) in non-decreasing order.

The following characterization of score sequences and losing score sequences in \( k \)-hypertournaments are due to Zhou, Yao and Zhang [12]. These results are analogous to Landau’s theorem [4] on tournament scores.

**Theorem 1.** Given two non-negative integers \( n \) and \( k \) with \( n \geq k > 1 \), a non-decreasing sequence \( R = [r_i]^n \) of non-negative integers is a losing score sequence of some \( k \)-hypertournament if and only if for each \( p \),

\[
\sum_{i=1}^{p} r_i \geq \binom{p}{k},
\]

with equality when \( p = n \).
Theorem 2. Given non-negative integers \( n \) and \( k \) with \( n \geq k > 1 \), a non-decreasing sequence \( S = [s_i]_1^n \) of non-negative integers is a score sequence of some \( k \)-hypertournament if and only if for each \( p \),
\[
\sum_{i=1}^{p} s_i \geq p \binom{n-1}{k-1} + \binom{n-p}{k} - \binom{n}{k};
\]
with equality when \( p = n \).

A new and short proof of Theorem 1 can be seen in Pirzada and Zhou [9]. Koh and Ree [2, 3] have given some properties of the hypertournament matrix and have also found some conditions for the existence of \( k \)-hypertournament matrix with constant score sequence. Various results on scores in bipartite, tripartite and multipartite hypertournament scores can be found in [5, 6, 7, 10]. The extension of score structure to oriented hypergraphs can be found in Pirzada and Zhou [8]. Further, various results on degrees and degree sequences in hypertournaments can be found in Wang and Zhou [11], and the extension of degrees to oriented hypergraphs can be seen in Zhou and Pirzada [13].

2 Scores in \([\alpha_i]^k\)-multipartite hypertournaments

Given non-negative integers \( n_i \) and \( \alpha_i(i = 1, 2, \ldots, k) \) with \( n_i \geq \alpha_i \geq 1 \) for each \( i \), an \([\alpha_1, \alpha_2, \ldots, \alpha_k]\)-partite hypertournament (or \([\alpha_i]^k\)-multipartite hypertournament or \([\alpha_i]^k\)-MHT) of order \( \sum_{i=1}^{k} n_i \) consists of \( k \) vertex set \( U_i \) with \( |U_i| = n_i \) for each \( i \) (1 \( \leq i \leq k \)) together with an arc set \( E \), a set of \( \sum_{i=1}^{k} \alpha_i \) tuples of vertices, with exactly \( \alpha_i \) vertices from \( U_i \), called arcs such that any \( \sum_{i=1}^{k} \alpha_i \) subset \( \bigcup_{i=1}^{k} U_i \) of \( \bigcup_{i=1}^{k} U_i \), \( E \) contains exactly one of the \( (\sum_{i=1}^{k} \alpha_i)! \) \( \sum_{i=1}^{k} \alpha_i \)-tuples whose \( \alpha_i \) entries belong to \( U_i \). The score and losing score of a vertex in \([\alpha_i]^k\)-hypertournament is defined in the same way as in \( k \)-hypertournaments. The following two results [7] characterize losing score lists and score lists in \([\alpha_i]^k\)-MHT.

Theorem 3. Given \( k \) non-negative integers \( n_i \) and \( \alpha_i \) with \( n_i \geq \alpha_i \geq 1 \), for each \( i (1 \leq i \leq k) \), the \( k \) non-decreasing lists \( R_i = [r_{ij}]_{ji}^{n_i} \) of non-negative integers are the losing score lists of \([\alpha_i]^k\)-MHT if and only if for each \( p_i (1 \leq i \leq k) \) with \( p_i \leq n_i \),
\[
\sum_{i=1}^{k} \sum_{j=1}^{p_i} r_{ij} \geq \prod_{i=1}^{k} \left( \frac{p_i}{\alpha_i} \right) \quad (2.1)
\]
with equality when \( p_i = n_i \).

Theorem 4. Given non-negative integers \( n_i \) and \( \alpha_i \) with \( n_i \geq \alpha_i \geq 1 \), for each \( i (1 \leq i \leq k) \), the \( k \) non-decreasing lists \( S_i = [s_{ij}]_{ji}^{n_i} \) of non-negative integers are the losing score lists of \([\alpha_i]^k\)-MHT if and only if for each \( p_i \leq n_i (1 \leq i \leq k) \) with \( p_i \leq n_i \),
\[
\sum_{i=1}^{k} \sum_{j=1}^{p_i} s_{ij} \geq \left( \sum_{i=1}^{k} \frac{\alpha_i p_i}{n_i} \right) \left( \prod_{i=1}^{k} \left( \frac{n_i}{\alpha_i} \right) \right) + \prod_{i=1}^{k} \left( \frac{n_i - p_i}{\alpha_i} \right) - \prod_{i=1}^{k} \left( \frac{n_i}{\alpha_i} \right) \quad (2.2)
\]
with equality when \( p_i = n_i \).
If in $[\alpha_j]^k$-MHT $H$, the arc set $E$ contains at most $(\sum_{i=1}^k \alpha_i)! \sum_{i=1}^k \alpha_i$-tuples (but at least one), then it is called $[\alpha_j]^k$-multipartite multihypertournament (or briefly $[\alpha_j]^k$-MMHT). If $E$ contains exactly $(\sum_{i=1}^k \alpha_i)! \sum_{i=1}^k \alpha_i$-tuples, then $[\alpha_j]^k$-MMHT is said to be complete. The score and losing score of a vertex in an $[\alpha_j]^k$-MMHT is defined in the same way as in $[\alpha_j]^1$-MHT. The losing score lists $R_i$, $i = 1, 2, \cdots, k$ of an $[\alpha_j]^k$-MMHT $H$ are the $k$ non-decreasing sequences $R_i = [r_{ij}]_{j=1}^{n_i}$, where $r_{ij}$ is the losing score of a vertex $u_{ij}$ in $U_i$. Similarly the score lists $S_i$, $i = 1, 2, \cdots, k$ of an $[\alpha_j]^k$-MMHT $H$ are the $k$ non-decreasing sequences $S_i = [s_{ij}]_{j=1}^{n_i}$, where $s_{ij}$ is the score of a vertex $u_{ij}$ in $U_i$. We note that there are exactly $(\sum_{i=1}^k \alpha_i)! \prod_{i=1}^k (n_i)$ arcs in a complete $[\alpha_j]^k$-MMHT, and therefore

$$\sum_{i=1}^k \sum_{j=1}^{n_i} d_H^-(u_{ij}) = \left( \sum_{i=1}^k \alpha_i \right)! \prod_{i=1}^k \left( \binom{n_i}{\alpha_i} \right),$$

where $d_H^-(u_{ij})$ denotes the number of arcs in which $u_{ij}$ is at the last entry and thus

$$\sum_{i=1}^k \sum_{j=1}^{n_i} r_{ij} = \left( \sum_{i=1}^k \alpha_i \right)! \prod_{i=1}^k \left( \binom{n_i}{\alpha_i} \right). \quad (2.3)$$

Now we have the following result.

**Theorem 5.** If $H$ is a complete $[\alpha_j]^k$-MMHT of order $\sum_{i=1}^k n_i$ with score lists $S_i = [s_{ij}]_{j=1}^{n_i}$ for each $i$, $1 \leq i \leq k$, then

$$\sum_{i=1}^k \sum_{j=1}^{n_i} s_{ij} = \left( \sum_{i=1}^k \alpha_i \right) - 1 \left( \sum_{i=1}^k \alpha_i \right)! \prod_{i=1}^k \left( \binom{n_i}{\alpha_i} \right).$$

**Proof.** Since there are $(\sum_{i=1}^k \alpha_i)! \prod_{i=1}^k (n_i)$ arcs containing a vertex $u_{ij} \in U_i$, for each $i$, $1 \leq i \leq k$ and $1 \leq j_i \leq n_i$, using equation (3), we get

$$\sum_{i=1}^k \sum_{j=1}^{n_i} s_{ij} = \sum_{i=1}^k \left( \sum_{j=1}^{n_i} \alpha_i \right)! \left( \binom{n_i}{\alpha_i} - 1 \right) \left( \prod_{i=1}^k \left( \binom{n_t}{\alpha_t} \right) - \left( \sum_{i=1}^k \alpha_i \right)! \prod_{i=1}^k \left( \binom{n_i}{\alpha_i} \right) \right) - \left( \sum_{i=1}^k \alpha_i \right)! \prod_{i=1}^k \left( \binom{n_i}{\alpha_i} \right)

= \left( \sum_{i=1}^k \alpha_i \right)! \prod_{i=1}^k \left( \binom{n_t}{\alpha_t} \right) \left( \binom{n_i}{\alpha_i} - 1 \right) \prod_{i=1}^k \left( \binom{n_i}{\alpha_i} \right)

= \left( \sum_{i=1}^k \alpha_i \right)! \prod_{i=1}^k \left( \binom{n_t}{\alpha_t} \right) \prod_{i=1}^k \left( \binom{n_i}{\alpha_i} \right)

= \left( \sum_{i=1}^k \alpha_i \right)! \prod_{i=1}^k \left( \binom{n_t}{\alpha_t} \right) \prod_{i=1}^k \left( \binom{n_i}{\alpha_i} \right),$$

and the proof is complete. \qed
The following two results are immediate consequences of the above observations.

**Theorem 6.** The lists $R_i = [r_{ij}]_{j=1}^{n_i}$ are the losing score lists of a complete $[\alpha_i]^k$-MMHT if and only if for each $p_i$, $1 \leq i \leq k$

$$\sum_{i=1}^{k} \sum_{j=1}^{p_i} r_{ij} = \left( \sum_{i=1}^{k} \alpha_i \right) - 1 \prod_{i=1, i \neq j}^{k} \left( \frac{n_i - 1}{\alpha_i} \right).$$

**Theorem 7.** The lists $S_i = [s_{ij}]_{j=1}^{n_i}$ are the score lists of a complete $[\alpha_i]^k$-MMHT if and only if for each $p_i$, $1 \leq i \leq k$

$$\sum_{i=1}^{k} \sum_{j=1}^{p_i} s_{ij} = \left( \sum_{i=1}^{k} \alpha_i \right) - 1 \prod_{i=1, i \neq j}^{k} \left( \frac{n_i - 1}{\alpha_i} \right).$$

3 **Scores in $(\alpha_i)^k$-multipartite hypertournaments**

Let $n_i$ and $\alpha_i$ ($i = 1, 2, \cdots, k$) be non-negative integers with $n_i \geq \alpha_i \geq 1$ for each $i$. An $(\alpha_1, \alpha_2, \cdots, \alpha_k)$-k-paritite hypertournament (or $(\alpha_i)^k$-multipartite hypertournament or $(\alpha_i)^n$-MHT) of order $\sum_{i=1}^{k} n_i$ consists of $k$ vertex sets $U_i$ with $|U_i| = n_i$ for each $i$ ($1 \leq i \leq k$) together with an arc set $E$, a set of $\sum_{i=1}^{k} \alpha_i$ tuples of vertices, with $\beta_i$ vertices from $U_i$, for every $\beta_i = 1, 2, \cdots, \alpha_i$ and for every $i = 1, 2, \cdots, n$, called arcs such that any $\sum_{i=1}^{k} \beta_i$ subset $\bigcup_{i=1}^{k} U_i'$ of $\bigcup_{i=1}^{k} U_i$, $E$ contains exactly one of the $(\sum_{i=1}^{k} \beta_i)!$ $\sum_{i=1}^{k} \beta_i$-tuples whose $\beta_i$ entries belong to $U_i'$. We denote an $(\sum_{i=1}^{k} \beta_i)$-arc by $e$.

If $H$ is an $(\alpha_i)^k$-MHT, for a given vertex $u_{ij} \in U_i$ for each $i$ ($1 \leq i \leq k$, $1 \leq j \leq \alpha_i$), the score $d_{\alpha_i}^+(u_{ij})$ or simply $d^+(u_{ij})$ respectively, is the number of $(\sum_{i=1}^{k} \beta_i)$-arcs containing $u_{ij}$ as not the last element (as the last element). The score lists (losing score lists) of $(\alpha_i)^k$-MHT are $k$ non-decreasing sequences of non-negative integers $Q_i = [q_{ij}]_{j=1}^{n_i}$ ($T_i = [t_{ij}]_{j=1}^{n_i}$), where $q_{ij}$, $t_{ij}$ is the score (losing score) of some vertex $u_{ij} \in U_i$, for each $i$ ($1 \leq i \leq k$).

Evidently, in $(\alpha_i)^k$-MHT $H$ there are exactly

$$\sum_{t_1=1}^{\alpha_1} \sum_{t_2=1}^{\alpha_2} \cdots \sum_{t_k=1}^{\alpha_k} \left( \begin{array}{c} n_1 \\ t_1 \end{array} \right) \left( \begin{array}{c} n_2 \\ t_2 \end{array} \right) \cdots \left( \begin{array}{c} n_k \\ t_k \end{array} \right)$$

arcs, and in each arc only one vertex is at the last entry. Therefore,

$$\sum_{t_1=1}^{\alpha_1} d_{\alpha_1}(u_{t_1}) + \sum_{t_2=1}^{\alpha_2} d_{\alpha_2}(u_{t_2}) + \cdots + \sum_{t_k=1}^{\alpha_k} d_{\alpha_k}(u_{t_k})$$

$$= \sum_{t_1=1}^{\alpha_1} \sum_{t_2=1}^{\alpha_2} \cdots \sum_{t_k=1}^{\alpha_k} \left( \begin{array}{c} n_1 \\ t_1 \end{array} \right) \left( \begin{array}{c} n_2 \\ t_2 \end{array} \right) \cdots \left( \begin{array}{c} n_k \\ t_k \end{array} \right).$$

Now, we have the following observation about the score lists of $(\alpha_i)^k$-MHT of order $\sum_{i=1}^{k} n_i$. 

Theorem 8. If $H$ is $(\alpha_i)_{1}^{k}$-MHT of order $\sum_{i=1}^{k} n_i$ with score lists $Q_i = [q_{ij}]_{n_i}$ for each $i, 1 \leq i \leq k$, then

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} q_{ij} = \left( \sum_{\beta_1=1}^{\alpha_1} \cdots \sum_{\beta_k=1}^{\alpha_k} \left( \left( \sum_{i=1}^{k} \beta_i \right) - 1 \right) \left( \prod_{i=1}^{k} \left( \frac{n_i}{\beta_i} \right) \right) \right).$$

Proof. Let $H$ be an $(\alpha_i)_{1}^{k}$-MHT of order $\sum_{i=1}^{k} n_i$ with score lists $Q_i = [q_{ij}]_{n_i}$ with $1 \leq i \leq k, 1 \leq j \leq \alpha_i$. Therefore, clearly

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} q_{ij} = \sum_{A} \left( \sum_{i=1}^{k} \sum_{j=1}^{n_i} s_{ij} \right),$$

where $\sum_{A}$ is the sum of scores of $[\beta_i]_{1}^{k}$-MHT for all $\beta_i = 1, 2, \cdots, \alpha_i$ and for all $i = 1, 2, \cdots, k$.

Therefore,

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} q_{ij} = \sum_{A} \left( \left( \sum_{i=1}^{k} \beta_i \right) - 1 \right) \left( \prod_{i=1}^{k} \left( \frac{n_i}{\beta_i} \right) \right) = \sum_{\beta_1=1}^{\alpha_1} \sum_{\beta_2=1}^{\alpha_2} \cdots \sum_{\beta_k=1}^{\alpha_k} \left( \left( \sum_{i=1}^{k} \beta_i \right) - 1 \right) \left( \prod_{i=1}^{k} \left( \frac{n_i}{\beta_i} \right) \right),$$

completing the proof. \hfill \Box

The following result provides a characterization of losing score lists in $(\alpha_i)_{1}^{k}$-multipartite hypertournaments.

Theorem 9. Given non-negative integers $n_i$ and $\alpha_i$ with $n_i \geq \alpha_i \geq 1$, for each $i$ ($1 \leq i \leq k$), the $k$ non-decreasing lists $T_i = [t_{ij}]_{n_i}$ of non-negative integers are the losing score lists of $[\alpha_i]_{1}^{k}$-MHT $H$ if and only if for each $p_i \leq n_i$ ($1 \leq i \leq k$) with $p_i \leq n_i$,

$$\sum_{i=1}^{k} \sum_{j=1}^{p_i} t_{ij} \geq \sum_{\beta_1=1}^{\alpha_1} \sum_{\beta_2=1}^{\alpha_2} \cdots \sum_{\beta_k=1}^{\alpha_k} \left( \prod_{i=1}^{k} \left( \frac{p_i}{\beta_i} \right) \right),$$

with equality when $p_i = n_i$.

Proof. Obviously for $[\alpha_i]_{1}^{k}$-MHT $H$, we have

$$\sum_{i=1}^{k} \sum_{j=1}^{p_i} t_{ij} = \sum_{A} \left( \sum_{i=1}^{k} \sum_{j=1}^{p_i} r_{ij} \right),$$

where $\sum_{A}$ is the sum of losing scores of all $[\beta_i]_{1}^{k}$-MHT for all $\beta_i = 1, 2, \cdots, \alpha_i$ and for all $i = 1, 2, \cdots, k$. By Theorem 3, $T_i$ are score lists of $(\alpha_i)_{1}^{k}$-MHT if and only if

$$\sum_{i=1}^{k} \sum_{j=1}^{p_i} t_{ij} \geq \sum_{A} \left( \prod_{i=1}^{k} \left( \frac{p_i}{\beta_i} \right) \right),$$
that is if and only if

$$\sum_{i=1}^{k} \sum_{j_i=1}^{p_i} t_{ij_i} \geq \sum_{\beta_1=1}^{\alpha_1} \sum_{\beta_2=1}^{\alpha_2} \cdots \sum_{\beta_k=1}^{\alpha_k} \left( \prod_{i=1}^{\alpha_i} \left( \frac{p_i}{\beta_i} \right) \right),$$

and the proof is complete. \qed

By using the same argument as in Theorem 9, we have the following result on scores in \((\alpha_i)_{1}^{k}\)-MHT.

**Theorem 10.** Given non-negative integers \(n_i\) and \(\alpha_i\) with \(n_i \geq \alpha_i \geq 1\), \(i = 1, 2, \ldots, k\), the \(k\) non-decreasing lists \(Q_i = [q_{ij_i}]_{j_i=1}^{n_i}\) of non-negative integers are the score lists of some \((\alpha_i)_{1}^{k}\)-MHT if and only if for each \(p_i \leq n_i\) (\(1 \leq i \leq k\)) with \(p_i \leq n_i\)

$$\sum_{i=1}^{k} \sum_{j_i=1}^{p_i} q_{ij_i} \geq \sum_{\beta_1=1}^{\alpha_1} \sum_{\beta_2=1}^{\alpha_2} \cdots \sum_{\beta_k=1}^{\alpha_k} \left[ \left( \sum_{i=1}^{\alpha_i} \beta_i p_i \right) \prod_{i=1}^{\alpha_i} \left( \frac{p_i}{\beta_i} \right) \right]$$

$$+ \prod_{i=1}^{\alpha_i} \left( \frac{n_i - p_i}{\beta_i} \right) - \prod_{i=1}^{\alpha_i} \left( \frac{p_i}{\beta_i} \right)$$

with equality when \(p_i = n_i\).

If in \((\alpha_i)_{1}^{k}\)-multipartite hypertournament \(H\), the arc set \(E\) contains at most (at least one) \((\sum_{i=1}^{k} \alpha_i)\) \(\sum_{i=1}^{k} \alpha_i\)-tuples, then \(H\) is said to be \([\alpha_i]_{1}^{k}\)-multipartite multi hypertournament (or briefly \((\alpha_i)_{1}^{k}\)-MMHT). In case \(E\) contains exactly \((\sum_{i=1}^{k} \alpha_i)\) \(\sum_{i=1}^{k} \alpha_i\)-tuples, then \(H\) is said to be complete \((\alpha_i)_{1}^{k}\)-MMHT. The score and losing score of a vertex in an \((\alpha_i)_{1}^{k}\)-MMHT is defined in the same way as in \((\alpha_i)_{1}^{k}\)-MHT. The scores (losing scores) arranged in \(k\) lists in non-decreasing order are then the score (losing score) lists of \((\alpha_i)_{1}^{k}\)-MMHT and are denoted by \(Q_i = [q_{ij_i}]_{j_i=1}^{n_i}\) and \(T_i = [t_{ij_i}]_{j_i=1}^{n_i}\). Evidently there are exactly

$$\left( \sum_{i=1}^{k} \alpha_i \right)! \left[ \sum_{\beta_1=1}^{\alpha_1} \sum_{\beta_2=1}^{\alpha_2} \cdots \sum_{\beta_k=1}^{\alpha_k} \prod_{i=1}^{\alpha_i} \left( \frac{n_i}{\beta_i} \right) \right]$$

arcs in a complete \((\alpha_i)_{1}^{k}\)-MMHT \(H\), and therefore

$$\sum_{i=1}^{k} \sum_{j_i=1}^{n_i} d_H(u_{ij_i}) = \left( \sum_{i=1}^{k} \alpha_i \right)! \left[ \sum_{\beta_1=1}^{\alpha_1} \sum_{\beta_2=1}^{\alpha_2} \cdots \sum_{\beta_k=1}^{\alpha_k} \prod_{i=1}^{\alpha_i} \left( \frac{n_i}{\beta_i} \right) \right].$$

So,

$$\sum_{i=1}^{k} \sum_{j_i=1}^{n_i} t_{ij_i} = \left[ \left( \sum_{i=1}^{k} \alpha_i \right) - 1 \right] \left( \sum_{i=1}^{k} \alpha_i \right)! \left[ \sum_{\beta_1=1}^{\alpha_1} \sum_{\beta_2=1}^{\alpha_2} \cdots \sum_{\beta_k=1}^{\alpha_k} \prod_{i=1}^{\alpha_i} \left( \frac{n_i}{\beta_i} \right) \right]. \quad (3.1)$$

Now, since there are

$$\left( \sum_{i=1}^{k} \alpha_i \right)! \left[ \sum_{t_1=1}^{\alpha_1} \sum_{t_2=1}^{\alpha_2} \cdots \sum_{t_k=1}^{\alpha_k} \left( \alpha_i - 1 \right) \prod_{i=1, t \neq i}^{k} \left( \frac{n_i}{\alpha_i} \right) \right]$$
arcs containing a vertex \( u_{ij} \in U_i \), by using equation (4), we obtain

\[
\sum_{i=1}^{k} \sum_{j_i=1}^{n_i} q_{ij_i} = \left( \sum_{i=1}^{k} \alpha_i \right)! \left[ \sum_{\beta_1=1}^{\alpha_1} \sum_{\beta_2=1}^{\alpha_2} \cdots \sum_{\beta_k=1}^{\alpha_k} \left( \sum_{i=1}^{k} \beta_i \right) - 1 \right] \prod_{i=1}^{k} \left( \frac{n_i}{\beta_i} \right). 
\]

The above observations lead to the following results, the proofs are immediate consequences.

**Theorem 11.** The lists \( T_i = [t_{ij_i}]_{j_i=1}^{n_i} \) are the losing score lists of a complete \((\alpha_i)_{i=1}^{k}\)-MMHT if and only if

\[
\sum_{i=1}^{k} \sum_{j_i=1}^{n_i} t_{ij_i} = \left[ \left( \sum_{i=1}^{k} \alpha_i \right) - 1 \right]! \left[ \sum_{\beta_1=1}^{\alpha_1} \sum_{\beta_2=1}^{\alpha_2} \cdots \sum_{\beta_k=1}^{\alpha_k} p_i \left( \frac{n_i - 1}{\beta_i} \right) \prod_{t=1, t \neq i}^{k} \left( \frac{n_t}{\beta_t} \right) \right],
\]

for each \( p_i, 1 \leq i \leq k \).

**Theorem 12.** The lists \( Q_i = [q_{ij_i}]_{j_i=1}^{n_i} \) are the score lists of a complete \((\alpha_i)_{i=1}^{k}\)-MMHT if and only if

\[
\sum_{i=1}^{k} \sum_{j_i=1}^{n_i} q_{ij_i} = \left[ \left( \sum_{i=1}^{k} \alpha_i \right) - 1 \right]! \left[ \left( \sum_{i=1}^{k} \alpha_i \right) - 1 \right] \cdot \left[ \sum_{\beta_1=1}^{\alpha_1} \sum_{\beta_2=1}^{\alpha_2} \cdots \sum_{\beta_k=1}^{\alpha_k} p_i \left( \frac{n_i - 1}{\beta_i} \right) \prod_{t=1, t \neq i}^{k} \left( \frac{n_t}{\beta_t} \right) \right],
\]

for each \( p_i, 1 \leq i \leq k \).
References


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