ON THE SECOND COHOMOLOGY GROUPS OF EXCEPTIONAL LIE ALGEBRAS IN POSITIVE CHARACTERISTIC

S.S. Ibraev

Communicated by A.S. Dzumadildaev

Key words: exceptional Lie algebra, Weyl module, algebraic group.

AMS Mathematics Subject Classification: 17B50, 17B56.

Abstract. The second cohomology groups of exceptional Lie algebras $E_n(n = 6, 7, 8)$, $F_4$, $G_2$ over an algebraically closed field of characteristic $p \geq h + 5$ with coefficients in modules, dual to Weyl modules, are explicitly described. Here $h$ is the Coxeter number.

1 Introduction

Examples of modules, dual to Weyl modules, over classical modular Lie algebras with non-vanishing second cohomology, were found, for the first time, in [11]. In that paper, the non-triviality of the second cohomology groups with the coefficients in the module, dual to a Weyl module, associated with $L((p - 2)\lambda_1 + \lambda_2)$ for Lie algebras of type $A_n$ (Theorem on p. 324) was established. Here $L((p - 2)\lambda_1 + \lambda_2)$ is an irreducible module with the highest weight $(p - 2)\lambda_1 + \lambda_2$ over a Lie algebra of type $A_n$. However, the question of finding all dual to Weyl modules with non-vanishing second cohomology group for classical Lie algebras, remains open.

In the present paper we want to study this question. The aim is to find and calculate all non-vanishing second cohomology groups with coefficients in the modules, dual to Weyl modules, for simple exceptional Lie algebras $E_n(n = 6, 7, 8)$, $F_4$, $G_2$ over an algebraically closed field of characteristic $p \geq h + 5$. This restriction on $p$ follows from the decomposability condition of the second cohomology groups. According to our result (Theorem 2.1), the number of the modules dual to Weyl modules with non-vanishing second cohomology is equal to $n + \frac{1}{2}(n + 2)(n - 1) = \frac{1}{2}(n^2 + 3n - 2)$, where $n$ is the rank of the corresponding exceptional Lie algebra. In the non-vanishing case the structure of the second cohomology group is described as a rational decomposable module over an algebraic group of the corresponding exceptional Lie algebra.

In Section 2 we introduce the basic notation and formulate our main result. Section 3 is devoted to the proof of Theorem 2.1.

2 Notation and the main result

Let $g$ be a classical Lie algebra of simple and simply connected algebraic group $G$ over an algebraically closed field $k$ of characteristic $p > 0$. We fix a maximal torus $T$ and
the Borel subgroup $B$ of $G$ corresponding to the negative roots. By $G_1$ we denote the kernel of the Frobenius morphism $F$ of $G$.

Choose the root system $R$ associated to $(G,T)$ with the maximal short root $\alpha_0$ and the maximal root $\bar{\alpha}$. The Weyl group $W$ of $R$ acts on the character group $X(T)$ of $T$ by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, where $s_\alpha \in W$, $\alpha \in R$ and $\alpha^\vee$ is the coroot of $\alpha$. If $\rho$ is the half of the sum of the positive roots, then the dot action is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $w \in W$, $\lambda \in X(T)$. We denote by $l(w)$ the length of $w \in W$, and by $w_0$ the longest element in $W$.

The affine Weyl group $W_p$ is the group generated by all $s_{\alpha,n}p$ for $\alpha \in R^+$ and $n \in \mathbb{Z}$, where $R^+$ is the set of positive roots. We will use the dot action of $W_p$ on $X(T)$:

$$s_{\alpha,n} \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha + np\alpha.$$  

Let

$$X_+(T) = \{ \lambda \in X(T) | \langle \lambda + \rho, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in S \}$$

is the set of dominant weights, where $S$ is the set of simple roots, and

$$X_1(T) = \{ \lambda \in X(T) | 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in S \}$$

is the set of restricted weights.

We denote by $\alpha_1, ..., \alpha_n$ the simple roots (numbering corresponds to Bourbaki’s tables [3]), and by $\lambda_1, ..., \lambda_n$, the fundamental weights.

For any $\lambda \in X(T)$ we define the one-dimensional $B$-module $k_\lambda$ via the isomorphism $T \cong B/U$, and the induced $G$-module $H^0(\lambda) = \text{Ind}_B^G(k_\lambda)$. $H^0(\lambda) \neq 0$ if and only if $\lambda \in X_+(T)$. If $V(\lambda)$ is a Weyl module with the highest weight $\lambda$ over $G$, then $H^0(\lambda) \cong V(-w_0\lambda)^*$. Hence, $H^0(\lambda)$ can also be considered as a dual to a Weyl module with the highest weight $-w_0(\lambda)$. A simple $G$-module $L(\lambda)$ with the highest weight $\lambda$ can be defined in terms of the $G$-modules $H^0(\lambda)$ and $V(\lambda)$. It is a simple socle of $H^0(\lambda)$, and also a unique simple factor of $V(\lambda)$ ([4], 5.7).

Any classical Lie algebra is a restricted Lie algebra with the $p$-map $x \mapsto x^p$. Since for $G_1$, the theory of $G_1$-modules is the same with the theory of $U^p$-modules, where $U^p$ is a restricted enveloping algebra for $g$, hence it is equal to the representation theory of $g$ considered as a restricted Lie algebra ([9], I.8.6). Therefore $H^0(\lambda)$, $V(\lambda)$, $L(\lambda)$ can be considered as $g$-modules, which are denoted by the same symbols.

The Hochschild cohomology group $H^k(G_1, V)$ of a restricted $g$-module $V$ coincides with the restricted cohomology group $H^k_B(g, V)$ ([4], 6.10).

The composition of a representation of $G$ on a vector space $V$ with the Frobenius morphism $F$ defines a new representation on which $G_1$ (and, therefore, $g$) acts trivially. The so obtained representation is denoted by $V^{(1)}$. On the other hand, if $V$ is a $G$-module on which $G_1$ (and, therefore, $g$) acts trivially, then there is a unique $G$-module $V^{(-1)}$ such that $V = (V^{(-1)})^{(1)}$.

Suppose now $g = E_n$ ($n = 6, 7, 8$), $F_4$, or $G_2$. For each $i \in \{1, 2, \cdots, n\}$, where $n$ is a rank of $g$, we introduce a set of the highest weights $\Lambda_i$ of the irreducible $G$-modules:

$$g = E_n : \Lambda_1^1 = \{\lambda_1\}, \Lambda_2^1 = \{\lambda_2, 0\}, \Lambda_3^1 = \{\lambda_3, \lambda_6\}, \Lambda_4^1 = \{\lambda_4, \lambda_1 + \lambda_6, \lambda_2\}, \Lambda_5^1 = \{\lambda_5, \lambda_1\}, \Lambda_6^1 = \{\lambda_6\};$$
\(g = E_7: \Lambda^1_1 = \{\lambda_1, 0\}, \Lambda^1_2 = \{\lambda_2, \lambda_7\}, \Lambda^1_3 = \{\lambda_3, \lambda_6, \lambda_1\}, \Lambda^1_4 = \{\lambda_4, \lambda_1 + \lambda_6, \lambda_2 + \lambda_7, \lambda_2\}, \Lambda^1_5 = \{\lambda_5, \lambda_1 + \lambda_7, \lambda_2\}, \Lambda^1_6 = \{\lambda_6, \lambda_1\}, \Lambda^1_7 = \{\lambda_7\}\)

\(g = E_8: \Lambda^1_1 = \{\lambda_1, \lambda_8\}, \Lambda^1_2 = \{\lambda_2, \lambda_7, \lambda_1\}, \Lambda^1_3 = \{\lambda_3, \lambda_6, \lambda_1 + \lambda_8, \lambda_2\}, \Lambda^1_4 = \{\lambda_4, \lambda_1 + \lambda_6, \lambda_2 + \lambda_7, \lambda_3 + \lambda_8, \lambda_1 + \lambda_2, \lambda_5\}, \Lambda^1_5 = \{\lambda_5, \lambda_1 + \lambda_7, \lambda_2 + \lambda_8, \lambda_3, \lambda_6\}, \Lambda^1_6 = \{\lambda_6, \lambda_1 + \lambda_7, \lambda_1, \lambda_8\}, \Lambda^1_7 = \{\lambda_7, \lambda_1, \lambda_8\}, \Lambda^8 = \{\lambda_8, 0\}\)

\(g = F_4: \Lambda^1_1 = \{\lambda_1, 0\}, \Lambda^1_2 = \{\lambda_2, 2\lambda_4, \lambda_1\}, \Lambda^1_3 = \{\lambda_3, \lambda_4\}, \Lambda^1_4 = \{\lambda_4\}\)

\(g = G_2: \Lambda^1_1 = \{\lambda_1\}, \Lambda^1_2 = \{\lambda_2, 0\}\)

The main result is following

**Theorem 2.1.** Let \(g\) be a classical Lie algebra over an algebraically closed field \(k\) of characteristics \(p\), and \(H^0(\lambda)\) be a dual to a Weyl module. If \(g = E_n(n = 6, 7, 8)\), \(F_4, G_2\)
and \(p \geq h + 5\), then \(H^2(g, H^0(\lambda))^{-1}\) is trivial, except in the following cases:

(a) \(H^2(g, H^0(p\nu + w_2 \cdot 0))^{-1} \cong H^0(\nu)\), for all \(w_2 \in \{w \in W | l(w) = 2\}\);

(b) \(H^2(g, H^0(p\lambda_i - \alpha_i))^{-1} \cong \bigoplus_{\nu \in \Lambda^1_1} H^0(\nu)\), for all \(i \in \{1, 2, \ldots, n\}\).

By Theorem 2.1 the number of peculiar duals to Weyl modules is equal to the sum the rank of \(g\) and the number of elements with length 2 in the Weyl group. The Weyl group of \(g\) has exactly \(\frac{n}{4}(n + 2)(n - 1)\) elements with length 2 hence a number of a peculiar dual to Weyl modules is equal to \(n + \frac{1}{4}(n + 2)(n - 1) = \frac{1}{2}(n^2 + 3n - 2)\).

### 3 The proof of Theorem 2.1

The proof is based on connection between Hochschild cohomology groups of \(G_1\) and Chevalley-Eilenberg cohomology groups of the Lie algebra \(g\). For a restricted module \(V\), this connection is defined by the following exact sequence \([8], [9], [5]:\)

\[
0 \to H^1(G_1, V) \to H^1(g, V) \to H^0(g(V)) \otimes g^* \to \]

\[
H^2(G_1, V) \to \cdots \to H^3(G_1, V).
\]

(3.1)

Obviously that \(H^2(g, H^0(0)) \cong H^2(g, k) = 0\).

Now, let \(V = H^0(\lambda)\) and \(\lambda \neq 0\). Since \(H^0(\lambda)^G = 0\) at \(\lambda \neq 0\), then \(H^0(g, H^0(\lambda)) \otimes g^* = 0\). It is well known that the first Hochschild cohomology group of \(G_1\) coincides with the first usual Lie algebra cohomology group of \(g\) \([8]\). Then, from (3.1) we get the following exact sequence

\[
0 \to H^2(G_1, H^0(\lambda)) \to H^2(g, H^0(\lambda)) \to H^1(G_1, H^0(\lambda)) \otimes g^* \to H^3(G_1, H^0(\lambda)).
\]

(3.2)

Thus, the calculation of \(H^2(g, H^0(\lambda))\) is reduced to the calculations of \(H^2(G_1, H^0(\lambda))\) and the image of the map \(f\) in the exact sequence (3.2). The cohomology groups \(H^1(G_1, H^0(\lambda))\) and \(H^2(G_1, H^0(\lambda))\) are well known. We use these results \([1], [10], [2]\).

**Lemma 3.1.** If \(p > 3\) and \(\lambda \in X_1(T)\). Then

\[
H^1(G_1, H^0(\lambda))^{-1} \cong \left\{
\begin{array}{ll}
H^0(\lambda_i), & \text{for all } \lambda = p\lambda_i - \alpha_i \text{ with } i \in \{1, 2, \ldots, n\}; \\
0, & \text{in other cases}.
\end{array}
\right.
\]
Lemma 3.2. Let $p > 5$ and $\lambda \in \mathcal{X}_1(T) \setminus \{0\}$. Then

$$H^2(G_1, H^0(\lambda))^{(-1)} \cong \begin{cases} H^0(\nu), & \text{for all } \lambda = p\nu + w_2 \cdot 0 \text{ with } w_2 \in \{w \in W \mid l(w) = 2\} \\ 0, & \text{in other cases.} \end{cases}$$

Now we prove the following

Proposition 3.1. Let $p \geq h + 5$ and $\lambda \in \mathcal{X}_1(T) \setminus \{0\}$. Then

$$\text{Im } f \cong \begin{cases} \bigoplus_{\nu \in \Lambda_1^+} H^0(\nu)^{(1)}, & \text{for all } \lambda = p\lambda_i - \alpha_i \text{ with } i \in \{1, 2, \ldots, n\}; \\ 0, & \text{in other cases.} \end{cases}$$

Proof. First, we prove that the $G$-modules $H^1(G_1, H^0(\lambda))^{(-1)} \otimes g^*$ and $H^3(G_1, H^0(\lambda))^{(-1)}$ are decomposable.

By Lemma 3.1, $H^1(G_1, H^0(\lambda))^{(-1)} \otimes g^*$ is not trivial if and only if

$$\lambda \in \{p\lambda_i - \alpha_i \in \mathcal{X}_1(T) \mid i \in \{1, 2, \ldots, n\}\},$$

and in this case $H^1(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)} \cong H^0(\lambda_i)$ for all $i \in I$. The isomorphism $g^* \cong H^0(\tilde{\alpha})$ yields the isomorphism

$$H^1(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)} \otimes g^* \cong H^0(\lambda_i) \otimes H^0(\tilde{\alpha}).$$

Then, assuming the usual partial order on the set of weight, the last tensor product and $H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$ have the same greatest weights, and it is equal to $\tilde{\alpha} + \lambda_i$. Since

$$\max_i \{\tilde{\alpha} + \lambda_i + \rho, \alpha_i^\vee\} = h + 5 \leq p,$$

then the highest weights of all composition factors of $H^0(\lambda_i) \otimes H^0(\tilde{\alpha})$ and of $H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$ lie in the bottom $p$-alcove of the affine Weyl group. So, they are decomposable as $G$-modules [6, 7].

By Lemma 3.2, $H^2(G_1, H^0(p\lambda_i - \alpha_i)) = 0$ for all $i \in \{1, 2, \ldots, n\}$. Thus, it follows from the exactness of (3.2) and from Lemma 3.2, that to establish the isomorphisms claimed in Proposition 3.1, it is enough to compare the composition factors of $H^0(\lambda_i) \otimes H^0(\tilde{\alpha})$ with the composition factors of $H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$.

We will determine the composition factors of $H^0(\lambda_i) \otimes H^0(\tilde{\alpha})$ using the table 5 in [12], because $H^0(\lambda_i) \otimes H^0(\tilde{\alpha})$ is decomposable as a $G$-module. For the calculation of $H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$ we use the general Andersen-Jantzen’s formula [1]:

$$H^k(G_1, H^0(w \cdot 0 + p\nu))^{(-1)} \cong \begin{cases} H^0(S(k-l(w))/2(u^* \otimes k_\nu), & \text{if } k - l(w) \text{ is even;} \\ 0, & \text{in other cases,} \end{cases}$$

(3.3)

where $S(u^*)$ is the symmetric algebra of $u^*$, $u^*$ is a maximal nilpotent subalgebra of $g$ corresponding to the negative roots.
The results of calculations are gathered in the following tables.

### Table 1. Weights of the composition factors for $g = E_6$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$H^u(\lambda_i) \otimes H^u(\bar{\alpha})$</th>
<th>$H^3(G_1, H^u(p\lambda_i - \alpha_i))^{(-1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\lambda_1 + \lambda_2, \lambda_5, \lambda_1$</td>
<td>$\lambda_1 + \lambda_2, \lambda_5$</td>
</tr>
<tr>
<td>2</td>
<td>$2\lambda_2, \lambda_4, \lambda_1 + \lambda_6, \lambda_2, 0$</td>
<td>$2\lambda_2, \lambda_4, \lambda_1 + \lambda_6$</td>
</tr>
<tr>
<td>3</td>
<td>$\lambda_2 + \lambda_3, \lambda_1 + \lambda_5, \lambda_2 + \lambda_6$</td>
<td>$\lambda_2 + \lambda_3, \lambda_1 + \lambda_5, \lambda_2 + \lambda_6, 2\lambda_1$</td>
</tr>
<tr>
<td>4</td>
<td>$\lambda_2 + \lambda_4, \lambda_3 + \lambda_5,$</td>
<td>$\lambda_2 + \lambda_4, \lambda_3 + \lambda_5,$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 + \lambda_2 + \lambda_6, \lambda_1 + \lambda_3,$</td>
<td>$\lambda_1 + \lambda_2 + \lambda_6, \lambda_1 + \lambda_3,$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_5 + \lambda_6, 2\lambda_2 \lambda_4, \lambda_1 + \lambda_6, \lambda_2$</td>
<td>$\lambda_5 + \lambda_6, 2\lambda_2 \lambda_4, \lambda_1 + \lambda_6, \lambda_2$</td>
</tr>
<tr>
<td>5</td>
<td>$\lambda_2 + \lambda_5, \lambda_3 + \lambda_6,$</td>
<td>$\lambda_2 + \lambda_5, \lambda_3 + \lambda_6,$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 + \lambda_2, 2\lambda_6, \lambda_5, \lambda_1$</td>
<td>$\lambda_1 + \lambda_2, 2\lambda_6, \lambda_5, \lambda_1$</td>
</tr>
<tr>
<td>6</td>
<td>$\lambda_2 + \lambda_6, \lambda_3, \lambda_6$</td>
<td>$\lambda_2 + \lambda_6, \lambda_3$</td>
</tr>
</tbody>
</table>

### Table 2. Weights of the composition factors for $g = E_7$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$H^u(\lambda_i) \otimes H^u(\bar{\alpha})$</th>
<th>$H^3(G_1, H^u(p\lambda_i - \alpha_i))^{(-1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2\lambda_1, \lambda_3, \lambda_6, \lambda_1, 0$</td>
<td>$2\lambda_1, \lambda_3, \lambda_6$</td>
</tr>
<tr>
<td>2</td>
<td>$\lambda_1 + \lambda_2, \lambda_5, \lambda_1 + \lambda_7, \lambda_2, \lambda_7$</td>
<td>$\lambda_1 + \lambda_2, \lambda_5, \lambda_1 + \lambda_7$</td>
</tr>
<tr>
<td>3</td>
<td>$\lambda_1 + \lambda_3, \lambda_4, \lambda_1 + \lambda_6,$</td>
<td>$\lambda_1 + \lambda_3, \lambda_1 + \lambda_6,$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2 + \lambda_7, 2\lambda_1, \lambda_3, \lambda_6, \lambda_1$</td>
<td>$\lambda_2 + \lambda_7, 2\lambda_1$</td>
</tr>
<tr>
<td>4</td>
<td>$\lambda_1 + \lambda_4, \lambda_2 + \lambda_5,$</td>
<td>$\lambda_1 + \lambda_4, \lambda_2 + \lambda_5,$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3 + \lambda_6, \lambda_1 + \lambda_2 + \lambda_7,$</td>
<td>$\lambda_1 + \lambda_2 + \lambda_7, \lambda_3 + \lambda_6,$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_5 + \lambda_7, \lambda_1 + \lambda_3, 2\lambda_2$</td>
<td>$\lambda_5 + \lambda_7, \lambda_1 + \lambda_3,$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_4, \lambda_1 + \lambda_6, \lambda_2 + \lambda_7, \lambda_3$</td>
<td>$\lambda_4, \lambda_1 + \lambda_6, \lambda_2 + \lambda_7, \lambda_3$</td>
</tr>
<tr>
<td>5</td>
<td>$\lambda_1 + \lambda_5, \lambda_2 + \lambda_6,$</td>
<td>$\lambda_1 + \lambda_5, \lambda_2 + \lambda_6,$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3 + \lambda_7, \lambda_1 + \lambda_2,$</td>
<td>$\lambda_3 + \lambda_7, \lambda_1 + \lambda_2,$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_6 + \lambda_7, \lambda_5, \lambda_1 + \lambda_7, \lambda_2$</td>
<td>$\lambda_6 + \lambda_7$</td>
</tr>
<tr>
<td>6</td>
<td>$\lambda_1 + \lambda_6, \lambda_2 + \lambda_7, \lambda_3, 2\lambda_7, \lambda_6, \lambda_1$</td>
<td>$\lambda_1 + \lambda_6, \lambda_2 + \lambda_7, \lambda_3, 2\lambda_7, \lambda_6, \lambda_1$</td>
</tr>
<tr>
<td>7</td>
<td>$\lambda_1 + \lambda_7, \lambda_2, \lambda_7$</td>
<td>$\lambda_1 + \lambda_7, \lambda_2$</td>
</tr>
</tbody>
</table>
Table 3. Weights of the composition factors for $g = E_8$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$H^0(\lambda_i) \otimes H^0(\bar{\alpha})$</th>
<th>$H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\lambda_1 + \lambda_8, \lambda_2, \lambda_7, \lambda_1, \lambda_8$</td>
<td>$\lambda_1 + \lambda_8, \lambda_2, \lambda_7$</td>
</tr>
<tr>
<td>2</td>
<td>$\lambda_3 + \lambda_8, \lambda_6, \lambda_7, \lambda_1 + \lambda_8$</td>
<td>$\lambda_2 + \lambda_8, \lambda_3, \lambda_6, \lambda_1 + \lambda_8$</td>
</tr>
<tr>
<td>3</td>
<td>$\lambda_3 + \lambda_8, \lambda_2 + \lambda_5, \lambda_1 + \lambda_7$</td>
<td>$\lambda_3 + \lambda_8, \lambda_1 + \lambda_2, \lambda_5$</td>
</tr>
<tr>
<td>4</td>
<td>$\lambda_2 + \lambda_6, \lambda_3 + \lambda_7, \lambda_1 + \lambda_8$</td>
<td>$\lambda_1 + \lambda_7, \lambda_2 + \lambda_8, 2\lambda_1$</td>
</tr>
<tr>
<td>5</td>
<td>$\lambda_6 + \lambda_8, \lambda_3 + \lambda_7, \lambda_2 + \lambda_6, \lambda_6 + \lambda_8$</td>
<td>$\lambda_2 + \lambda_6, \lambda_3 + \lambda_7, \lambda_1 + \lambda_2, \lambda_5$</td>
</tr>
<tr>
<td>6</td>
<td>$\lambda_2 + \lambda_8, \lambda_3 + \lambda_7, \lambda_5 + \lambda_8$</td>
<td>$\lambda_1 + \lambda_7, \lambda_2 + \lambda_8, 2\lambda_1$</td>
</tr>
<tr>
<td>7</td>
<td>$\lambda_3 + \lambda_8, \lambda_1 + \lambda_2, \lambda_6 + \lambda_8, \lambda_5$</td>
<td>$\lambda_4 + \lambda_8, \lambda_2 + \lambda_3, \lambda_1 + \lambda_8$</td>
</tr>
<tr>
<td>8</td>
<td>$\lambda_3 + \lambda_8, \lambda_1 + \lambda_8$</td>
<td>$\lambda_3 + \lambda_8, \lambda_1 + \lambda_2, \lambda_6 + \lambda_8, \lambda_5$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 + \lambda_7, \lambda_2 + \lambda_6, \lambda_3, \lambda_6$</td>
<td>$\lambda_2 + \lambda_6, \lambda_3 + \lambda_7, \lambda_1 + \lambda_2, \lambda_5$</td>
</tr>
</tbody>
</table>

Table 4. Weights of the composition factors for $g = F_4$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$H^0(\lambda_i) \otimes H^0(\bar{\alpha})$</th>
<th>$H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2\lambda_1, 2\lambda_2, 2\lambda_4, \lambda_1, 0$</td>
<td>$2\lambda_1, 2\lambda_2, 2\lambda_4, \lambda_1 + \lambda_4, \lambda_3$</td>
</tr>
<tr>
<td>2</td>
<td>$\lambda_3 + \lambda_4, 2\lambda_6, \lambda_2 + \lambda_4, \lambda_1$</td>
<td>$\lambda_2 + \lambda_3, 2\lambda_4, \lambda_1 + \lambda_2, \lambda_4, \lambda_1 + \lambda_3$</td>
</tr>
<tr>
<td>3</td>
<td>$\lambda_3 + \lambda_4, 2\lambda_6, \lambda_2 + \lambda_4, \lambda_1$</td>
<td>$\lambda_2 + \lambda_3, 2\lambda_4, \lambda_1 + \lambda_2, \lambda_4, \lambda_1 + \lambda_3$</td>
</tr>
<tr>
<td>4</td>
<td>$\lambda_1 + \lambda_4, \lambda_3, \lambda_4$</td>
<td>$\lambda_1 + \lambda_4, \lambda_3, \lambda_4$</td>
</tr>
</tbody>
</table>

Table 5. Weights of the composition factors for $g = G_2$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$H^0(\lambda_i) \otimes H^0(\bar{\alpha})$</th>
<th>$H^3(G_1, H^0(p\lambda_i - \alpha_i))^{(-1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\lambda_1 + \lambda_2, 2\lambda_1, \lambda_1$</td>
<td>$\lambda_1 + \lambda_2, 2\lambda_1, \lambda_2$</td>
</tr>
<tr>
<td>2</td>
<td>$2\lambda_2, 3\lambda_1, 2\lambda_1, \lambda_2, 0$</td>
<td>$2\lambda_2, 3\lambda_1, \lambda_1 + \lambda_2, 2\lambda_1$</td>
</tr>
</tbody>
</table>

Comparing the composition factors of $H^0(\lambda_i) \otimes H^0(\bar{\alpha})$ with the composition factors of $H^3(G_1, H^0(\mu_i))^{(-1)}$, listed in the tables 1-5, we obtain the statements of Proposition 3.1. This completes the proof of Proposition 3.1. □
Finally, we can finish the proof of Theorem 2.1.

By (3.3), $H^1(G_1, H^0(p\nu + w_2 \cdot 0) = 0$ for all $w_2 \in \{w \in W \mid l(w) = 2\}$. Then, from the exactness of (3.2) it follows that

$$H^2(G_1, H^0(p\nu + w \cdot 0)) \cong H^2(g, H^0(p\nu + w \cdot 0))$$

for all $w_2 \in \{w \in W \mid l(w) = 2\}$.

Thus, combining the statements of Lemma 3.2 and Proposition 3.1 we get the statement of Theorem 2.1. This completes the proof of Theorem 2.1. \qed

Acknowledgements. This paper was partially supported by the State Program of Fundamental Research F.0508.
On the second cohomology groups of exceptional Lie algebras in positive characteristic

References


Shenali Shapatayevich Ibraev
Bolashak University
31A Abai St.
120000 Kyzylorda, Kazakhstan
E-mail: ibrayevsh@mail.ru

Received: 12.09.2010