ON MAXIMAL SUBGROUP OF A FINITE SOLVABLE GROUP

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Abstract. Let $H$ be a non-normal maximal subgroup of a finite solvable group $G$, and let $q \in \pi(F(H/\text{Core}_{G}H))$. It is proved that $G$ has a Sylow $q$-subgroup $Q$ such that $N_{G}(Q) \subseteq H$.

1 Introduction

All groups considered in this paper are finite. All notation and definitions correspond to those in [1, 2].

In 1986, V. A. Vedernikov obtained the following result:

Theorem A. [3, Corollary 2.1] If $H$ is a non-normal maximal subgroup of a solvable group $G$ then $N_{G}(Q) \subseteq H$ for some Sylow subgroup $Q$ of $G$.

Here $N_{G}(Q)$ is the normalizer of $Q$ in $G$.

In this paper we consider the following problem:

What is a Sylow subgroup such that its normalizer is contained in a non-normal maximal subgroup of a solvable group?

Answering the question we prove the following theorem:

Theorem 1.1. Let $H$ be a non-normal maximal subgroup of a solvable group $G$, and let $q \in \pi(F(H/\text{Core}_{G}H))$. Then $G$ has a Sylow $q$-subgroup $Q$ such that $N_{G}(Q) \subseteq H$.

Here $F(X)$ is the Fitting subgroup of $X$, $\pi(Y)$ is the set of all prime divisors of $|Y|$, $\text{Core}_{G}H = \cap_{g \in G}H^{g}$ is the core of $H$ in $G$, i.e., is the largest normal subgroup of $G$ contained in $H$.

Corollary 1.1. Let $H$ be a non-normal maximal subgroup of a solvable group $G$, and let $q \in \pi(F(H/\text{Core}_{G}H))$. Then $H$ has a Sylow $q$-subgroup $Q$ such that $N_{G}(H_{1}) \subseteq H$ for each subgroup $H_{1}$ of $H$ satisfying $Q \subseteq H_{1} \subseteq H$. 
Corollary 1.2. Let $H$ be a non-normal maximal subgroup of a solvable group $G$, and let $\omega \subseteq \pi(F(H/\text{Core}_G H))$. Then $G$ has a Hall $\omega$-subgroup $G_\omega$ such that $N_G(G_\omega) \subseteq H$.

For non-solvable groups, this result is false. For example, $PSL(2, 17)$ has the order $2^4 \cdot 3^2 \cdot 17$, and the symmetric group $S_4$ is a maximal subgroup in $PSL(2, 17)$, see [3]. Since $|S_4| = 2^4 \cdot 3$, it follows that $S_4$ does not contain a Sylow subgroup of $PSL(2, 17)$. Thus it is not possible to extend theorem of V. A. Vedernikov and Theorem 1 to non-solvable groups.

The following question is contained in [3]:

Question. (V. A. Vedernikov, [3]) Is it possible to extend Theorem A to a $p$-solvable group $G$ containing a maximal subgroup $M$ such that $|G : M| = p^a$, $a \in \mathbb{N}$?

Answering this question we prove the following theorem:

Theorem 1.2. Let $G$ be a $p$-solvable group. Let $M$ be a non-normal maximal subgroup of $G$, and let $|G : M| = p^a$, $a \in \mathbb{N}$. Then:

1) if $F(M/\text{Core}_G M) \neq 1$ and $q \in \pi(F(M/\text{Core}_G M))$ then $G$ has a Sylow $q$-subgroup $Q$ such that $N_G(Q) \subseteq M$;

2) if $F(M/\text{Core}_G M) = 1$ then $N_G(K) \subseteq M$ for some Hall $p'$-subgroup $K$ of $G$.

2 Notations and preliminary results

In this section we give some definitions and basic results that will be used in our paper.

Let $\mathbb{P}$ be a set of all prime numbers, and let $\pi$ be a set of primes, i.e., $\pi \subseteq \mathbb{P}$. In the paper, $\pi'$ is the set of all primes not contained in $\pi$, i.e., $\pi' = \mathbb{P} \setminus \pi$, $\pi(m)$ is the set of all prime divisors of $m$. If $\pi(m) \subseteq \pi$ then $m$ is called a $\pi$-number.

A subgroup $H$ of $G$ is called a $\pi$-subgroup, if $|H|$ is a $\pi$-number. A subgroup $H$ of $G$ is called a Hall $\pi$-subgroup, if $|H|$ is a $\pi$-number and $|G : H|$ is a $\pi'$-number. As usual, $O_\pi(X)$ is the largest normal $\pi$-subgroup of $X$. A group is called $\pi$-separable if it has a normal series whose factors are $\pi$-groups or $\pi'$-groups.

A group $G$ is called $\pi$-solvable if it is $\pi$-separable and it has a solvable Hall $\pi$-subgroup.

Lemma 2.1. [3, Theorem 1] Let $G$ be a $\pi$-separable group, and let $H$ be a subgroup of $G$. If $|G : H|$ is a $\pi$-number then $O_\pi(H) \subseteq O_\pi(G)$.

Lemma 2.2. Let $R$ be a Hall $\pi$-subgroup of a $\pi$-separable group $G$, and let $N$ be a normal subgroup of $G$. Then $N_G(R)N/N = N_{G/N}(RN/N)$.

Proof. For $x \in N_G(R)$ we have:

$$(x^{-1}N)(RN/N)(xN) = R^xN/N = RN/N,$$

i.e., $N_G(R)N/N \subseteq N_{G/N}(RN/N)$.

Conversely, if $yN \in N_{G/N}(RN/N)$ then $R^yN = RN$. Next $R$ and $R^y$ are Hall subgroups of $RN$ which are conjugate, i.e., $R^y = R^{ak} = R^k$ for some $ak \in RN$. Therefore...
a ∈ R, k ∈ N. Then \( yk^{-1} \in N_G(R) \), whence \( y \in N_G(R)N \), i.e., \( N_G/N(RN/N) \leq N_G/RN/N \).

**Lemma 2.3.** Let \( G \) be a \( \pi \)-solvable group containing a nilpotent Hall \( \pi \)-subgroup. If \( H \) is a maximal subgroup of \( G \) and \( |G : H| \) is \( \pi \)-number then \( O_\pi(H) \) is a normal subgroup of \( G \).

**Proof.** By Lemma 2.1, \( O_\pi(H) \subseteq O_\pi(G) \). If \( O_\pi(H) = O_\pi(G) \) then \( O_\pi(H) \) is a normal subgroup of \( G \). Let \( O_\pi(H) \) be a proper subgroup of \( O_\pi(G) \), and let \( G_\pi \) be a Hall \( \pi \)-subgroup of \( G \). Clearly, \( O_\pi(G) \) is a proper subgroup of \( G_\pi \). Since \( G_\pi \) is a nilpotent subgroup we have that \( O_\pi(H) \) is a proper subgroup of \( D = N_{G_\pi}(O_\pi(H)) \). Since \( O_\pi(H) = H \cap O_\pi(G) \) it follows that \( D \) is not contained in \( H \), so \( N_G(O_\pi(H)) \supseteq \langle H, D \rangle = G \) and \( O_\pi(H) \) is a normal subgroup of \( G \).

3 Main results

**Theorem 3.1.** Let \( G \) be a \( \pi \)-solvable group containing a nilpotent Hall \( \pi \)-subgroup. Let \( M \) be a non-normal maximal subgroup of \( G \), and let \( |G : M| \) be an \( \pi \)-number. Then:

1) if \( F(M/\text{Core}_G M) \neq 1 \) and \( q \in \pi(F(M/\text{Core}_G M)) \) then \( G \) has a Sylow \( q \)-subgroup \( Q \) such that \( N_G(Q) \subseteq M \);

2) if \( F(M/\text{Core}_G M) = 1 \) then \( N_G(K) \subseteq M \) for some Hall \( \pi' \)-subgroup \( K \) of \( G \).

**Proof.** Case 1: \( \text{Core}_G M = 1 \). Since \( M \) is a \( \pi \)-solvable group we have that \( M \) contains a Hall \( \pi' \)-subgroup \( K \). Thus

\[ |G : K| = |G : M||M : K| \]

Since \( |G : M| \) is an \( \pi \)-number we have that \( K \) is a Hall \( \pi' \)-subgroup of \( G \). Hence \( O_{\pi'}(G) \leq K \leq M \), so \( O_{\pi'}(G) \leq \text{Core}_G M = 1 \). Since \( G \) is a \( \pi \)-solvable group we have that \( O_{\pi'}(G) \neq 1 \). However, \( G \) is a primitive group with a maximal subgroup \( M \) such that \( \text{Core}_G M = 1 \). Therefore, for some \( p \in \pi(G) \) we have the following:

\[ N = O_p(G) = F(G) = C_G(O_p(G)) \neq 1, \ G = M[O_p(G)], \ \Phi(G) = 1, \]

and \( O_p(M) = 1 \).

Assume that \( F(M) \neq 1 \), \( q \in \pi(F(M)) \), and let \( Q \) be a Sylow \( q \)-subgroup of \( M \). Then \( O_q(M) \neq 1 \), \( O_q(M) \subseteq Q \), and since \( \text{Core}_G M = 1 \) we have \( N_G(O_q(M)) = M \).

Since \( O_p(M) = 1 \), it follows that \( p \) does not belong to \( \pi(F(M)) \) and \( p \neq q \). A subgroup \( D = N_G(Q) \cap O_p(G) \) is a normal subgroup of \( N_G(Q) \), and

\[ D \subseteq C_G(Q) \subseteq C_G(O_q(M)) \subseteq N_G(O_q(M)) = M. \]

Next \( D = N_G(Q) \cap O_p(G) \subseteq M \cap O_p(G) = 1 \). We consider the subgroup \( L = N_G(Q)O_p(G) \). Since \( N_G(Q) \cap O_p(G) = 1 \) we have \( L = [O_p(G)]N_G(Q) \). It follows from \( G = [O_p(G)]M \) and Dedekind’s identity that

\[ L = [O_p(G)]N_G(Q) = [O_p(G)](L \cap M), \ L/O_p(G) \simeq N_G(Q) \simeq L \cap M, \]
and $L \cap M$ is a $q$-closed subgroup. The inclusion $Q \subseteq L \cap M$ implies that $Q$ is a normal subgroup of $L \cap M$. It follows from $|N_G(Q)| = |L \cap M|$ that $L \cap M = N_G(Q)$ and $N_G(Q) \subseteq M$.

Assume that $F(M) = 1$, and let $K$ be a Hall $\pi'$-subgroup of $M$. By Lemma 2.3, $O_{\pi}(M)$ is a normal subgroup of $G$. Hence $O_{\pi}(M) \leq Core_G M = 1$. Since $G$ is a $\pi$-solvable group we have $O_{\pi'}(M) \neq 1$ and $O_{\pi'}(M) \subseteq K$. Since $Core_G M = 1$ we have $N_G(O_{\pi'}(M)) = M$. A subgroup $B = N_G(K) \cap O_p(G)$ is a normal subgroup of $N_G(K)$ and

$$\forall C \subseteq \pi(G) \subseteq \pi(G) \subseteq N_G(O_{\pi'}(M)) = M.$$ 

Next $B = N_G(K) \cap O_p(G) \subseteq M \cap O_p(G) = 1$. We consider the subgroup $T = N_G(K)/O_p(G)$. Since $N_G(K) \cap O_p(G) = 1$ we have $T = [O_p(G)]N_G(K)$. It follows from $G = [O_p(G)]M$ and Dedekind’s identity that

$$T = [O_p(G)]N_G(K) = [O_p(G)](T \cap M), \quad T/O_p(G) \simeq N_G(K) \simeq T \cap M,$$

and that $T \cap M$ is a $\pi'$-closed subgroup. Since the Hall $\pi'$-subgroup $K$ is contained in $T \cap M$, $K$ is a normal subgroup of $T \cap M$ and $T \cap M \subseteq N_G(K)$. It follows from isomorphism $N_G(K) \simeq T \cap M$ that $|N_G(K)| = |T \cap M|$, so $T \cap M = N_G(K)$ and $N_G(K) \subseteq M$.

Thus the theorem in Case 1 is proved.

Case 2: $N = Core_G M \neq 1$. We consider the quotient group $G = G/N$. Clearly, $G$ is a $\pi$-solvable group and $|G : M| = |G : M|$ is a $\pi$-number, where $M = M/N$. Since $Core_G M = 1$, for the group $G$ with a non-normal maximal subgroup $M$, we can apply Case 1.

Assume that $F(M) \neq 1$, and let $q \in \pi(F(M))$. By Case 1, $G$ has a Sylow $q$-subgroup $Q$ such that $N_G(Q) \subseteq M$. Let $Q = A/N$, and let $Q$ be a Sylow $q$-subgroup of $A$. Then $Q/A = A/N$ and $Q = Q$.

$$N_G(Q) = N_G(A)N/Q = N_G(Q)N/N.$$ 

By the condition $N_G(Q)N/N \subseteq M = M/N$ it follows that $N_G(Q) \subseteq M$.

Assume that $F(M/Core_G M) = 1$. By Case 1, $N_G(K) \subseteq M$ for some Hall $\pi'$-subgroup $K$ of $G$. Let $K = B/N$, and let $R$ be a Hall $\pi'$-subgroup of $B$. It exists because $B$ is a $\pi$-solvable group. Therefore $R$ is a Hall $\pi'$-subgroup of $G$, $RN/N = B/N = K$, and by Lemma 2.2,

$$N_G(R)N/N \subseteq N_G(RN/N) = N_G(K).$$

By induction, $N_G(R)N/N \subseteq M = M/N$, so $N_G(R) \subseteq M$.

Thus the theorem is proved in Case 2.

Note that Theorem 1.2 follows from Theorem 3.1 if $\pi = \{p\}$. If $G$ is a solvable group then under the assumptions of Theorem 3.1 we have $\pi(F(M/Core_G M)) \neq \emptyset$. So Theorem 1.1 follows from Theorem 3.1.
Proof of Corollary 1.1. By Theorem 1.1, $G$ has a Sylow $q$-subgroup $Q$ such that $N_G(Q) \subseteq H$. Let $H_1$ be any subgroup such that $Q \subseteq H_1 \subseteq H$, and let $T = N_G(H_1)$. By Frattini’s argument, we have

$$T = N_T(Q)H_1 \subseteq N_G(Q)H_1 \subseteq H, \quad T = N_G(H_1) \subseteq H.$$

In the case in which $H_1$ is a Hall $\omega$-subgroup of $H$ we obtain Corollary 1.2.
References


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