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# Dispersionless limit of the (1+1)-dimensional Fokas-Lenells equation

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**Abstract.** Nonlinear dispersionless equations can be obtained as dispersionless limits (quasiclassical limits) of integrable hierarchies of equations or by constructing a system of hydrodynamic type. In this paper, the dispersionless limit of the (1+1)-dimensional Fokas-Lenells equation is found. We present Lax pair for this dispersionless equation. As well known, the Fokas-Lenells equation characterizes the propagation of ultrashort nonlinear light pulses in optical fibers. Exact solutions of the dispersionless version of the Fokas-Lenells equation describe nonlinear shock waves in optical fiber systems. By using the obtained results, one can find shock wave solutions of the dispersionless equation, which have different physical applications.

## 1. Introduction

After the discovery of the inverse scattering problem method (IST), the study of integrable models or nonlinear partial differential equations became an active area of the research. These models in some sense are universals since they manifest themselves in many areas of physics, such as solid state, nonlinear optics, hydrodynamics, field theory, and many others. In addition, integrable models are associated with many areas of mathematics and have peculiar structures [1]-[4]. One of such models is the recently proposed Fokas-Lenells (FL) equation, which describes the propagation of ultrashort nonlinear light pulses in optical fibers. The FL equation is as follows [5, 6]:

$$iq_{xt} - iq_{xx} + 2q_x - \delta |q|^2 q_x + iq = 0, \quad (1)$$

where  $q(x, t)$  is complex envelope of the field,  $x$  is the propagation distance and  $t$  the retarded time, which are also denoted partial differentiation by arguments  $x$ ,  $t$  and  $i$  - imaginary unit. Also  $\delta$  ( $\delta = \pm 1$ ) denotes self-focusing with  $\delta = 1$  or self-defocusing with  $\delta = -1$  [7].

In this paper, we consider the case of  $\delta = 1$ , that is, the equation (1) with self-focusing has the form

$$iq_{xt} - iq_{xx} + 2q_x - |q|^2 q_x + iq = 0. \quad (2)$$

Since equation (2) is integrable, it has a Lax pair, which plays a major role in the theory of integrable systems. She allows you to apply IST to construct exact solutions of integrable systems and to study the asymptotics of problems with initial conditions. For the studied equation, the Lax representation (LR) has the form [5, 6]

$$\Phi_x(x, t, \lambda) = U(x, t, \lambda)\Phi(x, t, \lambda), \quad (3)$$

$$\Phi_t(x, t, \lambda) = V(x, t, \lambda)\Phi(x, t, \lambda), \quad (4)$$



where  $\Phi = (\Phi_1, \Phi_2)^T$  is called  $2 \times 2$  the matrix eigenfunction of the eigenvalue  $\lambda$  (or spectral parameter) and the matrix operators  $U$  and  $V$  are given in the following form:

$$U(x, t, \lambda) = -i\lambda^2\sigma_3 + \lambda Q_x(x, t, \lambda), \quad (5)$$

$$V(x, t, \lambda) = -i\lambda^2\sigma_3 + \lambda Q_x(x, t, \lambda) + V_0(x, t, \lambda) + \frac{1}{\lambda}V_{-1}(x, t, \lambda) - \frac{i}{4\lambda^2}\sigma_3. \quad (6)$$

Here

$$Q = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \quad V_0 = i\sigma_3 - \frac{i|q|^2}{2}\sigma_3, \quad V_{-1} = \frac{i}{2} \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where  $\bar{q}$  means complex conjugate  $q$ . Also, can easily verify that the equations (3)-(4) satisfy the compatibility condition  $\Phi_{xt} = \Phi_{tx}$ , that is, the equation of zero curvature  $U_t - V_x + [U, V] = 0$  gives the equation (2).

For the first time, dispersionless equations were introduced independently by Lebedev, Manin, and Zakharov in 1980 and appear in the development of low-dimensional quantum field theory. The main example of such equations is the dispersionless Kadomtsev-Petviashvili equation, which describes wave propagation in the absence of dispersion.

In this paper, we present the dispersionless (1+1)-dimensional FL equation and its LR, which satisfies the compatibility condition, that is, it means that the desired equation is integrable.

## 2. Dispersionless (1+1)-dimensional Fokas-Lenells equation

We turn to find the dispersionless limit (semiclassical limit) of the equation (2). To do this, we make in the equation (2) a scale transformation with  $x$  and  $t$ , that is, [8, 9]

$$\frac{\partial}{\partial t} \rightarrow \epsilon \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x} \rightarrow \epsilon \frac{\partial}{\partial x}.$$

and the limit transition is  $\epsilon \rightarrow 0$ . Here  $\epsilon$  is the scaled Planck's constant. In this case, the equation (2) takes the form

$$i\epsilon^2 q_{xt} - i\epsilon^2 q_{xx} + 2\epsilon q_x - \epsilon |q|^2 q_x + iq = 0. \quad (7)$$

We perform a change of variables in the following form:

$$q = \sqrt{u} e^{\frac{iS}{\epsilon}}, \quad (8)$$

where  $u(x, t)$  is the amplitude and  $S(x, t)$  is the classical action, which are real functions, also,  $|q|^2 = u$ .

Now, first differentiate the equation (8) with respect to  $x$

$$q_x = \left( \frac{u_x}{\sqrt{2u}} + \frac{i\sqrt{u}S_x}{\epsilon} \right) e^{\frac{iS}{\epsilon}}, \quad (9)$$

then the equation (9) is differentiable with respect to  $x$  and  $t$ , that is,

$$q_{xx} = \left\{ \left( \frac{u_x}{2\sqrt{u}} \right)_x - \frac{S_x^2 \sqrt{u}}{\epsilon^2} + \frac{i}{\epsilon} \left[ S_{xx} \sqrt{u} + \frac{S_x u_x}{\sqrt{u}} \right] \right\} e^{\frac{iS}{\epsilon}}, \quad (10)$$

$$q_{xt} = \left\{ \left( \frac{u_x}{2\sqrt{u}} \right)_t - \frac{S_x S_t \sqrt{u}}{\epsilon^2} + \frac{i}{\epsilon} \left[ S_{xt} \sqrt{u} + \frac{S_x u_t + S_t u_x}{2\sqrt{u}} \right] \right\} e^{\frac{iS}{\epsilon}}. \quad (11)$$

Expressions (9)-(11) are substituted into the equation (7), which gives

$$\begin{aligned} i\epsilon^2 \left( \frac{u_x}{2\sqrt{u}} \right)_t - iS_x S_t \sqrt{u} - \epsilon \left( S_{xt} \sqrt{u} + \frac{S_x u_t}{2\sqrt{u}} + \frac{S_t u_x}{2\sqrt{u}} \right) - i\epsilon^2 \left( \frac{u_x}{2\sqrt{u}} \right)_x + iS_x^2 \sqrt{u} + \\ + \epsilon (S_{xx} \sqrt{u} + \frac{S_x u_x}{\sqrt{u}}) + \epsilon \frac{u_x}{\sqrt{u}} + 2iS_x \sqrt{u} - \epsilon \frac{\delta \sqrt{u} u_x}{2} - iS_x u \sqrt{u} + i\sqrt{u} = 0. \end{aligned} \quad (12)$$

Having considered the equation (12) for various degrees  $\epsilon$ , we obtain the following equations:

$$S_t - S_x + u - \frac{1}{S_x} - 2 = 0, \quad (13)$$

$$2S_{xt}u + S_x u_t + S_t u_x - 2(S_x u_x)_x - 2u_x + uu_x = 0, \quad (14)$$

where  $S = \partial_x^{-1}v$  ( $v = S_x$ ).

Then the equations (13) and (14) take the following final form:

$$v_t - v_x + u + \frac{v_x}{v^2} = 0, \quad (15)$$

$$u_t + (2v_t u + \gamma u_x - 2(uv)_x - 2u_x + uu_x)v^{-1} = 0, \quad (16)$$

here  $\gamma = \partial_x^{-1}v_t$ .

Thus, the equations (15) and (16) are the desired dispersionless (1+1)-dimensional FL equation and it is obvious that there are no dispersion terms in them.

### 3. Lax representation of the dispersionless Fokas-Lenells equation

To construct the LR of the equation (15) and (16), consider the LR of the FL equation as

$$\Phi_{1x} = -i\lambda^2 \Phi_1 + \lambda q_x \Phi_2, \quad (17)$$

$$\Phi_{2x} = \lambda \bar{q}_x \Phi_1 + i\lambda^2 \Phi_2 \quad (18)$$

and

$$\Phi_{1t} = \left( -i\lambda^2 + i - \frac{i|q|^2}{2} - \frac{i}{4\lambda^2} \right) \Phi_1 + \left( \lambda q_x + \frac{i}{2\lambda} q \right) \Phi_2, \quad (19)$$

$$\Phi_{2t} = \left( \lambda \bar{q}_x - \frac{i\bar{q}}{2\lambda} \right) \Phi_1 + \left( i\lambda^2 - i + \frac{i|q|^2}{2} + \frac{i}{4\lambda^2} \right) \Phi_2. \quad (20)$$

We set the substitution in the following form:

$$\Psi_1 = \xi_1 e^{\frac{i}{\epsilon}[F + \lambda^2 x]}, \quad (21)$$

$$\Psi_2 = \xi_2 e^{\frac{i}{\epsilon}[F + \lambda^2 x - \partial_x^{-1}v]} \quad (22)$$

and

$$q = \sqrt{u} e^{\frac{i}{\epsilon} \partial_x^{-1}v}, \quad (23)$$

where  $F, u, v, \xi_n$  are real functions. We apply scale transformation

$$\frac{\partial}{\partial t} \rightarrow \epsilon \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x} \rightarrow \epsilon \frac{\partial}{\partial x},$$

after which the equations (17)-(20) take the form

$$\epsilon\Phi_{1x} = -i\lambda^2\Phi_1 + \epsilon\lambda q_x\Phi_2, \quad (24)$$

$$\epsilon\Phi_{2x} = \epsilon\lambda\bar{q}_x\Phi_1 + i\lambda^2\Phi_2. \quad (25)$$

and

$$\epsilon\Phi_{1t} = \left(-i\lambda^2 + i - \frac{i|q|^2}{2} - \frac{i}{4\lambda^2}\right)\Psi_1 + \left(\epsilon\lambda q_x + \frac{i}{2\lambda}q\right)\Phi_2, \quad (26)$$

$$\epsilon\Phi_{2t} = \left(\epsilon\lambda\bar{q}_x - \frac{i\bar{q}}{2\lambda}\right)\Psi_1 + \left(i\lambda^2 - i + \frac{i|q|^2}{2} + \frac{i}{4\lambda^2}\right)\Psi_2. \quad (27)$$

Substituting the equations (21)-(23) into the equations (24) and (25), we get

$$F_x + 2\lambda^2 - \lambda\sqrt{uv}\frac{\xi_2}{\xi_1} = 0, \quad (28)$$

$$\xi_2 = -\frac{\lambda\sqrt{uv}\xi_1}{F_x - v}. \quad (29)$$

Then, in the obtained equation (29) we substitute the equation (28), and we get the first equation of the LR

$$F_x - \frac{\lambda^2 uv^2}{F_x - v} + 2\lambda^2 = 0 \quad (30)$$

or

$$p + \frac{\lambda^2 uv^2}{p - v} + 2\lambda^2 = 0, \quad (31)$$

where  $p = F_x$ .

Now we turn to find the second LR equation. We substituted equations (21)-(23) is into the equation (26)

$$F_t + \lambda^2 - 1 + \frac{u}{2} + \frac{1}{4\lambda^2} = -\frac{\lambda^2 uv^2}{F_x - v} - \frac{uv}{2(F_x - v)}. \quad (32)$$

We differentiate the equation (32) with respect to  $x$

$$F_{tx} + \frac{u_x}{2} + \left(\frac{\lambda^2 uv^2}{F_x - v}\right)_x + \left[\frac{uv}{2(F_x - v)}\right]_x = 0 \quad (33)$$

or

$$p_t - p_x + \frac{1}{2}\left(\frac{uv}{p - v}\right)_x + \frac{u_x}{2} = 0. \quad (34)$$

Integrating the equation (34) over  $x$  and entering the notation  $\partial_x^{-1}p_t = F_t$ , we obtain

$$F_t - F_x + \frac{1}{2}\left(\frac{uv}{F_x - v}\right) + \frac{u}{2} = E, \quad (35)$$

where  $E = \text{const.}$

Thus, we have obtained the Lax pair of the equation (31)-(34), that is

$$p + \frac{\lambda^2 uv^2}{p - v} + 2\lambda^2 = 0,$$

$$p_t - p_x + \frac{1}{2}\left(\frac{uv}{p - v}\right)_x + \frac{u_x}{2} = 0.$$

#### 4. Conclusion

Thus, the dispersionless (1+1)-dimensional Fokas-Lenells equation is found and its the Lax representation is constructed, which satisfies the compatibility condition. The next step is to find the shock wave solution for obtained the equation.

#### 5. References

- [1] Yersultanova Z, Zhassyybayeva M, Yesmakhanova K, Nugmanova G and Myrzakulov R 2016 Darboux transformation and exact solutions of the integrable Heisenberg ferromagnetic equation with self-consistent potentials *Geom. Methods Mod. Phys.* **13** 01
- [2] Myrzakulov R, Mamyrbekova G, Nugmanova G and Lakshmanan M 2015 Integrable (2+1)-dimensional spin models with self-consistent potentials *Symmetry* **7** 3
- [3] Myrzakul A and Myrzakulov R 2017 Integrable motion of two interacting curves, spin systems and the Manakov system *Geom. Methods Mod. Phys.* **14** 07
- [4] Mamyrbekova G, Nugmanova G, Yesmakhanova K and Lakshmanan M 2014 Integrable motion of curves in self-consistent potentials: Relation to spin systems and soliton equations *Phys. Lett. A* **378**, 2118-2123
- [5] Jingsong H, Shuwei X and Kuppuswamy P 2012 Rogue Waves of the Fokas-Lenells Equation *J. Phys. Soc. Jpn.* **81** 12
- [6] Yesmakhanova K and Zhassyybayeva M 2018 On the equivalence of the Myrzakulov-LXXII equation and the Fokas-Lenells equation *Preprint rg*
- [7] Baronio F, Chen S, Grellu P, Wabnitz S and Conforti M 2015 Baseband modulation instability as the origin of rogue waves *Phys. Rev. A* **91** 033804
- [8] Myrzakulova Z, Nugmanova G, Yesmakhanova K and Myrzakulov R 2019 Dispersionless Limits of Integrable Generalized Heisenberg Ferromagnet Equations *Preprint gr-qc/1903.09195*
- [9] Myrzakulova Z and Myrzakulov R 2019 Dispersionless Limits of Some Integrable Equations *Preprint gr-qc/1902.07806*