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The differential-q-difference 2D Toda equation: bilinear form and soliton solutions

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Abstract. In this paper, the differential-q-difference 2D Toda lattice is studied. Hirota's bilinear technique is applied to it. Soliton solutions are constructed through the resulting bilinear form for the differential-q-difference 2D Toda equation.

1. Introduction

During the last decades seeking exact solutions of nonlinear evolution equations have flourished into a research area of great importance and interest [1-8]. Among the methods of finding the solutions of soliton equations such as the inverse scattering (spectral) transform method [9], the Backlund and Darboux transformation technique [10-15], etc., the bilinear transformation method of Hirota is the most effective technique and has been widely used for many soliton equations, see [16-17].

Discrete integrable system has been getting a lot of attention from the viewpoints of difference scheme and algorithm. For example, the Toda lattice equation was derived by Toda as a model of one-dimensional chain of masses connected by springs with nonlinear interaction force [18-19]. The Toda lattice is one of the completely integrable systems with multi-soliton solutions. The equation of motion of the Toda lattice is

$$\ddot{q}_n = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}}, \quad (1)$$

where q_n is the position of the n-th particle. The time-discretization of the Toda lattice was obtained by Hirota [20] as a bilinear form. The two-dimensional Toda lattice equation was introduced in works [21-24].

In recent years a great of attention has been paid to the quantum group. Some kinds of q-special functions naturally appear in the representation theory of the quantum groups. Jackson proposed various q-special functions and introduced the notion of the q-difference equation and q-integration as analogue of the ordinary differential equation [25]. The q-difference version of the cylindrical Toda lattice equation is studied in [26]. A general framework for integrable discrete systems on \mathbb{R} , in particular, containing lattice soliton systems and their q-deformed analogs are presented in [27]. Basis of solutions of the scalar equation describing the spectrum of the q-Toda chain by using auxiliary non-linear integral equations are constructed in [28]. In [29] authors are constructed the Sato theory including the Hirota bilinear equations and tau function of a new q-deformed Toda hierarchy.



The purpose of this paper is to study the differential-q-difference 2D Toda equation by using Hirota's method [30-31]. We apply the main idea of work [32] and obtain the differential-q-difference 2D Toda equation [33]. The one-q-soliton solution is found by us in [33]. In this paper, we find two-soliton and three-soliton solutions of the differential-q-difference 2D Toda lattice.

2. The differential-q-difference 2D Toda lattice

2.1. Bilinear forms

The differential-q-difference 2D Toda lattice is proposed [33] as follows:

$$\frac{d^2}{dxdt} \ln(1 + V(x, y, t)) = V(x, qy, t) + V\left(x, \frac{y}{q}, t\right) - 2V(x, y, t). \quad (2)$$

Let us introduce the dependent variable transformation as

$$V(x, y, t) = \frac{d^2}{dxdt} \ln f(x, y, t). \quad (3)$$

By substituting (3) in (2) we can get

$$\frac{d^2}{dxdt} \ln \left(1 + \frac{d^2}{dxdt} \ln f(x, y, t) \right) = \frac{d^2}{dxdt} \left(\ln f(x, qy, t) + \ln f\left(x, \frac{y}{q}, t\right) - 2 \ln f(x, y, t) \right). \quad (4)$$

We integrate equation (4) by x and t , and then we obtain

$$\ln \left(1 + \frac{d^2}{dxdt} \ln f(x, y, t) \right) = \ln f(x, qy, t) + \ln f\left(x, \frac{y}{q}, t\right) - 2 \ln f(x, y, t). \quad (5)$$

By simplification equation (5) we can get

$$f_{xt}f - f_x f_t = f(x, qy, t)f\left(x, \frac{y}{q}, t\right) - f^2. \quad (6)$$

Left part of equation (6) can be rewritten by property of Hirota's operator. It is mean

$$f_{xt}f - f_x f_t = \frac{1}{2} 2(f_{xt}f - f_x f_t) = \frac{1}{2} D_x D_t (f \cdot f). \quad (7)$$

Right part of equation (6) we can rewrite in Hirota operator as

$$f(x, qy, t)f\left(x, \frac{y}{q}, t\right) - f^2 = \left[\frac{1}{2} (e^{hyD_y} + e^{-hyD_y}) - 1 \right] (f \cdot f). \quad (8)$$

With Hirota form (7) and (8) we have bilinear form for the differential-q-difference 2D Toda lattice as

$$\left[D_x D_t - (e^{hyD_y} + e^{-hyD_y} - 2) \right] (f(x, y, t) \cdot f(x, y, t)) = 0. \quad (9)$$

2.2. Soliton solutions

In order to obtain multy soliton solutions, we make to use of finite perturbation expansion around a formal perturbation parameter ϵ as

$$f(x, y, t) = 1 + \epsilon f_1(x, y, t) + \epsilon^2 f_2(x, y, t) + \epsilon^3 f_3(x, y, t) + \dots \quad (10)$$

Substituting (10) in (9) we have

$$\begin{aligned} P[D](f(x, y, t) \cdot f(x, y, t)) &= P(D)[1 \cdot 1 + \epsilon(1 \cdot f_1 + f_1 \cdot 1) + \\ &+ \epsilon^2(1 \cdot f_2 + f_2 \cdot 1 + f_1 \cdot f_1) + \epsilon^3(1 \cdot f_3 + f_3 \cdot 1 + f_1 \cdot f_2 + f_2 \cdot f_1) + \\ &+ \epsilon^4(1 \cdot f_4 + f_4 \cdot 1 + 1 \cdot f_3 + f_3 \cdot 1 + f_2 \cdot f_2) + \dots], \end{aligned} \quad (11)$$

where $P[D] \equiv D_x D_t - (e^{hyD_y} + e^{-hyD_y} - 2)$ - polynomial of operator D . We rewrite equation (11) by collecting some powers of ϵ

$$\epsilon^0 : P[D](1 \cdot 1) = 0, \quad (12)$$

$$\epsilon^1 : P[D](1 \cdot f_1 + f_1 \cdot 1) = 0, \quad (13)$$

$$\epsilon^2 : P[D](1 \cdot f_2 + f_2 \cdot 1 + f_1 \cdot f_1) = 0, \quad (14)$$

$$\epsilon^3 : P[D](1 \cdot f_3 + f_3 \cdot 1 + f_1 \cdot f_2 + f_2 \cdot f_1) = 0, \quad (15)$$

$$\epsilon^4 : P[D](1 \cdot f_4 + f_4 \cdot 1 + f_1 \cdot f_3 + f_3 \cdot f_1 + f_2 \cdot f_2) = 0, \quad (16)$$

...

We can obtain multy soliton solutions by solving the system of equations (12)-(16).

2.2.1. One-q-soliton solutions. In order to obtain one-q-soliton solution we take (10) as

$$f(x, y, t) = 1 + f_1(x, y, t). \quad (17)$$

Then system (12)-(16) can be rewritten in next form

$$\epsilon^1 : P[D](1 \cdot f_1 + f_1 \cdot 1) = 0, \quad (18)$$

$$\epsilon^2 : P[D](1 \cdot f_2 + f_2 \cdot 1 + f_1 \cdot f_1) = 0. \quad (19)$$

In order to solve equations (18)-(19) we take

$$f_1(x, y, t) = y^\alpha e^{\beta t + \gamma x + \eta}, \quad (20)$$

where α, γ are arbitrary constant. Substituting (20) in (18)-(19) we can get next relation

$$\beta = \frac{1}{\gamma}(q^\alpha + q^{-\alpha} - 2), \quad (21)$$

which is dispersion relation. Thus, by substituting (20) in (17) and then in (3) one-q-soliton solution can obtained in the next form [33]

$$V(x, y, t) = \frac{y^\alpha \beta \gamma e^{\alpha t + \beta x + \eta}}{(1 + y^\alpha e^{\alpha t + \beta x + \eta})^2}. \quad (22)$$

2.2.2. *Two-q-soliton solutions.* Two-q-soliton solution can be constructed by taking (10) as

$$f(x, y, t) = 1 + f_1(x, y, t) + f_2(x, y, t). \quad (23)$$

Then system (12)-(16) takes next form

$$\epsilon^1 : P[D](1 \cdot f_1 + f_1 \cdot 1) = 0, \quad (24)$$

$$\epsilon^2 : P[D](1 \cdot f_2 + f_2 \cdot 1 + f_1 \cdot f_1) = 0, \quad (25)$$

$$\epsilon^3 : P[D](1 \cdot f_3 + f_3 \cdot f_1 \cdot f_2 + f_2 \cdot f_1) = 0. \quad (26)$$

Let's solve system (24)-(26). By taking starting solution as

$$f_1(x, y, t) = y^{\alpha_1} e^{\beta_1 t + \gamma_1 x + \eta_1} + y^{\alpha_2} e^{\beta_2 t + \gamma_2 x + \eta_2}, \quad (27)$$

and substituting (27) in (24)-(26) and then obtained expressions in (3) we can get the two-q-soliton solution in following form

$$V(x, y, t) = \frac{d^2}{dxdt} \ln(1 + f_1 + f_2), \quad (28)$$

where

$$f_1(x, y, t) = y^{\alpha_1} e^{\beta_1 t + \gamma_1 x + \eta_1} + y^{\alpha_2} e^{\beta_2 t + \gamma_2 x + \eta_2}, \quad (29)$$

$$f_2(x, y, t) = A_{12} y^{\alpha_1 + \alpha_2} e^{(\beta_1 + \beta_2)t + (\gamma_1 + \gamma_2)x + (\eta_1 + \eta_2)}, \quad (30)$$

with

$$A_{12} = -\frac{(\beta_1 - \beta_2)(\gamma_1 - \gamma_2) - (q^{\alpha_1 - \alpha_2} + q^{\alpha_2 - \alpha_1} - 2)}{(\beta_1 + \beta_2)(\gamma_1 + \gamma_2) - (q^{\alpha_1 + \alpha_2} + q^{-\alpha_1 - \alpha_2} - 2)}, \quad (31)$$

$$\beta_1 = \frac{1}{\gamma_1} (q^{\alpha_1} + q^{-\alpha_1} - 2) \quad \text{and} \quad \beta_2 = \frac{1}{\gamma_2} (q^{\alpha_2} + q^{-\alpha_2} - 2). \quad (32)$$

2.2.3. *Three-q-soliton solutions.* According to Hirota method in this case we take

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3. \quad (33)$$

Then system (12)-(16) takes form

$$\epsilon^1 : P[D](1 \cdot f_1 + f_1 \cdot 1) = 0, \quad (34)$$

$$\epsilon^2 : P[D](1 \cdot f_2 + f_2 \cdot 1 + f_1 \cdot f_1) = 0, \quad (35)$$

$$\epsilon^3 : P[D](1 \cdot f_3 + f_3 \cdot f_1 \cdot f_2 + f_2 \cdot f_1) = 0, \quad (36)$$

$$\epsilon^4 : P[D](1 \cdot f_4 + f_4 \cdot f_1 \cdot f_3 + f_3 \cdot f_1 + f_2 \cdot f_2) = 0. \quad (37)$$

In order to solve system (34)-(37) we take starting solution as

$$f_1 = y^{\alpha_1} e^{\beta_1 t + \gamma_1 x + \eta_1} + y^{\alpha_2} e^{\beta_2 t + \gamma_2 x + \eta_2} + y^{\alpha_3} e^{\beta_3 t + \gamma_3 x + \eta_3}. \quad (38)$$

By substituting (38) into system of equation (34)-(37) and doing some calculation we can obtain three-q-soliton solutions in next form

$$V(x, y, t) = \frac{d^2}{dxdt} \ln(1 + f_1 + f_2 + f_3), \quad (39)$$

where

$$f_1 = y^{\alpha_1} e^{\beta_1 t + \gamma_1 x + \eta_1} + y^{\alpha_2} e^{\beta_2 t + \gamma_2 x + \eta_2} + y^{\alpha_3} e^{\beta_3 t + \gamma_3 x + \eta_3},$$

$$\begin{aligned} f_2 = & A_{12} e^{(\beta_1 + \beta_2)t + (\gamma_1 + \gamma_2)x + (\eta_1 + \eta_2)} y^{\alpha_1 + \alpha_2} + \\ & + A_{13} e^{(\beta_1 + \beta_3)t + (\gamma_1 + \gamma_3)x + (\eta_1 + \eta_3)} y^{\alpha_1 + \alpha_3} + \\ & + A_{23} e^{(\beta_2 + \beta_3)t + (\gamma_2 + \gamma_3)x + (\eta_2 + \eta_3)} y^{\alpha_2 + \alpha_3}, \end{aligned}$$

$$f_3 = A_{123} y^{\alpha_1 + \alpha_2 + \alpha_3} e^{(\beta_1 + \beta_2 + \beta_3)t + (\gamma_1 + \gamma_2 + \gamma_3)x + (\eta_1 + \eta_2 + \eta_3)},$$

with

$$A_{12} = \frac{(\beta_1 - \beta_2)(\gamma_1 - \gamma_2) - (q^{\alpha_1 - \alpha_2} + q^{\alpha_2 - \alpha_1} - 2)}{(\beta_1 + \beta_2)(\gamma_1 + \gamma_2) - (q^{\alpha_1 + \alpha_2} + q^{-\alpha_1 - \alpha_2} - 2)},$$

$$A_{13} = \frac{(\beta_1 - \beta_3)(\gamma_1 - \gamma_3) - (q^{\alpha_1 - \alpha_3} + q^{\alpha_3 - \alpha_1} - 2)}{(\beta_1 + \beta_3)(\gamma_1 + \gamma_3) - (q^{\alpha_1 + \alpha_3} + q^{-\alpha_1 - \alpha_3} - 2)},$$

$$A_{23} = \frac{(\beta_2 - \beta_3)(\gamma_2 - \gamma_3) - (q^{\alpha_2 - \alpha_3} + q^{\alpha_3 - \alpha_2} - 2)}{(\beta_2 + \beta_3)(\gamma_2 + \gamma_3) - (q^{\alpha_2 + \alpha_3} + q^{-\alpha_2 - \alpha_3} - 2)},$$

$$A_{123} = A_{12} A_{13} A_{23},$$

and dispersion relations

$$\beta_1 = \frac{1}{\gamma_1} (q^{\alpha_1} + q^{-\alpha_1} - 2), \quad \beta_2 = \frac{1}{\gamma_2} (q^{\alpha_2} + q^{-\alpha_2} - 2), \quad \beta_3 = \frac{1}{\gamma_3} (q^{\alpha_3} + q^{-\alpha_3} - 2).$$

3. Conclusion

In summary, we obtain one-q-soliton solution, two-q-soliton solution, three-q-soliton solution of the differential-q-difference 2D Toda lattice via the Hirota method. Using the proposed Hirota method one can obtain other kind wave solutions of nonlinear differential-difference equations. We hope that obtained results will be useful in the further perturbative and numerical analysis of various solutions Toda lattice. Additional applications of this method to other nonlinear differential-difference systems deserve further investigation.

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