



Unifying an asymmetric bounce to the dark energy in Chern–Simons $F(R)$ gravity

Sergei D. Odintsov^{a,b,c}, Tanmoy Paul^{d,e,*}, Indrani Banerjee^f, Ratbay Myrzakulov^{c,g}, Soumitra SenGupta^h

^a ICREA, Passeig Luis Companys, 23, 08010 Barcelona, Spain

^b Institute of Space Sciences (IEEC-CSIC) C. Can Magrans s/n, 08193 Barcelona, Spain

^c Ratbay Myrzakulov Eurasian International Centre for Theoretical Physics, Nur-Sultan 010009, Kazakhstan

^d Department of Physics, Chandernagore College, Hooghly 712 136, India

^e International Laboratory for Theoretical Cosmology, TUSUR, 634050 Tomsk, Russia

^f Department of Physics and Astronomy, National Institute of Technology, Rourkela 769008, India

^g Eurasian National University, Nur-Sultan 010008, Kazakhstan

^h School of Physical Sciences, Indian Association for the Cultivation of Science, Kolkata 700032, India



ARTICLE INFO

Article history:

Received 20 May 2021

Received in revised form 25 July 2021

Accepted 29 July 2021

Keywords:

Unification of cosmological epochs

Bouncing cosmology

Dark energy epoch

Cosmological perturbation

Higher curvature gravity theory

Planck data

ABSTRACT

We propose a cosmological scenario in which the universe undergoes through a non-singular bounce, and after the bounce, it decelerates having a matter-like dominated evolution during some regime of the deceleration era, and finally at the present epoch it evolves through an accelerating stage. Our aim is to study such evolution in the context of Chern–Simons corrected $F(R)$ gravity theory and confront the model with various observational data. Using the reconstruction technique, and in addition by employing suitable boundary conditions, we determine the form of $F(R)$ for the entire possible range of the cosmic time. The form of $F(R)$ seems to unify a non-singular bounce with a dark energy epoch, in particular, from a non-singular bounce to a deceleration epoch and from a deceleration epoch to a late time acceleration era. It is important to mention that the bouncing scenario in the present context is an asymmetric bounce, in particular, the Hubble radius monotonically increases and asymptotically diverges at the late contracting era, while it seems to decrease with time at the present epoch. The decreasing behaviour of the Hubble radius ensures a late time acceleration era of the universe. Moreover, due to the aforesaid evolution of the Hubble radius, the primordial perturbation modes generate at the deep contracting era far away from the bounce when all the perturbation modes lie within the horizon. Correspondingly we calculate the scalar and tensor power spectra, and accordingly, we evaluate the primordial observable quantities like the spectral index of the scalar curvature perturbation, the tensor-to-scalar ratio, and as a result, they are found to be in agreement with the latest Planck 2018 constraints. In this regard, the Chern–Simons term seems to have considerable effects on the tensor perturbation evolution, however keeping intact the scalar part of the perturbation with that of in the case of a vacuum $F(R)$ model, and as a result, the Chern–Simons term proves to play an important role in making the observable quantities consistent with the Planck results. Furthermore the theoretical expectation of the effective equation of state parameter of the dark energy epoch is confronted with the Planck+SNe+BAO data.

© 2021 Elsevier B.V. All rights reserved.

1. Introduction

Cosmology today is largely data-driven which opens up the opportunity to construct a consistent history of the universe that explains the observations. In this light, it is intriguing that the

current observations and experiments delimits the evolution of the universe in terms of the well established laws of physics, e.g. general relativity or some modified theories of gravity, the Standard Model of particle physics, fluid dynamics etc. However, direct experimental results probing physics above the TeV scale ceases to exist which also turns out to be a major impediment towards an unambiguous understanding of the physics of the very early universe. This has given birth to diversified early universe scenarios, e.g. the *inflationary scenario* [1–7], the *bouncing universe* [8–59], the *emergent universe scenario* [60–62] etc., all of

* Corresponding author at: Department of Physics, Chandernagore College, Hooghly 712 136, India.

E-mail addresses: odintsov@ieec.uab.es (S.D. Odintsov), pul.tnmy9@gmail.com (T. Paul), banerjeein@nitrrkl.ac.in (I. Banerjee), rmyrzakulov@gmail.com (R. Myrzakulov), tpssg@iacs.res.in (S. SenGupta).

which can consistently explain the nearly scale-invariant power spectrum or the low tensor to scalar ratio observed by the Planck satellite [63].

In the present work, we take the route of bouncing scenario which comprises of an era of contraction followed by an era of expansion of the scale factor, both the epochs being connected by a non-singular bounce. Such a scenario is interesting, particularly in the absence of a successful quantum theory of gravity, as it evades the Big Bang singularity which is indeed an unavoidable feature of inflationary scenario in the realm of classical gravity when extrapolated backwards in time.

The bouncing scenario has been studied extensively in the context of scalar tensor models and other modified gravity theories [8–59], some of which are often inspired from string theory. Among various bounce models proposed so far, the matter bounce scenario (MBS) [14,22,23,45–59] earned a lot of attention as it generates an almost scale invariant primordial power spectrum and also leads to a matter dominated epoch during the late expanding phase. Moreover in the matter bounce theory, the Universe evolved from an epoch at large negative time in the contracting era where the primordial spacetime perturbations are generated deeply inside the Hubble radius, and is thus able to solve the horizon problem. Despite these successes, the matter bounce scenario hinges with some serious problems, in particular, an exact MBS characterized by a single scalar field leads to an *exactly* scale invariant power spectrum or equivalently a vanishing running of the spectral index and a tensor-to-scalar ratio of order unity, all of which are inconsistent with the latest Planck results. Such problematic issues was also confirmed from a different point of view, in particular in the context of F(R) gravity in [52,56] (for early and late time cosmology in F(R) gravity, see [64–66]). In this regard, we would like to mention that a scalar-tensor model can be equivalently mapped to F(R) gravity by a suitable conformal transformation of the spacetime metric and thus the inconsistencies of MBS with the Planck observations in both the scalar-tensor and F(R) model are well justified. Furthermore in both the matter bounce or quasi-matter bounce scenarios, the comoving Hubble radius (defined by $r_h = 1/(aH)$) monotonically increases with time and diverges to infinity at the distant future, which in turn leads to a deceleration stage of the universe at the late expanding phase and thus fails to explain the dark energy epoch of the universe as expected from SNe-Ia+BAO+H(z)+CMB observations [67–69].

Some of the above mentioned problems seem to be cured when the background bounce scenario is considered to be a quasi-matter bounce scenario where the FRW scale factor behaves as quasi-matter dominated epoch during the asymptotic time. In particular, the scalar-tensor of F(R) quasi-matter bounce model yields a *nearly* scale invariant power spectrum (in accordance with the Planck results), although the amplitudes of tensor and scalar perturbations continue to be comparable, due to which the tensor-to-scalar ratio remains to be order of unity [14]. In order to resolve the issue of the high tensor-to-scalar ratio, some suitable higher curvature gravity theories come into the picture, with success in many of the cases [56,58] (for a general review on modified gravity, see [64,65,70,71]). However in most modified gravity theories describing the bouncing cosmology, the comoving Hubble radius increases with cosmic time and leads to a deceleration era of the universe at distant future, and thus the problem of describing a dark energy epoch still persists in such bouncing models.

Motivated by this problem, in the present work, we aim to study bouncing cosmology which is also compatible with the dark energy epoch of our current universe. For this purpose, we consider the Chern–Simons (CS) corrected F(R) gravity theory, where the presence of the CS coupling induces a parity violating

term in the gravitational action. The gravitational Chern–Simons term arises in the low-energy effective action of several string inspired models [72,73] and studying its role in explaining the primordial power spectrum may provide an indirect testbed for string theory. The parity violating Chern–Simons gravity distinguishes the evolution of the two polarization modes of primordial gravitational waves [74,75], which leads to the generation of chiral gravitational waves leaving non-trivial imprints in the Cosmic Microwave Background Radiation (CMBR). Such signatures if detected in the future generation of experiments may signal the presence of the string inspired Chern–Simons gravity in the early universe. This has motivated several works in this direction, see for example [75–83]. The astrophysical implications of the gravitational Chern–Simons (GCS) term has also been explored e.g. [84]. A further and important motivation to include the CS term in the present context of F(R) gravity stems from the fact that, as mentioned earlier, vacuum F(R) bounce models generally cannot reproduce the observed tensor-to-scalar ratio in respect to the Planck data. However the addition of the Chern–Simons term in the F(R) gravity will possibly resolve this issue as the CS term does not affect the evolution of the spatially flat FRW background or the scalar perturbations, but plays a pivotal role in the evolution of the tensor perturbations. Moreover some of our authors explored the importance of the Chern–Simons F(R) model in producing a viable *inflationary* scenario compatible with the Planck results [85]. This further motivates us to explore the relevance of such a model in inducing a *bouncing* universe and subsequently unifying it with the dark energy epoch. Based on these arguments, in the present paper, we try to provide a cosmological model which unifies certain cosmological era of the universe – from a non-singular bounce to a matter dominated era and from the matter dominated to the dark energy epoch. Here we would like to mention that the unification of bounce with dark energy era has been studied earlier in [17,18,28,86], however in a different context; for instance, in [28], the Hubble radius asymptotically goes to zero at both sides of the bounce and thus the primordial perturbation modes generate near the bounce, unlike to the present work where the perturbation modes generate at the distant past far away from the bounce.

The paper is organized as follows: in Section 2, we will briefly describe the essential features of Chern–Simons F(R) gravity theory. Having set the stage, Section 3 describes the background evolution and the constraints on various model parameters, while Section 4 is reserved for studying the perturbation evolution and their confrontation with the Planck results. The paper will end with some conclusions along with some scope for future work.

2. Essential features of Chern–Simons F(R) gravity

Let us briefly recall some basic features of Chern–Simons corrected F(R) gravity, which are necessary for our presentation [74,85]. The gravitational action of F(R) gravity generalized by Chern–Simons coupling is given by,

$$S = S = \int d^4x \sqrt{-g} \frac{1}{2\kappa^2} \left[F(R) + \frac{1}{8} v(R) \tilde{R}^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \right] \quad (1)$$

where $v(R)$ is known as Chern–Simons coupling function, $\tilde{R}^{\mu\nu\alpha\beta} = \epsilon^{\gamma\delta\mu\nu} R_{\gamma\delta}^{\alpha\beta}$, κ^2 stands for $\kappa^2 = 8\pi G = \frac{1}{M_{\text{Pl}}^2}$ and also M_{Pl} is the reduced Planck mass. By using the metric formalism, we vary the action with respect to the metric tensor $g_{\mu\nu}$, and the gravitational equations read,

$$F'(R)R_{\mu\nu} - \frac{1}{2}F(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F'(R) + g_{\mu\nu} \square F'(R) = T_{\mu\nu}^{(c)}, \quad (2)$$

with

$$T_{\mu\nu}^{(c)} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left\{ \frac{1}{8} \sqrt{-g} v(R) \tilde{R}^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \right\}$$

$$= v'(R)\tilde{R}^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\left(\frac{\delta R}{\delta g^{\mu\nu}}\right) + \epsilon_{\mu}{}^{cde}\left[v_{,e}fR_{,cd}^f - 2v_{,e}R_{,cd}^e\right] \quad (3)$$

is the energy-momentum tensor contributed from the Chern-Simons term [74]. Moreover $R_{\mu\nu}$ is the Ricci tensor constructed from $g_{\mu\nu}$ and $v'(R) = \frac{dv}{dR}$. Since the present article is devoted to cosmological context, the background metric of the Universe will be assumed to be a flat Friedmann-Robertson-Walker (FRW) metric,

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] \quad (4)$$

with $a(t)$ being the scale factor of the Universe. For such metric, the non-zero components of the Riemann tensor are given by $R_{0i}{}^{0j}$ and $R_{ij}{}^{kl}$ (with i, j, k, l denote the spatial indices), in particular, the FRW metric Eq. (4) leads to the Ricci scalar and the non-zero components of Ricci tensor, Riemann tensor as,

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right), \quad R_{00} = -3\ddot{a}/a, \quad R_{ij} = (a\ddot{a} + 2\dot{a}^2)\delta_{ij}$$

$$R_{0i}{}^{0j} = -\left(\frac{\ddot{a}}{a}\right)\delta_i^j, \quad R_{ij}{}^{kl} = \frac{\dot{a}^2}{a^2}(\delta_i^k\delta_j^l - \delta_i^l\delta_j^k)$$

respectively, where an overdot represents $\frac{d}{dt}$. In effect, the energy-momentum tensor $T_{\mu\nu}^{(c)}$ identically vanishes in the background of FRW spacetime, i.e we may argue that the Chern-Simons term does not affect the background Friedmann equations, as also stressed in [74]. However as we will see later that the Chern-Simons term indeed affects the perturbation evolution over the FRW spacetime, particularly the tensor type perturbation. Hence the temporal and spatial components of Eq. (2) become,

$$0 = -\frac{F(R)}{2} + 3(H^2 + \dot{H})F'(R) - 18(4H^2\dot{H} + H\ddot{H})F''(R)$$

$$0 = \frac{F(R)}{2} - (3H^2 + \dot{H})F'(R) + 6(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \ddot{H})F''(R)$$

$$+ 36(4H\dot{H} + \ddot{H})^2F'''(R), \quad (5)$$

where $H = \dot{a}/a$ denotes the Hubble parameter of the Universe. Comparing Eq. (5) with the usual Friedmann equations, it is easy to reveal that the F(R) gravity provides an effective energy-momentum tensor with the following forms of effective energy density (ρ_{eff}) and effective pressure (p_{eff}),

$$\rho_{\text{eff}} = \frac{1}{\kappa^2}\left[-\frac{f(R)}{2} + 3(H^2 + \dot{H})f'(R) - 18(4H^2\dot{H} + H\ddot{H})f''(R)\right] \quad (6)$$

$$p_{\text{eff}} = \frac{1}{\kappa^2}\left[\frac{f(R)}{2} - (3H^2 + \dot{H})f'(R) + 6(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \ddot{H})f''(R) + 36(4H\dot{H} + \ddot{H})^2f'''(R)\right] \quad (7)$$

respectively, where $f(R)$ is the deviation of F(R) gravity from the Einstein gravity, that is $F(R) = R + f(R)$. Thus, the effective energy-momentum tensor (EMT) depends on the form of $F(R)$, as expected. Therefore, different forms of $F(R)$ will lead to different evolution of the Hubble parameter. We will use such effective EMT of F(R) gravity to realize the cosmological evolution of the Universe.

In the present context, we are interested to study a unified scenario of a non-singular bounce and dark energy epoch, in particular, from a non-singular bounce to a deceleration epoch and from a deceleration epoch to a late time acceleration era. The bouncing scenario requires a violation of energy condition(s), which we incorporate here through a modified gravity theory in the form of a higher curvature gravity model. This is motivated from the fact that the spacetime curvature becomes large during the bounce and thus it is natural to generalize the Einstein-Hilbert action by adding higher order curvature terms in the

gravitational action. Such higher curvature terms may also naturally arise from the diffeomorphism property of the action. Furthermore, with the help of higher curvature gravity theories, the unification of inflation and dark energy epochs has been studied earlier by some of our authors [87,88], thereby we hope that higher curvature theories may have a significant role also in unifying the bounce with dark energy epoch. In particular, here we consider a Chern-Simons (CS) corrected F(R) theory in a four dimensional spacetime model. The importance of Chern-Simons Lagrangian from various perspectives has been reported in [89]. Here we would like to mention that in four dimensional spacetime, a special case of Lanczos-Lovelock theory, namely the Gauss-Bonnet (GB) theory becomes a total surface term and thus has no contributions in the field equations. The scenario however becomes different in higher dimensional spacetime where the GB term affects the field equations non-trivially [89]. Coming back to our present model, i.e the Chern-Simons corrected F(R) model in four dimensional FRW spacetime, the Chern-Simons term shows no contributions in the background Friedmann equations, and thus the background evolution is entirely controlled by the vacuum F(R) term (see Eq. (5)). This is in contrast to [89] where the authors considered an AdS invariant Chern-Simons Lagrangian in *five dimensional* spacetime, which can be recovered from the Lovelock theory, and found the corresponding cosmological as well as spherically symmetric solutions. On contrary, in our present scenario (which is a four dimensional spacetime model), the effective energy density and pressure arises from the higher curvature F(R) degrees of freedom drive the background evolution. However the vacuum F(R) theory is known to predict a large tensor-to-scalar ratio in the bouncing cosmology, which is not consistent with the Planck data. Thus in order to reduce the tensor-to-scalar ratio, we consider the Chern-Simons term along with the F(R) model. Although, the Chern-Simons term does not affect the background FRW equations, it indeed modifies the perturbation evolution considerably. Moreover it has been shown earlier that the Chern-Simons term helps to reduce the tensor-to-scalar ratio in an *inflationary* spacetime [85]. This motivates us to consider the Chern-Simons corrected F(R) model to explore the unification of a non-singular bounce to the dark energy epoch, with a hope that due to the effect of the CS term, the tensor-to-scalar ratio gets reduced (and fit within the Planck constraint) compared to that of in the vacuum F(R) case.

3. Background evolution

As mentioned earlier, we are interested in getting an unified cosmological picture of a non-singular bounce to a late time dark energy epoch. In this regard, the background scale factor present in the FRW metric is considered as [86],

$$a(t) = \left[1 + a_0\left(\frac{t}{t_0}\right)^2\right]^n \exp\left[\frac{1}{(\alpha-1)}\left(\frac{t_s-t}{t_0}\right)^{1-\alpha}\right], \quad (8)$$

where a_0, n, α are positive valued dimensionless parameters, while the other ones like t_s and t_0 have the dimensions of time. The parameter t_0 is taken to scale the cosmic time in billion years, so we take $t_0 = 1\text{By}$ (the By stands for 'billion years' throughout the paper) and consequently, the scale factor becomes,

$$a(t) = [1 + a_0t^2]^n \times \exp\left[\frac{1}{(\alpha-1)}(t_s-t)^{1-\alpha}\right]$$

$$= a_1(t) \times a_2(t) \text{ (say)}. \quad (9)$$

The scale factor is taken as a product of two factors- $a_1(t)$ and $a_2(t)$ respectively, where the factor $a_2(t)$ is motivated in getting a viable dark energy epoch at late time. Actually $a(t) = a_1(t)$ is sufficient for getting a non-singular bouncing universe where

the bounce occurs at $t = 0$. However at late expanding phase of the universe, the scale factor $a_1(t)$ goes as $\sim t^{2n}$, which, in turn, does not lead to a viable dark energy model according to the Planck results. Thereby, in order to get a bounce along with a viable dark energy epoch, we consider the scale factor as of Eq. (9) where $a_1(t)$ is multiplied by $a_2(t)$. We will show that the presence of $a_2(t)$ does not harm the bouncing character of the universe, however it slightly shifts the bouncing time from $t = 0$ to a negative time and moreover the scale factor Eq. (9) leads to an asymmetric bounce scenario (as $a(t) \neq a(-t)$). In particular, due to the exponential term, the $a_2(t)$ seems to have negligible effects at negative values of t i.e during the contracting universe; however it shows considerable effects during the expanding phase, which, along with $a_1(t)$, leads to a viable dark energy epoch. We will come back to this point in details at some stage.

The scale factor of Eq. (9) immediately leads to the Hubble parameter ($H = \dot{a}/a$) and the Ricci scalar ($R(t) = 12H^2 + 6\dot{H}$) as (an overdot represents $\frac{d}{dt}$ of the respective quantity),

$$H(t) = \frac{1}{a} \frac{da}{dt} = \frac{2a_0 n t}{(1 + a_0 t^2)} + \frac{1}{(t_s - t)^\alpha} \quad (10)$$

and

$$R(t) = \frac{12a_0 n}{(1 + a_0 t^2)^2} \{1 - a_0 t^2 (1 - 4n)\} + \frac{12}{(t_s - t)^{2\alpha}} + \frac{6\alpha}{(t_s - t)^{1+\alpha}} + \frac{48a_0 n t}{(1 + a_0 t^2)(t_s - t)^\alpha} \quad (11)$$

respectively. Eq. (10) refers different types of finite time singularity at $t = t_s$ (see [90] for different types of finite time future singularity), where the singularity structure depends on the value of α . In particular,

- For $\alpha > 1$, a Type-I singularity appears at $t = t_s$, i.e the scale factor, the effective energy density (ρ_{eff}) and the effective pressure (p_{eff}) simultaneously diverge at $t = t_s$.
- For $0 < \alpha < 1$, a Type-III singularity occurs at $t = t_s$, i.e the scale factor tends to a finite value, while the ρ_{eff} and p_{eff} diverge at $t = t_s$.
- For $-1 < \alpha < 0$, a Type-II singularity appears at $t = t_s$. In this case, the scale factor and ρ_{eff} tend to a finite value, while p_{eff} diverges at $t = t_s$.
- For $\alpha < -1$ and non-integer, a Type-IV singularity appears at $t = t_s$, in which case, the scale factor, ρ_{eff} and p_{eff} tend to a finite value $t = t_s$, however the higher derivatives of the Hubble parameter diverge at the singularity point.

Therefore the finite time singularity at $t = t_s$ is almost inevitable in the present context. Thus in order to describe a singularity free universe's evolution up-to the present epoch ($\approx 13.5\text{By}$), we consider the parameter t_s to be greater than the present age of the universe, i.e $t_s > t_p \approx 13.5\text{By}$. Therefore with this condition, we may argue that the Hubble parameter of Eq. (10) describes a singularity free cosmological evolution up-to $t \gtrsim t_p$. During the cosmic time $t \gg t_p$: either the universe will hit to the finite time singularity at $t = t_s$ (predicted by the present model) or possibly some more fundamental theory will govern that regime by which the finite time singularity can be avoided.

Coming back to our present model, the Hubble horizon at distant past gets the evolution as $|1/aH| \sim |t|^{1-2n}$ which, for $n < 1/2$ (which is indeed compatible with the Planck results in regard to the observables like spectral index, tensor to scalar ratio – as we will demonstrate in Section 4), diverges to infinity at $t \rightarrow -\infty$. This indicates that the primordial perturbations generate in the deep contracting era when all the perturbation modes are within the Hubble radius. Such generation era of the primordial perturbation modes is similar to that of in the matter

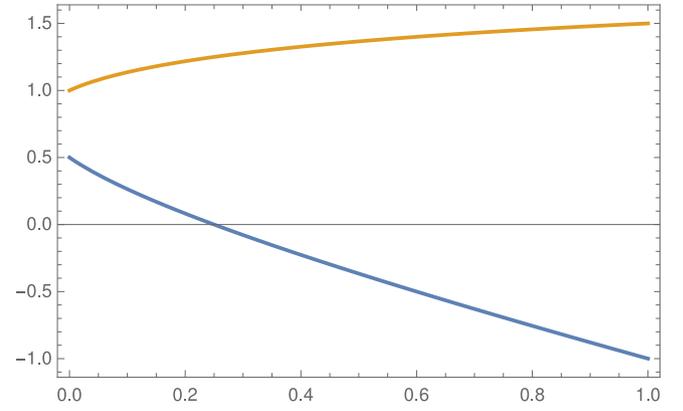


Fig. 1. Blue Curve: ρ vs. n and Yellow Curve: δ vs. n ; as per the Eq. (15). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

bounce scenario. However the matter bounce model leads to a deceleration phase at late expanding phase and thus is not consistent with dark energy model, unlike to the present bounce scenario which indeed leads to a dark energy era at the present epoch.

In regard to the background evolution at late contracting era – the scale factor, Hubble parameter and the Ricci scalar have the following expressions:

$$a(t) \approx a_0^n t^{2n}, \quad H(t) \approx \frac{2n}{t} \quad \text{and} \quad R(t) \approx -\frac{12n(1-4n)}{t^2}. \quad (12)$$

With these expressions, the F(R) gravitational Eq. (5) turns out to be,

$$\left(\frac{2}{1-4n}\right) R^2 \frac{d^2 F}{dR^2} - \left(\frac{1-2n}{1-4n}\right) R \frac{dF}{dR} + F(R) = 0, \quad (13)$$

on solving which, we get the form of F(R) at late contracting era as,

$$F(R) = R_0 \left[\left(\frac{R}{R_0}\right)^\rho + \left(\frac{R}{R_0}\right)^\delta \right] \quad (14)$$

where R_0 is a constant, and the exponents ρ, δ have the following forms (in terms of n),

$$\rho = \frac{1}{4} \left[3 - 2n - \sqrt{1 + 4n(5+n)} \right], \quad \delta = \frac{1}{4} \left[3 - 2n + \sqrt{1 + 4n(5+n)} \right] \quad (15)$$

respectively. From Eq. (14), we get the expression of $F'(R)$ which proves to be useful to investigate the stability of the primordial perturbations in the present context,

$$F'(R) = \frac{\rho}{(R/R_0)^{1-\rho}} + \delta (R/R_0)^{\delta-1} \quad (16)$$

In order to have a clear understand, we give the plots of ρ and δ with respect to n in Fig. 1 which clearly demonstrates that δ remains positive for all possible values of n , however in the case of ρ - it starts from a positive value at $n = 0$ and gets a zero crossing from positive to negative values at $n = 1/4$. Depending on the choices whether $n < 1/4$ or $n > 1/4$, we get two different physical pictures in regard to the sign of $F'(R)$. The demonstration goes as follows:

- *The case $n > 1/4$:* In this regime, ρ is negative. Therefore at late contracting era (at $t \rightarrow -\infty$) where $R \rightarrow 0$, the

Table 1
Comparison between the evolutions of $H_1(t)$ and $H_2(t)$ during $t \leq 0$.

Evolution of $H_1(t)$	Evolution of $H_2(t)$
$H_1(t) = 0$ at $t \rightarrow -\infty$	$H_2(t) = 0$ at $t \rightarrow -\infty$
$H_1(t) = 0$ at $t = 0$	$H_2(t) = 1/t_s^\alpha$ (i.e. positive) at $t = 0$
$H_1(t)$ has a maximum at $-\infty < t < 0$	$H_2(t)$ monotonically increases during $-\infty < t < 0$
$dH_1/dt \Big _{t \rightarrow -\infty} \sim 1/ t ^2$	$dH_2/dt \Big _{t \rightarrow -\infty} \sim 1/ t ^{(1+\alpha)}$

dominant term of $F'(R)$ is the first term of Eq. (16) which makes the $F'(R)$ negative at distant past. This in turn yields the instability of the primordial perturbations in the deep contracting regime when the perturbation modes generate inside the Hubble radius.

- *The case $n < 1/4$:* In this regime, $0 < \rho < 1$ and $\delta > 1$. Thereby both the terms present in the $F'(R)$ are positive and thus $F'(R) > 0$ in the deep sub-Hubble regime. This makes the primordial perturbations stable at their generation era.

Based on the above discussions, we consider $n < 1/4$ in the present work. However for $n < 1/4$, the Ricci scalar at distant past, from Eq. (12), becomes negative and thus in order to get real valued function of $F(R)$, we consider the constant R_0 (appeared in the solution of $F(R)$ in Eq. (14)) to be negative as well. Here it deserves mentioning that Eq. (12) is valid at distant past, from which it seems that the Ricci scalar is negative, only at late contracting era. In particular, by using the expression of $R(t)$ in Eq. (11) which is valid for entire cosmic time, we will show that the $R(t)$ indeed has a zero crossing from negative to positive values at the contracting era. In particular, the zero crossing (from negative to positive values) of $R(t)$ occurs before the instant of bounce, and after that zero crossing, the Ricci scalar remains positive throughout the cosmic time. Before moving to the full evolution of $R(t)$, let us demonstrate the bouncing behaviour of the $a(t)$ of Eq. (9).

3.1. Realization of a non-singular asymmetric bounce

In this section, we will show that the scale factor Eq. (9) allows a non-singular bounce at a finite time. The universe's evolution in a general bouncing cosmology consists of two eras, an era of contraction where the Hubble parameter is negative and an era of expansion having positive Hubble parameter. In particular, the bounce phenomena is defined by the conditions $H(t) = 0$ and $\dot{H} > 0$ respectively. To realize this in the present context, we borrow the expression of $H(t)$ from Eq. (10), i.e

$$H(t) = \frac{2a_0 n t}{(1 + a_0 t^2)} + \frac{1}{(t_s - t)^\alpha} \quad (17)$$

As the parameters a_0 , n and α are positive, the Hubble parameter during $t > 0$ remains positive. However during negative time, i.e for $t < 0$, the first term of Eq. (17) becomes negative while the second term remains positive, thus there is a possibility to have $H(t) = 0$ and $\dot{H} > 0$ at some negative t . Let us check it more explicitly.

For $t < 0$, we can write $t = -|t|$ and Eq. (17) can be expressed as,

$$H(t) = -\frac{2a_0 n |t|}{(1 + a_0 |t|^2)} + \frac{1}{(t_s + |t|)^\alpha} = -H_1(t) + H_2(t) \text{ (say)} \quad (18)$$

The term $H_1(t)$ starts from the value zero at $t \rightarrow -\infty$ and reaches to zero at $t = 0$, with an extremum (in particular, a maximum) at an intermediate stage of $-\infty < t < 0$. However the second term $H_2(t)$ starts from the value zero at $t \rightarrow -\infty$ and reaches to $1/t_s^\alpha$ at $t = 0$, with a monotonic increasing behaviour during

$-\infty < t < 0$. Furthermore both the $H_1(t)$ and $H_2(t)$ increase at $t \rightarrow -\infty$ with their respective rate of increasing are given by:

$$\frac{dH_1}{dt} \Big|_{t \rightarrow -\infty} \sim 1/|t|^2, \quad \frac{dH_2}{dt} \Big|_{t \rightarrow -\infty} = 1/|t|^{(1+\alpha)}$$

respectively, i.e $H_1(t)$ increases at a faster rate compared to that of $H_2(t)$ at $t \rightarrow -\infty$ for $\alpha > 1$. Here it may be mentioned that the condition $\alpha > 1$ is also related to the positivity of the Ricci scalar, as we will establish in Eq. (24), and thus $\alpha > 1$ is well justified in the present context. Such evolutions of $H_1(t)$ and $H_2(t)$ during $t \leq 0$ are given in the following Table 1, which are essential to realize a bouncing behaviour of the Hubble parameter:

Therefore at $t \rightarrow -\infty$: both the $H_1(t)$ and $H_2(t)$ start from the value zero with an increasing behaviour, however their rate of increasing are different to each other, in particular $H_1(t)$ increases at a faster rate compared to that of $H_2(t)$. Moreover at $t = 0$: the term $H_2(t)$ is positive while $H_1(t) = 0$, i.e $H_2(t)$ becomes larger than $H_1(t)$. These arguments clearly indicate that there exists a negative finite t , say $t = -\tau$ with τ being positive, for which the following statements of $H(t)$ hold true:

- $H_1(t) > H_2(t)$ or equivalently $H(t) < 0$ during $t < -\tau$,
- $H_1(t) = H_2(t)$ or equivalently $H(t) = 0$ at $t = -\tau$,
- $H_1(t) < H_2(t)$ or equivalently $H(t) > 0$ during $t > -\tau$.

Therefore, $t = -\tau$ (with $\tau > 0$) is the instant when the bounce occurs, and can be determined by the condition $H(-\tau) = 0$, i.e,

$$\frac{2a_0 n \tau}{(1 + a_0 \tau^2)} = \frac{1}{(t_s + \tau)^\alpha} \quad (19)$$

In order to see whether the above equation has a real solution (for τ), we need to put some definite values of the parameters, as Eq. (19) may not be solved in a closed form. In the following two subsections, we will determine the constraints on the model parameters from various requirements (like – to get a late time accelerating stage, the effective EoS parameter at present epoch matches with the Planck results [91] etc.). However for instance, let us choose some specific values of the parameters which indeed match with the constraints that we will determine after Eq. (34), in particular, we consider: $a_0 = 0.30$, $n = 0.185$, $t_s = 30$ and $\alpha = 4/3$, for which the algebraic Eq. (19) yields $\tau \approx 0.09$ (recall the bounce occurs at $t = -\tau$). Furthermore, we would like to mention that the scale factor remains positive at the instant of bounce, particularly the aforementioned parametric regime leads to $a(-\tau) \approx 2.62$, i.e the bounce in the present context is indeed a non-singular bounce.

In regard to the time evolution of the Ricci scalar, Eq. (12) clearly indicates that $R(t)$ behaves as $\sim -1/t^2$ at distant past, i.e the Ricci scalar starts from 0^- at $t \rightarrow -\infty$. However at the instant of bounce, the $R(t)$ becomes positive, due to the reason that the Hubble parameter vanishes and its derivative is positive at the bounce point. Therefore the Ricci scalar must undergo a zero crossing from negative to positive value before the bounce occurs. At this stage, we require that after that zero crossing, the Ricci scalar remains to be positive throughout the cosmic time, which can be realized by a more stronger condition that the Ricci scalar has to be positive during the expanding phase of

the universe, in particular,

$$R(t > -\tau) > 0, \quad (20)$$

where we may recall that $t = -\tau$ is the instant of bounce. Here we will determine the constraints on the model parameters such that the above requirement holds true. For this purpose, we borrow the expression of $R(t)$ from Eq. (11) as,

$$R(t) = \frac{12a_0n}{(1+a_0t^2)^2} \{1 - a_0t^2(1-4n)\} + \frac{12}{(t_s-t)^{2\alpha}} + \frac{6\alpha}{(t_s-t)^{1+\alpha}} + \frac{48a_0nt}{(1+a_0t^2)(t_s-t)^\alpha} \quad (21)$$

The only term (present in the above expression) due to which the Ricci scalar may acquire negative values during the *expanding phase* is given by the second term $a_0t^2(1-4n)$, in particular during $a_0t^2(1-4n) > 1$, the terms within the curly bracket provide negative contributions to $R(t)$. Thereby we consider the duration $\frac{1}{a_0(1-4n)} < t^2 < t_s^2$ where the Ricci scalar can be expressed as,

$$R(t) = \frac{12a_0n}{(1+a_0t^2)^2} \left\{ 1 + \frac{4}{t_s^\alpha}t - a_0t^2(1-4n) + \frac{4a_0}{t_s^\alpha}t^3 \right\} + \frac{12}{t_s^{2\alpha}} + \frac{6\alpha}{t_s^{1+\alpha}}. \quad (22)$$

The last two terms in the above expression contribute positive values to the Ricci scalar and thereby in order to determine the constraint(s) (on the model parameters) corresponds to the requirement $R(t > -\tau) > 0$, we can only consider the terms that are within the curly bracket of Eq. (22). Let us denote it by $\tilde{R}(t)$, i.e

$$\tilde{R}(t) = \left\{ 1 + \frac{4}{t_s^\alpha}t - a_0t^2(1-4n) + \frac{4a_0}{t_s^\alpha}t^3 \right\} \quad (23)$$

It is clear that the condition $\tilde{R} > 0$ during the expanding phase in turn leads to our requirement given in Eq. (20). One can check that for $n < 1/4$ (which is indeed our consideration to make the primordial perturbations stable, as discussed earlier after Eq. (16)), the $\tilde{R}(t)$ during the expanding phase becomes positive if the model parameters satisfy the following relations:

$$\alpha > 1 \quad \text{and} \quad \frac{\sqrt{a_0}(1-4n)}{8\sqrt{3}} < \frac{1}{t_s^\alpha} < \frac{\sqrt{a_0}(1-4n)}{4\sqrt{3}} \quad (24)$$

respectively. We determine the above constraints by finding the minimum of $\tilde{R}(t)$ (from its derivative with respect to t) and use the condition that $\tilde{R}_{min} > 0$: which is indeed a necessary condition to make a function positive valued. Thus as a whole, Eq. (24) confirms that the Ricci scalar remains positive after its zero crossing which, in fact, occurs before the instant of bounce. Hence we stick to these parameter constraints throughout the paper. Furthermore, here we would like to mention that one of the above constraints $\alpha > 1$ leads to a Type-I singularity at $t = t_s$, i.e the Hubble parameter as well as the ρ_{eff} and p_{eff} diverge at $t = t_s$, as mentioned after Eq. (11). However, since $t_s \gtrsim t_p$, the present model satisfactorily describes a singularity free cosmological evolution up-to $t \gtrsim t_p$ with $t_p \approx 13.5\text{By}$ being the present age of the universe.

3.2. Acceleration and deceleration stages of the expanding universe

The acceleration factor of the universe is given by $\ddot{a}/a = \dot{H} + H^2$ which, from Eq. (10), turns out to be,

$$\frac{\ddot{a}}{a} = \frac{2a_0n \{1 - a_0t^2(1-2n)\}}{(1+a_0t^2)^2} + \frac{\alpha}{(t_s-t)^{1+\alpha}}$$

$$+ \frac{4a_0nt}{(1+a_0t^2)(t_s-t)^\alpha} + \frac{1}{(t_s-t)^{2\alpha}}. \quad (25)$$

To understand the acceleration or deceleration stages of the universe, we need to give the plot of the above expression of \ddot{a}/a , which in turn requires the values of the model parameters present in Eq. (25). However before moving to such quantitative description and the corresponding plot of \ddot{a}/a , first we want to analyse that how much information(s) of \ddot{a}/a can obtain qualitatively from Eq. (25).

It is evident that near $t \approx 0$, $\frac{\ddot{a}}{a} \approx 2a_0n + \frac{\alpha}{t_s^{1+\alpha}} + \frac{1}{t_s^{2\alpha}}$, i.e \ddot{a} is positive. This is however expected, because $\dot{t} \approx 0$ is the bouncing regime where, due to the fact that $\dot{H} > 0$ near the bounce, the universe undergoes through an accelerating stage. However, as t increases particularly during $t^2 > \frac{1}{a_0(1-2n)}$, the first term of Eq. (25) becomes negative and hence the universe may expand through a decelerating phase. As t increases further, the terms containing $1/(t_s-t)$ starts to grow at a faster rate compared to the other terms (since α is positive) and possibly \ddot{a} becomes positive, i.e the universe may transit from a decelerating phase to an accelerating one. The transition of the universe from acceleration to deceleration or vice-versa can be described by $\ddot{a} = 0$ which, due to Eq. (25), is expressed as,

$$\frac{2a_0n \{1 - a_0t^2(1-2n)\}}{(1+a_0t^2)^2} + \frac{\alpha}{(t_s-t)^{1+\alpha}} + \frac{4a_0nt}{(1+a_0t^2)(t_s-t)^\alpha} + \frac{1}{(t_s-t)^{2\alpha}} = 0. \quad (26)$$

The above algebraic equation of t may not be solved in a closed form, however based on the above arguments, we consider two different regimes of the cosmic time to understand the transition from acceleration to deceleration (or vice-versa) of the expanding universe.

- During, $\frac{1}{a_0(1-2n)} < t^2 \ll t_s^2$: In this regime of t , Eq. (26) can be written as

$$\frac{4a_0n}{t_s^\alpha}t^3 - 2a_0n(1-2n)t^2 + 2n = 0 \quad (27)$$

where $t_s - t \approx t_s$ is considered. Here we need to investigate whether the above algebraic equation of t has a solution in the regime $\frac{1}{a_0(1-2n)} < t^2 \ll t_s^2$; thus we may consider the solution ansatz as $t_1 = \frac{1}{\sqrt{a_0(1-2n)}}(1 + \delta)$, with t_1 being the root of Eq. (27) and $\delta < 1$. Plugging back the ansatz into Eq. (27) and retaining up-to the first order in δ yields $\delta = \frac{a_0}{t_s^\alpha(a_0(1-2n))^{3/2}}$ and consequently t_1 is given by

$$t_1 = \frac{1}{\sqrt{a_0(1-2n)}} \left(1 + \frac{a_0}{t_s^\alpha(a_0(1-2n))^{3/2}} \right). \quad (28)$$

Thereby, we may argue that Eq. (27) contains a root in the aforesaid regime for at $t = t_1$. This along with the fact that the universe passes through an acceleration near the bounce, indicate that during $\frac{1}{a_0(1-2n)} < t^2 \ll t_s^2$ the universe makes a transition from the accelerating phase to a decelerating one. Later we will show that this is indeed the case when we give the plot of the full effective EoS in Fig. 4b.

- During, $\frac{1}{a_0(1-2n)} \ll t^2 < t_s^2$: In this regime of t , Eq. (26) can be expressed as,

$$\left(\frac{1}{t_s^{2\alpha}} \right) t^2 + \frac{2\alpha}{t_s} (1-2n)t - (1-2n) = 0 \quad (29)$$

where the term $\frac{1}{(t_s-t)^{2\alpha}}$ is considered to be the dominant piece compared to the others. According to the ‘‘Descartes

rule of signs”, Eq. (29) must contain one positive and one negative real root for t . The positive root (say at t_2) is obtained as,

$$t_2 = \alpha t_s^{2\alpha-1} (1-2n) \left\{ \sqrt{1 + \frac{1}{\alpha^2 (1-2n) t_s^{2\alpha-2}} - 1} \right\} \simeq \frac{t_s}{2\alpha}, \quad (30)$$

which, due to $\alpha > 1$, is less than t_s . The existence of the above root depicts that during $\frac{1}{a_0(1-2n)} \ll t^2 < t_s^2$, the universe makes another transition from the intermediate decelerating phase to an accelerating one and continues the expansion in that accelerating stage. Finally at $t = t_s$, \ddot{a} diverges, which is expected due to the occurrence of the Type-I singularity at $t = t_s$ in the present context.

Thereby Eq. (25) has two positive real roots for t during the regime $\frac{1}{a_0(1-2n)} < t^2 \ll t_s^2$ and $\frac{1}{a_0(1-2n)} \ll t^2 < t_s^2$ respectively. As a whole, the picture is following: (1) the universe undergoes through an accelerating stage near the bounce during the expanding era, (2) as t increases, particularly during $\frac{1}{a_0(1-2n)} < t^2 \ll t_s^2$, the universe gets a transition from the accelerating phase to a decelerating phase and (3) with further increases of t , the universe makes a second and final transition from the intermediate decelerating stage to an accelerating one. The second transition from $\ddot{a} < 0$ to $\ddot{a} > 0$ is identified with the *late time acceleration* epoch of the universe. Therefore we require $t_2 \lesssim t_p$, where t_2 is the instant of the second transition and recall, t_p represents the present age of the universe. In particular, Eq. (29) has the solution at,

$$t_2 = \alpha t_s^{2\alpha-1} (1-2n) \left\{ \sqrt{1 + \frac{1}{\alpha^2 (1-2n) t_s^{2\alpha-2}} - 1} \right\} \simeq \frac{t_s}{2\alpha} \quad (31)$$

where in getting the second equality, we expand the terms within the square root binomially (as $\alpha > 1$ and also $t_s \gtrsim t_p$) and retain upto the first order. Therefore the above solution of t_2 along with the requirement $t_2 \lesssim t_p$ put a constraint on the model parameters as,

$$t_s \lesssim 2\alpha t_p. \quad (32)$$

Here we need to recall that the present model predicts a Type-I singularity at $t = t_s$ (since $\alpha > 1$) and thus in order to get a singularity free cosmological evolution upto the cosmic time $t \gtrsim t_p$, the parameter t_s satisfies $t_s > t_p$. Combining this condition with Eq. (32), we get the allowed range of t_s as,

$$t_p < t_s \lesssim 2\alpha t_p. \quad (33)$$

The EoS parameter of the dark energy epoch is defined as $\omega_{\text{eff}}(t) = -1 - \frac{2\dot{H}}{3H^2}$, where $H(t)$ is shown in Eq. (10). With this expression of ω_{eff} , we confront the model with the latest Planck+SNe+BAO results which put a constraint on the dark energy EoS parameter as [91]

$$\omega_{\text{eff}}(t_p) = -0.957 \pm 0.080 \quad (34)$$

with $t_p \approx 13.5\text{By}$ being the present age of the universe. Thereby we choose the model parameters in such a way that the above constraint on $\omega_{\text{eff}}(t_p)$ holds true.

As a whole, we have four parameters in our hand: n , a_0 , t_s and α . Below is the list of their constraints that we found earlier from various requirements,

- C1: The parameter n is constrained by $n < 1/4$ in order to make the primordial perturbations stable at the deep sub-Hubble radius in the contracting era.

- C2: t_s is larger than the present age of the universe, i.e. $t_s > t_p \approx 13.5\text{By}$ to describe a singularity free evolution of the universe upto the cosmic time $t \gtrsim t_p$.
- C3: $t_s \lesssim 2\alpha t_p$ in order to have an accelerating stage of the present universe. This along with the previous condition C2 lead to $t_p < t_s \lesssim 2\alpha t_p$.
- C4: In regard to the parameters α and a_0 , they are found to be constrained as $\alpha > 1$ and $\frac{\sqrt{a_0(1-4n)}}{8\sqrt{3}} < \frac{1}{t_s} < \frac{\sqrt{a_0(1-4n)}}{4\sqrt{3}}$. These make the Ricci scalar positive after its zero crossing at the contracting era. In particular, the zero crossing (from negative to positive values) of the Ricci scalar occurs before the instant of the bounce.
- C5: $\omega_{\text{eff}}(t_p) = -0.957 \pm 0.080$, to confront the theoretical expectations of the dark energy EoS with the Planck+SNe+BAO results.

In order to better understand the above constraints on the model parameters, we give contour plots in Fig. 2 depicting the allowed regions of the parameters. The aforesaid constraints from C1 to C5 are taken care in Fig. 2 which actually demonstrates the variation of t_s vs. a_0 for three different set of values of α and n , in particular we take $(\alpha, n) = (1.2, 0.185)$, $(\alpha, n) = (\frac{4}{3}, 0.185)$ and $(\alpha, n) = (1.5, 0.185)$ respectively. The value $n = 0.185$ is motivated due to the fact that this certain value of n leads to the *central value* of the spectral index of scalar curvature perturbation, as we will show in Section 4 during the analysis the cosmological perturbation.

Keeping the parameter constraints in mind, we further give the plots of the background $H(t)$, $R(t)$ and $\omega_{\text{eff}}(t)$ (with respect to cosmic time) by using Eqs. (10) and (11), see Figs. 3 and 4. The parameter values are considered as $n = 0.185$, $\alpha = 4/3$ and $t_s = 30$ in getting the plots, which are within the green region of Fig. 2 and thus are allowed in the present context. With such values of n , α , t_s , the condition C4 gives $0.08 < a_0 < 0.33$ while the C5 leads to $a_0 > 0.12$. Combining these two, one may take $0.12 < a_0 < 0.33$ to satisfy both the C4 and C5, and consequently we consider $a_0 = 0.30$ in the plots. Actually this specific value of a_0 (along with $n = 0.185$, $\alpha = 4/3$ and $t_s = 30$) leads to $\omega_{\text{eff}}(t_p) = -0.997$, which is in agreement with [92] where the authors considered an exponential F(R) model to explain the dark energy model, (here it may be mentioned that such value of current ω_{eff} is also consistent with the holographic dark energy model, see [93]). Fig. 3a clearly depicts that the Hubble parameter becomes zero and shows an increasing behaviour with time (i.e. $\dot{H} > 0$) near $t \approx 0$, which indicates the instant of a non-singular bounce. In particular, $t \approx -0.09\text{By}$ is the time when the bounce occurs (see Fig. 3b), which is consistent with the arguments of Section 3.1 where we realized a non-singular bounce at some negative t . In regard to the evolution of the Ricci scalar, Fig. 4a demonstrates that $R(t)$ starts from 0^- at asymptotic past. The negative values of $R(t)$ during the asymptotic past is due to Eq. (12) with $n < 1/4$. However as the cosmic time increases, the Ricci scalar gets a zero crossing from negative to positive values at the contracting era. In particular, such zero crossing of $R(t)$ happens before the bounce occurs and after that zero crossing the Ricci scalar seems to be positive throughout the cosmic time. This, in fact, is in agreement with the constraint C4 as mentioned above. Here we would like to mention that both the Hubble parameter and the Ricci scalar diverge at $t = t_s = 30\text{By}$ (recall we consider $t_s = 30\text{By}$ in the plots), which in turn refers to a Type-I singularity at $t = t_s$. However as evident from the figures that the occurrence of the Type-I singularity is far away from the present age of the universe and thus we argue that the present model satisfactorily describes a singularity free cosmological evolution of the universe upto $t \gtrsim t_p$. Coming to the evolution of the effective EoS parameter, the red curve of

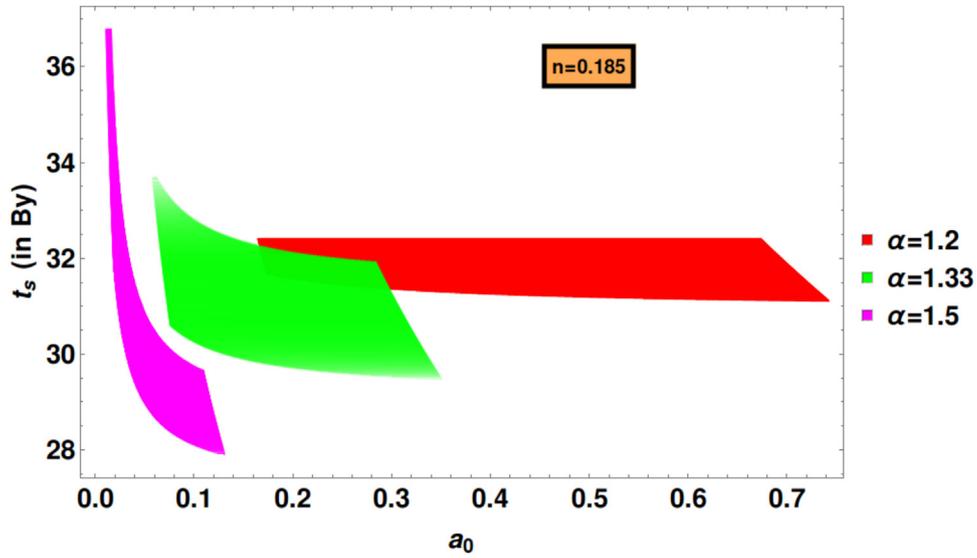


Fig. 2. The contour regions of t_s (in the unit of By.) vs. a_0 for three different set of values of α and n , in particular – $(\alpha, n) = (1.2, 0.185)$, $(\alpha, n) = (\frac{4}{3}, 0.185)$ and $(\alpha, n) = (1.5, 0.185)$ respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

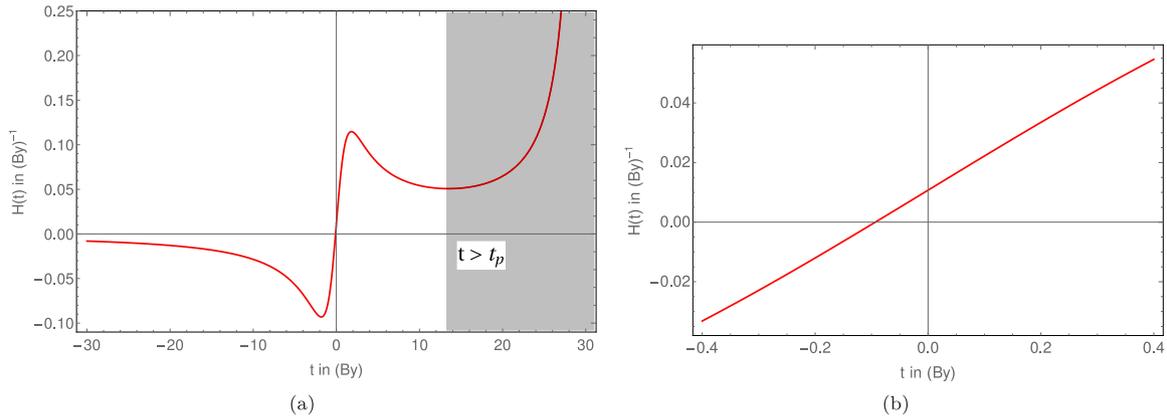


Fig. 3. The above figure depicts the time evolution of (a) the Hubble parameter $H(t)$ and (b) the zoomed-in version of $H(t)$ near the bounce. Both the plots correspond to $n = 0.185$, $\alpha = 4/3$, $t_s = 30$ and $a_0 = 0.30$. Moreover the shaded region in the left plot corresponds to the cosmic time larger than the present age of the universe, i.e $t > t_p \approx 13.5\text{By}$.

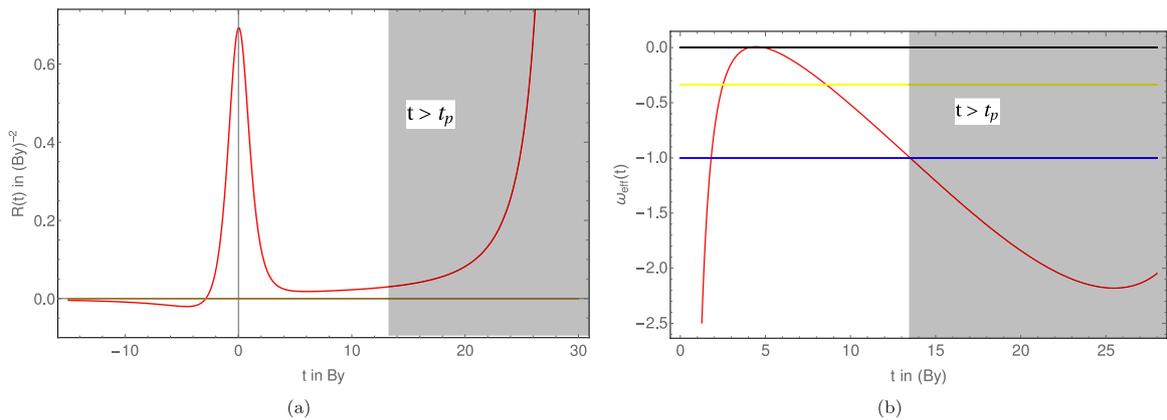


Fig. 4. The above figure depicts the time evolution of (a) the Ricci scalar $R(t)$ and (b) the EoS parameter $\omega_{\text{eff}}(t)$. Both the plots correspond to $n = 0.185$, $\alpha = 4/3$, $t_s = 30$ and $a_0 = 0.30$. Moreover the shaded region in the plots correspond to the cosmic time larger than the present age of the universe, i.e $t > t_p \approx 13.5\text{By}$. In the right plot, the black, yellow and blue curve correspond to $\omega_{\text{eff}} = 0, -\frac{1}{3}, -0.997$ respectively. The curve $\omega_{\text{eff}} = 0$ helps to investigate whether the present model exhibits a matter-like dominated epoch during some regime of cosmic time, the curve $\omega_{\text{eff}} = -1/3$ is to demonstrate the accelerating or decelerating stages of the universe and the $\omega_{\text{eff}} = -0.997$ curve reveals that the effective EoS of the present model matches with the Planck results at the present epoch i.e at $t = 13.5\text{By}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

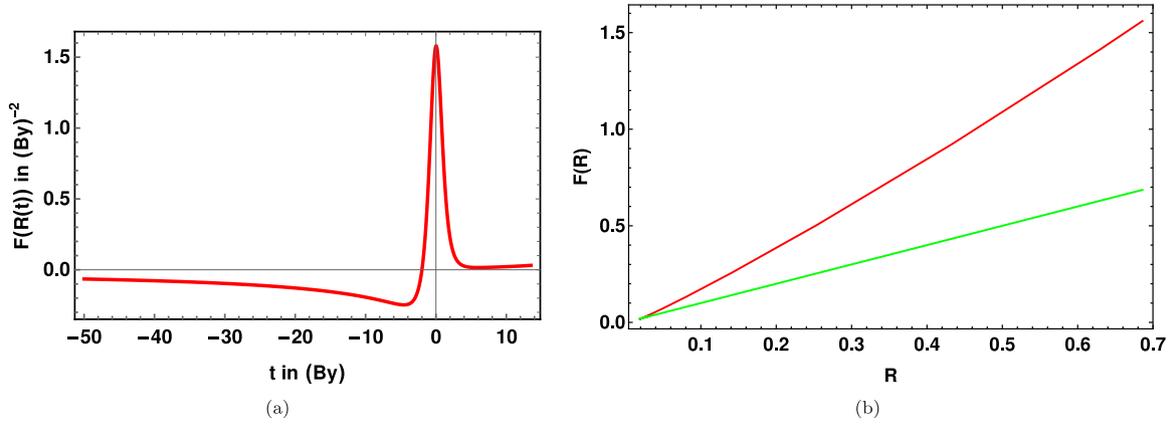


Fig. 5. The above figure depicts- (a) the time evolution of $F(R(t))$ and (b) the $F(R)$ vs. R . The initial condition during this numerical analysis is considered to be $F(R(t)) = (R(t)/R_0)^\rho + (R(t)/R_0)^\delta$ with $R_0 = -1(\text{By})^{-2}$, recall the parameter R_0 is negative, as discussed after Eq. (16). The right plot is obtained from the parametric plot of $F(R(t))$ and $R(t)$. Both the plots correspond to $n = 0.185$, $\alpha = 4/3$, $t_s = 30$ and $a_0 = 0.30$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 4b represents the $w_{\text{eff}}(t)$ for the present model while the yellow one of the same is for the constant value $-\frac{1}{3}$ (we will keep the yellow graph to investigate the accelerating or decelerating era of the Universe). Fig. 4b clearly demonstrates that near the bounce i.e. near $t \approx 0$, the EoS parameter diverges from the negative side, however this is expected because at the bounce the Hubble parameter itself becomes zero and in turn makes the $w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2} \rightarrow -\infty$. Then after the bounce, w_{eff} crosses the value $-\frac{1}{3}$ leading to a transition from a bounce to a decelerating phase of the Universe, and during the deceleration epoch, the EoS parameter becomes zero during an epoch indicating a matter-like dominated Universe. The deceleration era continues till $t \approx 8.5\text{By}$ when the EoS parameter again crosses the value $-\frac{1}{3}$ and thus the Universe transits from a stage of deceleration to a stage of acceleration which, in turn, is identified with the present dark energy epoch. Therefore, the present model may provide an unified scenario of certain cosmological epochs from bounce to late-time acceleration followed by a matter-like dominated epoch in the intermediate regime. Moreover, the present value of the dark energy EoS parameter seems to be $\omega_{\text{eff}}(t_p) \simeq -0.997$ from the Fig. 4b (the blue curve is for the constant value -0.997 , that coincides with the red one at $t = t_p \approx 13.5\text{By}$), which is indeed consistent with the Planck-2018+SNe+BAO results [91].

The remaining task is to determine the form of $F(R)$ from the gravitational Eq. (5), which leads to such background cosmological evolution of the universe. In accordance the form of $H(t)$ in Eq. (10), the $F(R)$ gravitational equation may not be solved analytically and thus we will solve it numerically. For this purpose, Eq. (5) is re-written in terms of cosmic time as,

$$H(t)\dot{R}(t)\frac{d^2F}{dt^2} - \{\dot{H}\dot{R} + H^2\dot{R} + H\ddot{R}\}\frac{dF}{dt} + \frac{1}{6}\dot{R}^2(t)F(t) = 0, \quad (35)$$

where $F(t) = F(R(t))$. Using the form of $H(t)$ from Eq. (10) along with the expression $R(t) = 12H^2 + 6\dot{H}$, we numerically solve Eq. (35) for $F = F(t)$ during a wide range of cosmic time, see Fig. 5a. The initial condition of this numerical analysis is considered to be $F(R(t)) = (R(t)/R_0)^\rho + (R(t)/R_0)^\delta$, i.e the analytic form of $F(R(t))$ during the late contracting era is taken as the initial condition of the numerical solution of Eq. (35). Such numerical solution of $F = F(t)$ along with the expression of $R = R(t)$ (see Eq. (11)) lead to the form of $F(R)$ (by using “parametric plot” in MATHEMETICA), see Fig. 5b. Actually the form of $F(R)$ is demonstrated by the red curve, while the green one represents the Einstein gravity. Fig. 5b clearly depicts that the $F(R)$ in the present context matches with the Einstein gravity

as the Ricci scalar approaches to the present value, while the $F(R)$ seems to deviate from the usual Einstein gravity, when the scalar curvature takes larger and larger values. It is evident that $F'(R)$ is positive, which, as we will show in Section 4, is connected to the stability of the primordial perturbation near the bounce; moreover $F'(R) > 0$ also indicates that the model is free from the Ostrogradsky instability.

Here we would like to mention that in regard to the background evolution, the effective EoS parameter at distant past is given by $\omega_{\text{eff}} = -1 + \frac{1}{3n}$ which is indeed less than unity due to the aforementioned range of n that makes the observable quantities viable with the Planck results. In effect, the anisotropic energy density grows as a^{-6} during the contracting era and thus the background evolution in the contracting stage becomes unstable to the growth of anisotropies, which is known as BKL instability [94]. Thereby like many other bounce models, the present model is suffered from the BKL instability. However on contrary, in the ekpyrotic bounce scenario, the bouncer field decays more faster than the anisotropic energy density, by which the BKL instability can be resolved [22,24,95,96]. Therefore it may be an interesting avenue to unify an ekpyrotic bounce with a dark energy epoch in some appropriate modified theories of gravity, which we expect to address in future work. Moreover the Chern-Simons $F(R)$ gravity can be extended by axion DM, as considered in [97,98] in the context of inflationary background spacetime. The possible effects of DM may also be included in the current unified scenario of bounce and dark energy epochs, which will be considered elsewhere.

4. Cosmological perturbation

In this section, we perform the cosmological perturbation of the background spacetime in the present context and consequently determine various observable quantities like scalar spectral index, tensor to scalar ratio etc. In a bouncing scenario, the Hubble parameter vanishes and thus the comoving Hubble radius (defined by $r_h = 1/(aH)$) diverges at the instant of bounce. However the asymptotic behaviour of the Hubble radius may be different in different bounce models, which qualitatively differ various bounce model(s) in regard to the generation era of the primordial perturbation. In some bounce models (for example, the matter bounce scenario), the Hubble radius monotonically increases with time at late contracting era and asymptotically diverges at $t \rightarrow -\infty$, due to which the perturbation modes in such bounce models generate at deep contracting era where all

the perturbation modes lie within the horizon. On other hand, there may exist some bounce models (see [28]) where the Hubble radius decreases with time and asymptotically goes to zero at $t \rightarrow -\infty$; in such scenario, the primordial perturbation generate near the bounce where the Hubble horizon has an infinite size to contain all the perturbation modes within it.

In the present context, the scale factor at late contracting era behaves as $a(t) \sim |t|^{2n}$ and thus the Hubble radius goes as $r_h \sim |t|^{1-2n}$. Here we need to recall that $n < 1/4$ (see the aforementioned condition C1), due to which the Hubble radius diverges at $t \rightarrow -\infty$. This makes the generation era of the primordial perturbation at the early contracting stage within the deep sub-Hubble radius. We would like to mention that the scale factor is asymmetric with respect to the bounce point, in particular, unlike to the fact that the Hubble radius diverges at $t \rightarrow -\infty$, it monotonically decreases with cosmic time at the late stage of the expanding era. Actually, such decreasing behaviour of the Hubble radius ensures a dark energy epoch of the present universe.

4.1. Scalar perturbations

The scalar perturbation of FRW background metric is defined as follows,

$$ds^2 = -(1 + 2\Psi)dt^2 + a(t)^2(1 - 2\Psi)\delta_{ij}dx^i dx^j, \quad (36)$$

where $\Psi(t, \vec{x})$ denotes the scalar perturbation. Here we are working in the comoving gauge, in which case the curvature perturbation $(\mathcal{R}(t, \vec{x}))$ becomes identical to the $\Psi(t, \vec{x})$ and thus we can proceed with the perturbation variable $\Psi(t, \vec{x})$. The perturbed action up to Ψ^2 order is [74,99,100],

$$\delta S_\Psi = \int dt d^3\vec{x} a(t) z(t)^2 \left[\dot{\Psi}^2 - \frac{1}{a^2} (\partial_i \Psi)^2 \right], \quad (37)$$

where $z^2(t)$, in the present context of Chern-Simons corrected F(R) gravity theory, has the following expression [74],

$$z^2(t) = \frac{a^2(t)}{\kappa^2 \left(H(t) + \frac{1}{2F'(R)} \frac{dF'(R)}{dt} \right)^2} \left\{ \frac{3}{2F'(R)} \left(\frac{dF'(R)}{dt} \right)^2 \right\}. \quad (38)$$

As it can be observed from the above form of $z(t)$, the Chern-Simons (CS) term does not affect the scalar perturbation; due to the reason that it is not possible to form a scalar energy momentum tensor nor vector or symmetric tensor, which contains the Levi-Civita tensor and scalar derivatives only [74]. However the CS term indeed affects the tensor type perturbation, which we will demonstrate in the next section. Coming back to Eq. (37), it is evident that the speed of the scalar perturbation waves (or the sound speed) is unity, which indicates the absence of the superluminal modes from the present model, or equivalently we may argue that the model is free from the gradient instability. The stability of the scalar perturbations is ensured from the condition $z^2(t) > 0$ which along with Eq. (38) leads to $F'(R) > 0$. We have the expression of $F'(R)$ at contracting stage in Eq. (16), which is indeed positive due to $n < 1/4$ – this indicates the stability of the scalar perturbation during the contracting stage at where the perturbation modes generate. Moreover the numerical solution of $F(R)$ in Fig. 5 clearly indicates that the $F'(R)$ is positive during wide range of cosmic time, which makes the scalar perturbations stable in the present context.

Here we are interested to determine various observable quantities like the scalar spectral index and tensor-to-scalar ratio which are eventually evaluated at the time of horizon crossing of the large scale modes. Due to the fact that the Hubble radius diverges at $t \rightarrow -\infty$, the large scale modes cross the horizon

at deep contracting stage (later we will explicitly calculate the horizon crossing instant of the large scale modes) where the Hubble parameter and $F(R)$ follow Eqs. (12) and (14) respectively. Thereby using such expressions of $H(t)$ and $F(R)$, we determine various terms present in the expression of $z(t)$ (see Eq. (38)) as,

$$\frac{a(t)}{\left(H(t) + \frac{1}{2F'(R)} \frac{dF'(R)}{dt} \right)} = \frac{a_0^n (12n(4n-1))^{n+1/2}}{(-R)^{n+1/2}} \times \left\{ 2n + \frac{(1-\rho) \left[1 + \frac{\delta(\delta-1)}{\rho(\rho-1)} \left(\frac{R}{R_0} \right)^{\delta-\rho} \right]}{\left[1 + \frac{\delta}{\rho} \left(\frac{R}{R_0} \right)^{\delta-\rho} \right]} \right\}^{-1},$$

and

$$\frac{3}{2F'(R)} \left(\frac{dF'(R)}{dt} \right)^2 = |R_0| \left(\frac{R}{R_0} \right)^\rho \times \left\{ \frac{\rho(1-\rho)^2 \left[1 + \frac{\delta(\delta-1)}{\rho(\rho-1)} \left(\frac{R}{R_0} \right)^{\delta-\rho} \right]^2}{2n(1-4n) \left[1 + \frac{\delta}{\rho} \left(\frac{R}{R_0} \right)^{\delta-\rho} \right]} \right\}.$$

where, recall, $R(t)$ is negative at late contracting era due to $n < 1/4$ and thus $(-R)^{n+1/2}$ present in the previous expression is real. Consequently $z(t)$ takes the following form,

$$z(t) = \left\{ \frac{a_0^n |R_0|^n [12n(1-4n)]^n}{\kappa (R/R_0)^{n+1/2-\rho/2}} \right\} \times \left(\frac{P(R)}{Q(R)} \right) \quad (39)$$

where $P(R)$ and $Q(R)$ are defined as follows,

$$P(R) = \left\{ \frac{\sqrt{\rho}(1-\rho) \left[1 + \frac{\delta(\delta-1)}{\rho(\rho-1)} \left(\frac{R}{R_0} \right)^{\delta-\rho} \right]}{\left[1 + \frac{\delta}{\rho} \left(\frac{R}{R_0} \right)^{\delta-\rho} \right]^{1/2}} \right\}, \quad (40)$$

and

$$Q(R) = \left\{ 2n + \frac{(1-\rho) \left[1 + \frac{\delta(\delta-1)}{\rho(\rho-1)} \left(\frac{R}{R_0} \right)^{\delta-\rho} \right]}{\left[1 + \frac{\delta}{\rho} \left(\frac{R}{R_0} \right)^{\delta-\rho} \right]} \right\}. \quad (41)$$

Eq. (37) clearly indicates that $\Psi(t, \vec{x})$ is not canonically normalized and to this end we introduce the well-known Mukhanov-Sasaki variable as $v = z\mathcal{R}$ ($= z\Psi$ as we are working in the comoving gauge). The corresponding Fourier mode of the Mukhanov-Sasaki variable satisfies,

$$\frac{d^2 v_k}{d\eta^2} + \left(k^2 - \frac{1}{z(\eta)} \frac{d^2 z}{d\eta^2} \right) v_k(\eta) = 0, \quad (42)$$

where $\eta = \int dt/a(t)$ is the conformal time and $v_k(\eta)$ is the Fourier transformed variable of $v(t, \vec{x})$ for the k th mode. Eq. (42) may not be solved analytically in general, as $z(\eta)$ depends on the background evolution. However the equation can be solved at late contracting era, as we now demonstrate. The conformal time η is related to the cosmic time as, $\eta(t) = \frac{1}{a_0^n (1-2n)} t^{1-2n}$ for $n \neq 1/2$; since the parameter n is constrained to be less than $1/4$, we can safely work with this expression of $\eta(t)$. Using the $\eta = \eta(t)$, we can express the Ricci scalar as a function of the conformal time,

$$R(\eta) = \frac{1}{\eta^{2/(1-2n)}} \left\{ \frac{12n(1-4n)}{|R_0| [a_0^n (1-2n)]^{2/(1-2n)}} \right\} \propto \frac{1}{\eta^{2/(1-2n)}} \quad (43)$$

Having this in mind, we can express $z(\eta)$ from Eq. (39) in terms of η as follows,

$$z(\eta) \propto \left(\frac{P(\eta)}{Q(\eta)} \right) \times \eta^{\frac{2n+1-\rho}{1-2n}} \quad (44)$$

where $P(\eta) = P(R(\eta))$ and $Q(\eta) = Q(R(\eta))$, with $P(R)$, $Q(R)$ are given in Eqs. (40), (41) respectively. The above expression of $z = z(\eta)$ yields the expression of $\frac{1}{z} \frac{d^2 z}{d\eta^2}$, which is essential for the Mukhanov–Sasaki equation,

$$\frac{1}{z} \frac{d^2 z}{d\eta^2} = \frac{\xi(\xi - 1)}{\eta^2} \times \left[1 + \frac{2\delta(\delta - \rho) \{ (1 - \rho)^2 + 2n(1 + \rho - \delta) \}}{\rho(1 - \rho)(4n - \rho)(2n + 1 - \rho)} \left(\frac{R}{R_0} \right)^{\delta - \rho} \right] \quad (45)$$

with $\xi = \frac{(2n+1-\rho)}{(1-2n)}$. Recall $\rho = \frac{1}{4} [3 - 2n - \sqrt{1 + 4n(5+n)}]$ and $\delta = \frac{1}{4} [3 - 2n + \sqrt{1 + 4n(5+n)}]$, which clearly indicate that $\delta - \rho$ is a positive quantity. Thus the term containing $(R/R_0)^{\delta - \rho}$ within the parenthesis in Eq. (45) can be safely considered to be small during the late contracting era as $R \rightarrow 0$ at $t \rightarrow -\infty$. As a result, $\frac{1}{z} \frac{d^2 z}{d\eta^2}$ becomes proportional to $1/\eta^2$ i.e., $\frac{1}{z} \frac{d^2 z}{d\eta^2} = \sigma/\eta^2$ with,

$$\sigma = \xi(\xi - 1) \left[1 + \frac{2\delta(\delta - \rho) \{ (1 - \rho)^2 + 2n(1 + \rho - \delta) \}}{\rho(1 - \rho)(4n - \rho)(2n + 1 - \rho)} \left(\frac{R}{R_0} \right)^{\delta - \rho} \right]. \quad (46)$$

which is approximately a constant in the era, when the primordial perturbation modes generate deep inside the Hubble radius. In effect along with the fact that $c_s^2 = 1$, the Mukhanov Eq. (42) can be solved as follows,

$$v(k, \eta) = \frac{\sqrt{\pi|\eta|}}{2} [c_1(k)H_\omega^{(1)}(k|\eta|) + c_2(k)H_\omega^{(2)}(k|\eta|)], \quad (47)$$

with $\omega = \sqrt{\sigma + \frac{1}{4}}$ and c_1 and c_2 are integration constants which can be determined from the initial Bunch–Davies condition. The consideration of Bunch–Davies vacuum initially, leads to these integration constants as $c_1 = 0$ and $c_2 = 1$ respectively. Using the solution of $v_k(\eta)$, we immediately evaluate the power spectrum (defined for the Bunch–Davies vacuum state) corresponding to the k th scalar perturbation mode, which is defined as follows,

$$P_\psi(k, \eta) = \frac{k^3}{2\pi^2} |\Psi_k(\eta)|^2 = \frac{k^3}{2\pi^2} \left| \frac{\sqrt{\pi|\eta|}}{2z(\eta)} H_\omega^{(2)}(k|\eta|) \right|^2 \quad (48)$$

The horizon crossing of the mode k is given by $k = |aH|$ which, due to Eq. (12), take the following form,

$$k = a_0^n \left| \frac{2n}{t_h^{1-2n}} \right| = \frac{1}{|\eta_h|} \left(\frac{2n}{1-2n} \right), \quad (49)$$

where the suffix ‘h’ denotes the horizon crossing instant and in the second expression, we use the aforementioned relation of $\eta = \eta(t)$. Eq. (49) leads to the horizon crossing time for the large scale modes, in particular for $k = 0.002 \text{ Mpc}^{-1}$ (around which we will determine the observable quantities), as

$$|\eta_h| = \left(\frac{2n}{1-2n} \right) \left(\frac{1}{0.002} \right) \text{Mpc} \approx 10^{17} \text{s} \approx -10 \text{By}. \quad (50)$$

Therefore, the large scale modes crosses the horizon at $\sim -10 \text{By}$, i.e at deep contracting era. This justifies our consideration to use the late contracting expressions of $H(t)$ and $F(R)$ (from Eqs. (12) and (14)) to evaluate the observable quantities. At the horizon crossing of the large scale modes, the Ricci scalar acquires $|R| \sim 10^{-3} \text{By}^{-2}$. Moreover, in the present context, the sub-Hubble and

super-Hubble scale from Eq. (49) are given by,

$$k|\eta| > \frac{2n}{(1-2n)} \quad ; \quad \text{sub Hubble scale} \\ k|\eta| < \frac{2n}{(1-2n)} \quad ; \quad \text{super Hubble scale} \quad (51)$$

respectively. Here we would like to mention that the factor $2n/(1-2n)$ is less than unity for $n < 1/4$ (see the condition C1), and thus the superhorizon limit can be equivalently expressed as $k|\eta| \ll 1$. In such superhorizon limit, the scalar power spectrum of Eq. (48) becomes,

$$P_\psi(k, \eta) = \left[\frac{1}{2\pi} \frac{1}{z|\eta|} \frac{\Gamma(\omega)}{\Gamma(3/2)} \right]^2 \left(\frac{k|\eta|}{2} \right)^{3-2\omega}. \quad (52)$$

By using Eq. (Eq. (52)), we can determine the spectral index of the primordial curvature perturbations (denoted by n_s). Before proceeding to calculate n_s , we will consider first the tensor power spectrum, which is necessary for evaluating the tensor-to-scalar ratio.

4.2. Tensor perturbation

In this section we consider the tensor perturbation on the FRW metric background which is defined as follows,

$$ds^2 = -dt^2 + a(t)^2 (\delta_{ij} + h_{ij}) dx^i dx^j, \quad (53)$$

where $h_{ij}(t, \vec{x})$ is the tensor perturbation. The variable $h_{ij}(t, \vec{x})$ is itself a gauge invariant quantity, and the tensor perturbed action up to quadratic order is given by [74,99,100],

$$\delta S_h = \sum_{L,R} \int dt d^3\vec{x} a(t) z_{L,R}(t)^2 \left[\dot{h}_{ij} \dot{h}^{ij} - \frac{1}{a^2} (\partial_l h_{ij})^2 \right], \quad (54)$$

where the suffix ‘L’ and ‘R’ characterize the polarization of the tensor perturbation, in particular the left and right polarization states respectively. The factor $z_{L,R}(t)$, in the Chern–Simons F(R) theory i.e the case of the present context, has the following form [74],

$$z_L^2(t) = \left(\frac{1}{\kappa} \right) a^2(t) F'(R) \left\{ 1 - \frac{2\dot{\nu}(R)k}{aF'(R)} \right\}, \\ z_R^2(t) = \left(\frac{1}{\kappa} \right) a^2(t) F'(R) \left\{ 1 + \frac{2\dot{\nu}(R)k}{aF'(R)} \right\}, \quad (55)$$

with $\nu(R)$ being the CS coupling function (see Eq. (1)) and an overdot denotes $\frac{d}{dt}$. It may be observed that the CS term has considerable effects on the tensor perturbed action or equivalently on the dynamical evolution of the tensor perturbation variable. In particular, due to the presence of $\nu(R)$, the left and right polarization modes of gravity waves evolve differently, unlike to the case of vacuum F(R) model where both the tensor polarization get similar evolution. Such difference of the tensor perturbation evolution between the CS corrected F(R) and the vacuum F(R) theory reflect on the primordial observable quantity, particularly on the tensor to scalar ratio, as we will demonstrate at some stage.

Eq. (54) depicts that the speed of the tensor perturbation propagation is $c_T^2 = 1$ for the polarization states. Here we would like to mention that the unit speed of the gravitational waves is consistent with the event GW170817, according to which the gravitational waves have same speed with the electromagnetic waves, i.e unity in natural units. In order to evaluate $z_{L,R}(t)$, we consider $\nu(R)$ (having the mass dimension $[-2]$) to be a power law form of the Ricci scalar, i.e

$$\nu(R) = \frac{1}{|R_0| (m+1)} \left(\frac{R}{R_0} \right)^{m+1}, \quad (56)$$

with m being a parameter. As demonstrated earlier, the large scale modes cross the Hubble horizon during the deep contracting era (in particular, $\sim -10^{17}$ s), due to which we consider the Hubble parameter and $F(R)$ from Eqs. (12) and (14) respectively. In effect and in conjunction with the above form of $\nu(R)$, we determine various terms present in $z_{L,R}(t)$ in Eq. (55):

$$a(t)\sqrt{F'(R)} = \frac{a_0^n [12n(1-4n)]^n}{|R_0|^n (R/R_0)^{n+1/2-\rho/2}} \left\{ 1 + \frac{\delta}{\rho} \left(\frac{R}{R_0} \right)^{\delta-\rho} \right\}^{1/2} \quad (57)$$

and

$$\frac{2\dot{\nu}(R)k}{aF'(R)} = \frac{8n}{\rho [12n(1-4n)]} \left(\frac{R}{R_0} \right)^{3-\rho+m} \quad (58)$$

respectively. To derive the above expression, the mode-momentum k is evaluated from its horizon crossing condition as,

$$k = |aH| = \frac{2na_0^n |R_0|^{-n+1/2}}{[12n(1-4n)]^{-n+1/2}} \left(\frac{R}{R_0} \right)^{-n+1/2}. \quad (59)$$

Eqs. (57) and (58) lead to the $z_{L,R}$ from Eq. (55) as,

$$z_{L,R} = \left(\frac{a_0^n [12n(1-4n)]^n}{|R_0|^n (R/R_0)^{n+1/2-\rho/2}} \right) \times \left\{ 1 + \frac{\delta}{2\rho} \left(\frac{R}{R_0} \right)^{\delta-\rho} \mp \frac{4n}{\rho [12n(1-4n)]} \left(\frac{R}{R_0} \right)^{3-\rho+m} \right\}, \quad (60)$$

where we consider the leading order terms of (R/R_0) as $\frac{R}{R_0} < 1$ during the horizon crossing of the primordial perturbation modes. For convenience, we parametrize $m = \rho - 3 + (\delta - \rho)(1 - g)$ in respect to a new parameter g . With such parametrization, Eq. (60) becomes,

$$z_{L,R} = \left(\frac{a_0^n [12n(1-4n)]^n}{|R_0|^n (R/R_0)^{n+1/2-\rho/2}} \right) \times \left\{ 1 + \frac{\delta}{2\rho} \left(\frac{R}{R_0} \right)^{\delta-\rho} \mp \frac{4n}{\rho [12n(1-4n)]} \left(\frac{R}{R_0} \right)^{(\delta-\rho)(1-g)} \right\}. \quad (61)$$

The parameter g reflects the possible effects of the CS coupling function in the above expression of $z_{L,R}$. In the vacuum F(R) theory, $z_{(F)} = \left(\frac{a_0^n [12n(1-4n)]^n}{|R_0|^n (R/R_0)^{n+1/2-\rho/2}} \right) \left\{ 1 + \frac{\delta}{2\rho} \left(\frac{R}{R_0} \right)^{\delta-\rho} \right\}$ which, by comparing with Eq. (61), clearly indicates that the difference between the CS corrected F(R) and the vacuum F(R) theory is controlled by g . Thereby it seems that the parameter g plays a crucial role in the present context and thus we need to scan it carefully before proceeding further. Depending on various values of g , below we give a list of $z_{L,R}$ in the leading order of R/R_0 ,

$$z_{L,R} = \left(\frac{a_0^n [12n(1-4n)]^n}{|R_0|^n (R/R_0)^{n+1/2-\rho/2}} \right) \times \left\{ 1 + \frac{\delta}{2\rho} \left(\frac{R}{R_0} \right)^{\delta-\rho} \right\}; \quad \text{for } g < 0 \quad (62)$$

$$z_{L,R} = \left(\frac{a_0^n [12n(1-4n)]^n}{|R_0|^n (R/R_0)^{n+1/2-\rho/2}} \right) \times \left\{ 1 + \left(\frac{\delta}{2\rho} \mp \frac{4n}{\rho [12n(1-4n)]} \right) \left(\frac{R}{R_0} \right)^{\delta-\rho} \right\}; \quad \text{for } g = 0 \quad (63)$$

$$z_{L,R} = \left(\frac{a_0^n [12n(1-4n)]^n}{|R_0|^n (R/R_0)^{n+1/2-\rho/2}} \right) \times \left\{ 1 \mp \frac{4n}{\rho [12n(1-4n)]} \left(\frac{R}{R_0} \right)^{(\delta-\rho)(1-g)} \right\}; \quad \text{for } 0 < g < 1 \quad (64)$$

Now we need to investigate qualitatively that which of the above expressions of $z_{L,R}$ will be appropriate for evaluating the observable quantities in the present context. For this purpose, we may

recall that the vacuum F(R) theory is not consistent with the Planck results of primordial observable quantities in the background of a non-singular bounce where $a(t) \sim t^{2n}$ during the early contracting era [52,56]. In particular, the scalar and tensor perturbation amplitudes in vacuum F(R) bounce scenario become comparable to each other and thus the tensor-to-scalar ratio comes as order of unity, which is not compatible with the Planck constraint. Thereby to get a viable bounce scenario, either the tensor perturbation amplitude needs to be suppressed or the scalar perturbation amplitude needs to be enhanced in comparison to the vacuum F(R) theory so that the tensor-to-scalar ratio becomes less than unity and comes within the Planck constraint. In this regard, comparing the above three expressions of $z_{L,R}$ with $z_{(F)}$, we may argue that in the present context of CS corrected F(R) theory, the $z_{L,R}$ and consequently the tensor perturbation evolution get considerably different than the vacuum F(R) case, when the parameter g lies within $0 < g < 1$ (i.e Eq. (64)); thus for $0 < g < 1$, there is a possibility to get viable observable quantities in the CS corrected F(R) model. Based on these arguments, we consider $0 < g < 1$, in which case the $z_{L,R}$ is given by Eq. (64).

Similar to scalar perturbation, the Mukhanov-Sasaki variable for tensor perturbation is defined as $v_\lambda = z_\lambda h_\lambda$ (with $\lambda = L, R$) which, upon performing the Fourier transformation, satisfies the following equation,

$$\frac{d^2 v_\lambda(k, \eta)}{d\eta^2} + \left(k^2 - \frac{1}{z_\lambda(\eta)} \frac{d^2 z_\lambda}{d\eta^2} \right) v_\lambda(k, \eta) = 0. \quad (65)$$

By using Eq. (Eq. (43)), we evaluate $z_\lambda(\eta)$ and $\frac{1}{z_\lambda(\eta)} \frac{d^2 z_\lambda}{d\eta^2}$ and these read,

$$z_{L,R} \propto \left\{ 1 \mp \frac{4n}{\rho [12n(1-4n)]} \left(\frac{R}{R_0} \right)^{(\delta-\rho)(1-g)} \right\} \times \eta^{(2n+1-\rho)/(1-2n)} \quad (66)$$

and

$$\frac{1}{z_{L,R}} \frac{d^2 z_{L,R}}{d\eta^2} = \frac{\xi(\xi-1)}{\eta^2} \times \left\{ 1 \mp \frac{16n(\delta-\rho)(1-g)}{\rho(4n-\rho)[12n(1-4n)]} \left(\frac{R}{R_0} \right)^{(\delta-\rho)(1-g)} \right\} \quad (67)$$

respectively. Due to the fact that $\delta - \rho$ is positive, the variation of the term in the parenthesis in Eq. (Eq. (67)), can be regarded to be small in the low-curvature regime or equivalently during the early contracting stage and thus $\frac{1}{z_\lambda} \frac{d^2 z_\lambda}{d\eta^2}$ becomes proportional to $1/\eta^2$ that is $\frac{1}{z_\lambda} \frac{d^2 z_\lambda}{d\eta^2} = \sigma_\lambda/\eta^2$ (with $\lambda = L, R$), where

$$\sigma_{L,R} = \xi(\xi-1) \left\{ 1 \mp \frac{16n(\delta-\rho)(1-g)}{\rho(4n-\rho)[12n(1-4n)]} \left(\frac{R}{R_0} \right)^{(\delta-\rho)(1-g)} \right\}, \quad (68)$$

and recall $\xi = \frac{(2n+1-\rho)}{(1-2n)}$. The above expressions yield the tensor power spectrum, defined with the initial Bunch-Davies vacuum state at the deep sub-Hubble radius, so we have,

$$P_h(k, \eta) = P_L(k, \eta) + P_R(k, \eta) \quad (69)$$

with

$$P_L(k, \eta) = \left[\frac{1}{2\pi} \frac{1}{z_L|\eta|} \frac{\Gamma(\Omega_L)}{\Gamma(3/2)} \right]^2 \left(\frac{k|\eta|}{2} \right)^{3-2\Omega_L},$$

$$P_R(k, \eta) = \left[\frac{1}{2\pi} \frac{1}{z_R|\eta|} \frac{\Gamma(\Omega_R)}{\Gamma(3/2)} \right]^2 \left(\frac{k|\eta|}{2} \right)^{3-2\Omega_R}. \quad (70)$$

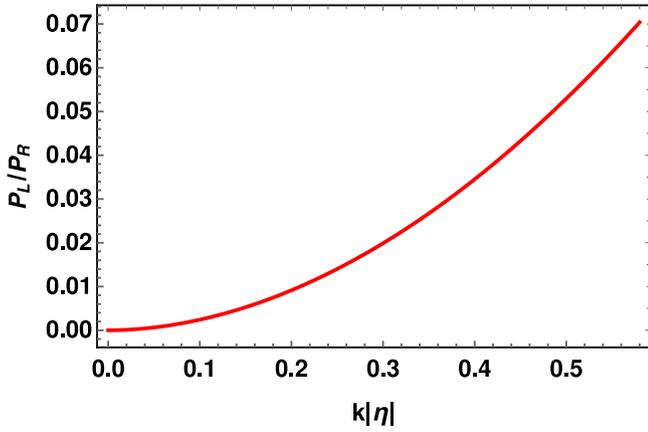


Fig. 6. P_L/P_R vs. $k|\eta|$ in the superhorizon scale, i.e during $k|\eta| < \frac{2n}{1-2n}$. The plot corresponds to $n = 0.185$ and $g = 0.5$.

The factor $\Omega_{L,R} = \sqrt{\sigma_{L,R} + \frac{1}{4}}$ where $\sigma_{L,R}$ is defined in Eq. (Eq. (68)). It may be observed that the left and right polarization modes of the tensor perturbation have different power spectra at a given (k, η) , due to the fact that $\Omega_L \neq \Omega_R$ which actually inherits from the CS coupling function. In particular, Eq. (68) indicates $\Omega_R > \Omega_L$, which in turn makes $P_L(k, \eta)$ suppressed compared to the $P_R(k, \eta)$ in the superhorizon scale. We give the plot of P_L/P_R vs. $k|\eta|$ in the superhorizon limit, i.e during $k|\eta| < \frac{2n}{1-2n}$, in Fig. 6 which corresponds to $n = 0.185$ and $g = 0.5$ (such values of n and g are compatible in respect to the Planck results of n_s and r , as we will demonstrate soon). Fig. 6 clearly depicts that the $P_L(k, \eta)$ is indeed suppressed than the $P_R(k, \eta)$.

Now we can explicitly confront the model at hand with the latest Planck observational data [63], so we calculate the spectral index of the primordial curvature perturbations n_s and the tensor-to-scalar ratio r , which are defined as follows,

$$n_s = 1 + \left. \frac{\partial \ln P_\psi}{\partial \ln k} \right|_{h.c.}, \quad r = \left. \frac{P_h(k, \eta)}{P_\psi(k, \eta)} \right|_{h.c.}, \quad (71)$$

where $P_\psi(k, \eta)$ and $P_h(k, \eta)$ are obtained in Eqs. (52) and (69) respectively, and the suffix 'h.c.' denotes the horizon crossing instant when the mode k satisfies $k = |aH|$. From Eq. (52), the scalar spectral index comes with the following expression,

$$n_s = 4 - \sqrt{1 + 4\sigma} \quad (72)$$

with σ being given in Eq. (46).

It may be noticed that n_s depends on $\frac{R_h}{R_0}$ and n , while r depends on $\frac{R_h}{R_0}$, n and g . The dependency of r on the parameter g comes from the fact that the CS coupling function, which contains the parameter g , affects the tensor perturbation only. The R_h denotes the Ricci scalar at the horizon crossing instant of the large scale modes (in particular $k = 0.002 \text{ Mpc}^{-1}$) on which we are interested to evaluate the observable quantities. As mentioned earlier, the mode $k = 0.002 \text{ Mpc}^{-1}$ crosses the horizon at $t_h \approx -10\text{By}$ and thereby the corresponding Ricci scalar is given by $R(t_h) = \frac{-12n(1-4n)}{t_h^2} \approx -0.12n(1-4n)$. Taking $R_0 = -1\text{By}^{-2}$ (recall R_0 is negative, see the discussion after Eq. (16)), we get $\frac{R_h}{R_0} = 0.12n(1-4n)$. Thus, as a whole, the spectral index depends only on n and the tensor-to-scalar ratio depends on n and g . With this information, we now directly confront the theoretical expressions of scalar spectral index Eq. (72) and tensor-to-scalar ratio Eq. (71) derived from the present model with the Planck 2018 constraints [63]. In particular, we estimate the allowed values of n and g which in turn can give rise to n_s and r in agreement with the

Planck data. This is presented in Fig. 7 where we compute n_s and r for three choices of n , viz. $n = 0.1855$ (blue point), $n = 0.185$ (black point) and $n = 0.1845$ (red point) with $g = 0.5$. The allowed values of n_s and r from Planck data within $1 - \sigma$ and $2 - \sigma$ constraints are illustrated by the yellow and the blue regions respectively in Fig. 7. We note that with $g = 0.5$ and all the three aforesaid values of n , the model estimated n_s and r are within the $1 - \sigma$ or $2 - \sigma$ constraints reported by Planck 2018 data. Thereby in the present context, the scalar spectral index and the tensor-to-scalar ratio are simultaneously compatible with the latest Planck 2018 constraints. On contrary, here we would like to mention that in the vacuum F(R) model, the observable quantities like n_s and r are not simultaneously compatible with the Planck results in the background of a non-singular bounce where $a(t) \sim t^{2n}$ during early contracting stage. In particular, the scalar and tensor perturbation amplitudes in the vacuum F(R) bounce model become comparable to each other and thus the tensor-to-scalar ratio comes as order of unity which is excluded from the Planck data. However, in the Chern-Simons corrected F(R) theory, the CS coupling function considerably affects the tensor perturbation evolution, keeping intact the scalar type perturbation with that of in the vacuum F(R) case. In effect, the tensor perturbation amplitude in the Chern-Simons generalized F(R) bounce model gets suppressed compared to the vacuum F(R) case, and as a result, the tensor-to-scalar ratio in the present context becomes less than unity and comes within the Planck constraints. Such effects of CS coupling function on the tensor type perturbation is in agreement with [85] where some of our authors showed the similar effects of the CS term in the context of an inflationary background spacetime. It has been showed in [85], that a simple power-law F(R) inflationary model, in particular $F(R) = R + \alpha R^n$ (with $n \approx 1.817$), without the CS term yields the correct value of the scalar spectral index, but the tensor-to-scalar ratio comes as $r \approx 0.24$ which is excluded from the Planck data [63]; however the inclusion of CS term in this F(R) model reduces the value of the tensor-to-scalar ratio without affecting the scalar spectral index, and thus the inflationary parameters of the said $R + \alpha R^n + \text{CS}$ model become compatible with the observations.

Before concluding, here we would like to mention that in regard to the observable parameters related to the early epoch of the universe, we have evaluated the scalar spectral index (n_s) and tensor-to-scalar ratio (r) of the primordial perturbations, which appears to be consistent with the Planck data, well within the $1 - \sigma$ and $2 - \sigma$ region, for $0.1845 \lesssim n \lesssim 0.1855$ (recall, n appears in the power exponent of the scale factor). Therefore, in future, if n_s and r are further improved by the Planck collaborations, the parameter n and hence the bouncing behaviour of the scale factor can be better constrained. Beside the scalar spectral index and the tensor-to-scalar ratio (that are related to the two point correlators of scalar and tensor perturbations respectively), the bounce scenario can also be examined from the corresponding higher point correlators of primordial perturbations, in particular, by estimating the theoretical expectations of various non-linear parameters (like f_{NL} , τ_{NL} etc, which represent the amplitudes of the bispectrum and trispectrum respectively [63,101,102]) with the respective Planck data. Furthermore, it is also important to study the signatures of lower scale modes of tensor perturbation, and investigate them with respect to the sensitivity of various gravitational waves (GWs) observatories. The proposed GWs observatories may include advanced LIGO ($10 - 10^3 \text{ Hz}$), ET ($1 - 10^4 \text{ Hz}$), BBO ($10^{-3} - 10 \text{ Hz}$), DECIGO ($10^{-3} - 1 \text{ Hz}$), eLISA ($10^{-5} - 1 \text{ Hz}$), and SKA ($10^{-9} - 10^{-6} \text{ Hz}$) [103-105]. The presence of the parity violating Chern-Simons term in the model distinguishes the evolution of the two polarization modes of tensor perturbation and leads to the generation of chiral gravitational waves, which may have non-trivial imprints on today's GWs spectrum.

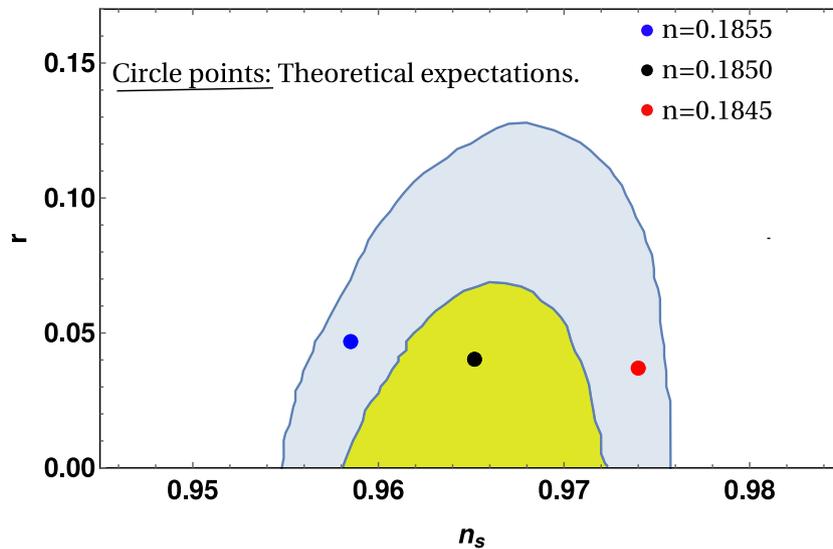


Fig. 7. 1σ (yellow) and 2σ (light blue) contours for Planck 2018 results [63], on $n_s - r$ plane. Additionally, we present the predictions of the present bounce scenario with $n = 0.1855$ (blue point), $n = 0.185$ (black point) and $n = 0.1845$ (red point).

Therefore the evolution of primordial gravitational waves in the background of the bounce scenario discussed here may be compared with the future GWs spectrum from various observatories to provide a unique way to constraint our model. We hope to address these issues in our future work.

5. Conclusion

In this work, we proposed an unified cosmological scenario of a non-singular bounce to a dark energy (DE) epoch in the context of Chern–Simons corrected $F(R)$ gravity theory, where the Chern–Simons coupling function is assumed to have a power law behaviour with the Ricci scalar. Using the reconstruction technique, we analytically determine the form of $F(R)$ during the late contracting era. Using such analytic solution, and in addition, by employing suitable boundary conditions, we numerically solve the gravitational equation and evaluate the $F(R)$ for the entire possible range of the cosmic time, which clearly depicts that the $F(R)$ matches with the Einstein gravity in the low curvature regime, while it deviates from the usual Einstein gravity as the scalar curvature acquires larger and larger values. The form of $F(R)$ leads to an unified cosmological scenario of a non-singular bounce to a dark energy epoch, in particular, from a bounce to a deceleration stage having a matter-like evolution during some regime of the deceleration stage and from the deceleration phase to a late time acceleration era. The effective EoS of the dark energy epoch acquires the value $\omega_{\text{eff}} = -0.997$ at present time, which is indeed compatible with the results provided by Planck+SNe+BAO data. Moreover the model predicts a finite time future singularity of the universe around 30By when the scale factor, the effective energy density and the effective pressure are found to diverge, and thus the singularity is a Type-I type of singularity. However, since the present age of our universe is $t_p \approx 13.5\text{By}$, i.e the Type-I singularity occurs at far future from the present age, we may argue that the current model satisfactorily describes a singular free evolution of the universe up-to the cosmic time $t \gtrsim t_p$. Here it deserves mentioning that the bounce in the present context is an asymmetric bounce, in particular, the comoving Hubble radius monotonically increases with cosmic time and asymptotically diverges at distant past, while it decreases with time at the present epoch of the universe. Due to

such evolution of the Hubble horizon, the primordial perturbation modes generate at distant past far away from the bounce when all the relevant perturbation modes lie within the horizon. Correspondingly the scalar and tensor perturbations power spectra are determined, which in turn leads to the primordial observable quantities like the spectral index of the scalar curvature perturbation (n_s) and the tensor-to-scalar ratio (r). The theoretical expectations of n_s and r in the present context are found to be simultaneously compatible with the latest Planck 2018 constraints. In this regard, the Chern–Simons term proves to play an important role in making the observable quantities, particularly the tensor-to-scalar ratio, consistent with the Planck data. Actually in the case of vacuum $F(R)$ model in the background of a non-singular bounce, the scalar and tensor perturbation amplitudes are comparable to each other and thus the tensor-to-scalar ratio becomes order of unity which is indeed excluded from the Planck results. However, in the $F(R)$ model generalized by the Chern–Simons term, the perturbations evolution get considerably affected due to the presence of the CS term, in particular the tensor perturbation amplitude gets suppressed and the scalar perturbation remains intact compared to that of in the vacuum $F(R)$ case. As a result, the tensor-to-scalar ratio in the Chern–Simons corrected $F(R)$ bounce model becomes less than unity and moreover it comes within the Planck constraints for a suitable parametric regime.

In summary, the present model provides an unified cosmological scenario of a non-singular bounce to viable dark energy epoch in the Chern–Simons generalized $F(R)$ theory, where the Chern–Simons term plays a crucial role in regard to the compatibility of the primordial observable quantities with the Planck results.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This work was supported in part by MINECO (Spain), project PID2019-104397GB-I00 (SDO). TP sincerely acknowledges the hospitality by ICE-CSIC/IEEC (Barcelona, Spain), where this work

was generated during his visit. The work was supported by the Ministry of Education and Science of the Republic of Kazakhstan, Grant AP09261147 (RM).

References

- [1] A.H. Guth, *Phys. Rev. D* 23 (1981) 347–356.
- [2] A.D. Linde, *Contemp. Concepts Phys.* 5 (1990) 1, [hep-th/0503203].
- [3] D. Langlois, hep-th/0405053.
- [4] A. Riotto, *ICTP Lect. Notes Ser.* 14 (2003) 317, [hep-ph/0210162].
- [5] J.D. Barrow, P. Saich, *Classical Quantum Gravity* 10 (1993) 279.
- [6] J.D. Barrow, J.P. Mimoso, *Phys. Rev. D* 50 (1994) 3746.
- [7] D. Baumann, <http://dx.doi.org/10.1142/97898143271830010> [arXiv:0907.5424 [hep-th]].
- [8] R.H. Brandenberger, arXiv:1206.4196 [astro-ph.CO].
- [9] R. Brandenberger, P. Peter, arXiv:1603.05834 [hep-th].
- [10] D. Battefeld, P. Peter, *Phys. Rep.* 571 (2015) 1, <http://dx.doi.org/10.1016/j.physrep.2014.12.004>, [arXiv:1406.2790 [astro-ph.CO]].
- [11] M. Novello, S.E.P. Bergliaffa, Bouncing cosmologies, *Phys. Rep.* 463 (2008) 127, <http://dx.doi.org/10.1016/j.physrep.2008.04.006>, [arXiv:0802.1634 [astro-ph]].
- [12] Y.F. Cai, *Sci. China Phys. Mech. Astron.* 57 (2014) 1414, <http://dx.doi.org/10.1007/s11433-014-5512-3>, [arXiv:1405.1369 [hep-th]].
- [13] Y. Cai, Y. Wan, H.G. Li, T. Qiu, Y.S. Piao, *J. High Energy Phys.* 01 (2017) 090, [http://dx.doi.org/10.1007/JHEP01\(2017\)090](http://dx.doi.org/10.1007/JHEP01(2017)090), [arXiv:1610.03400 [gr-qc]].
- [14] J. de Haro, Y.F. Cai, *Gen. Relativity Gravitation* 47 (8) (2015) 95, <http://dx.doi.org/10.1007/s10714-015-1936-y>, [arXiv:1502.03230 [gr-qc]].
- [15] J.L. Lehners, *Classical Quantum Gravity* 28 (2011) 204004, <http://dx.doi.org/10.1088/0264-9381/28/20/204004>, [arXiv:1106.0172 [hep-th]].
- [16] J.L. Lehners, *Phys. Rep.* 465 (2008) 223, <http://dx.doi.org/10.1016/j.physrep.2008.06.001>, [arXiv:0806.1245 [astro-ph]].
- [17] Y.F. Cai, A. Marciano, D.G. Wang, E. Wilson-Ewing, *Universe* 3 (1) (2016) 1, <http://dx.doi.org/10.3390/Universe3010001>, [arXiv:1610.00938 [astro-ph.CO]].
- [18] S. Colin, N. Pinto-Neto, *Phys. Rev. D* 96 (6) (2017) 063502, <http://dx.doi.org/10.1103/PhysRevD.96.063502>, [arXiv:1706.03037 [gr-qc]].
- [19] C. Cattoen, M. Visser, *Classical Quantum Gravity* 22 (2005) 4913, <http://dx.doi.org/10.1088/0264-9381/22/23/001>, [gr-qc/0508045].
- [20] C. Li, R.H. Brandenberger, Y.K.E. Cheung, *Phys. Rev. D* 90 (12) (2014) 123535, <http://dx.doi.org/10.1103/PhysRevD.90.123535>, [arXiv:1403.5625 [gr-qc]].
- [21] D. Brizuela, G.A.D. Mena Marugan, T. Pawłowski, *Classical Quantum Gravity* 27 (2010) 052001, <http://dx.doi.org/10.1088/0264-9381/27/5/052001>, [arXiv:0902.0697 [gr-qc]].
- [22] Y.F. Cai, E. McDonough, F. Duplessis, R.H. Brandenberger, *J. Cosmol. Astropart. Phys.* 1310 (2013) 024, <http://dx.doi.org/10.1088/1475-7516/2013/10/024>, [arXiv:1305.5259 [hep-th]].
- [23] J. Quintin, Y.F. Cai, R.H. Brandenberger, *Phys. Rev. D* 90 (6) (2014) 063507, <http://dx.doi.org/10.1103/PhysRevD.90.063507>, [arXiv:1406.6049 [gr-qc]].
- [24] Y.F. Cai, R. Brandenberger, P. Peter, *Classical Quantum Gravity* 30 (2013) 075019, <http://dx.doi.org/10.1088/0264-9381/30/7/075019>, [arXiv:1301.4703 [gr-qc]].
- [25] N. Pinto-Neto, Bouncing quantum cosmology, *Universe* 7 (2021) 110, <http://dx.doi.org/10.3390/universe7040110>.
- [26] M. Koehn, J.L. Lehners, B. Ovrut, *Phys. Rev. D* 93 (10) (2016) 103501, <http://dx.doi.org/10.1103/PhysRevD.93.103501>, [arXiv:1512.03807 [hep-th]].
- [27] S. Nojiri, S.D. Odintsov, V.K. Oikonomou, *Phys. Rev. D* 93 (8) (2016) 084050, <http://dx.doi.org/10.1103/PhysRevD.93.084050>, [arXiv:1601.04112 [gr-qc]].
- [28] S.D. Odintsov, V.K. Oikonomou, T. Paul, *Classical Quantum Gravity* 37 (23) (2020) 235005, <http://dx.doi.org/10.1088/1361-6382/abb47>, [arXiv:2009.09947 [gr-qc]].
- [29] M. Koehn, J.L. Lehners, B.A. Ovrut, *Phys. Rev. D* 90 (2) (2014) 025005, <http://dx.doi.org/10.1103/PhysRevD.90.025005>, [arXiv:1310.7577 [hep-th]].
- [30] L. Battarra, J.L. Lehners, *J. Cosmol. Astropart. Phys.* 1412 (12) (2014) 023, <http://dx.doi.org/10.1088/1475-7516/2014/12/023>, [arXiv:1407.4814 [hep-th]].
- [31] J. Martin, P. Peter, N. Pinto Neto, D.J. Schwarz, *Phys. Rev. D* 65 (2002) 123513, <http://dx.doi.org/10.1103/PhysRevD.65.123513>, [hep-th/0112128].
- [32] J. Khoury, B.A. Ovrut, P.J. Steinhardt, N. Turok, *Phys. Rev. D* 64 (2001) 123522, <http://dx.doi.org/10.1103/PhysRevD.64.123522>, [hep-th/0103239].
- [33] J.C. Hackworth, E.J. Weinberg, *Phys. Rev. D* 71 (2005) 044014, <http://dx.doi.org/10.1103/PhysRevD.71.044014>, [hep-th/0410142].
- [34] M.C. Johnson, J.L. Lehners, *Phys. Rev. D* 85 (2012) 103509, <http://dx.doi.org/10.1103/PhysRevD.85.103509>, [arXiv:1112.3360 [hep-th]].
- [35] P. Peter, N. Pinto-Neto, *Phys. Rev. D* 66 (2002) 063509, <http://dx.doi.org/10.1103/PhysRevD.66.063509>, [hep-th/0203013].
- [36] M. Gasperini, M. Giovannini, G. Veneziano, *Phys. Lett. B* 569 (2003) 113, <http://dx.doi.org/10.1016/j.physletb.2003.07.028>, [hep-th/0306113].
- [37] P. Creminelli, A. Nicolis, M. Zaldarriaga, *Phys. Rev. D* 71 (2005) 063505, <http://dx.doi.org/10.1103/PhysRevD.71.063505>, [hep-th/0411270].
- [38] J.L. Lehners, E. Wilson-Ewing, *J. Cosmol. Astropart. Phys.* 1510 (10) (2015) 038, <http://dx.doi.org/10.1088/1475-7516/2015/10/038>, [arXiv:1507.08112 [astro-ph.CO]].
- [39] J.L. Lehners, P.J. Steinhardt, *Phys. Rev. D* 87 (12) (2013) 123533, <http://dx.doi.org/10.1103/PhysRevD.87.123533>, [arXiv:1304.3122 [astro-ph.CO]].
- [40] Y.F. Cai, J. Quintin, E.N. Saridakis, E. Wilson-Ewing, *J. Cosmol. Astropart. Phys.* 1407 (2014) 033, <http://dx.doi.org/10.1088/1475-7516/2014/07/033>, [arXiv:1404.4364 [astro-ph.CO]].
- [41] Y.F. Cai, T. Qiu, Y.S. Piao, M. Li, X. Zhang, *J. High Energy Phys.* 0710 (2007) 071, <http://dx.doi.org/10.1088/1126-6708/2007/10/071>, [arXiv:0704.1090 [gr-qc]].
- [42] J.D. Barrow, D. Kimberly, J. Magueijo, *Classical Quantum Gravity* 21 (2004) 4289, <http://dx.doi.org/10.1088/0264-9381/21/18/001>, [astro-ph/0406369].
- [43] J. Haro, E. Elizalde, *J. Cosmol. Astropart. Phys.* 1510 (10) (2015) 028, <http://dx.doi.org/10.1088/1475-7516/2015/10/028>, [arXiv:1505.07948 [gr-qc]].
- [44] A. Das, D. Maity, T. Paul, S. SenGupta, *Eur. Phys. J. C* 77 (12) (2017) 813, <http://dx.doi.org/10.1140/epjc/s10052-017-5396-2>, [arXiv:1706.00950 [hep-th]].
- [45] Y.F. Cai, T. t. Qiu, R. Brandenberger, X. m. Zhang, *Phys. Rev. D* 80 (2009) 023511, <http://dx.doi.org/10.1103/PhysRevD.80.023511>, [arXiv:0810.4677 [hep-th]].
- [46] F. Finelli, R. Brandenberger, *Phys. Rev. D* 65 (2002) 103522, <http://dx.doi.org/10.1103/PhysRevD.65.103522>, [hep-th/0112249].
- [47] Y.F. Cai, R. Brandenberger, X. Zhang, *Phys. Lett. B* 703 (2011) 25, <http://dx.doi.org/10.1016/j.physletb.2011.07.074>, [arXiv:1105.4286 [hep-th]].
- [48] J. Haro, J. Amorós, *PoS FFP* 14 (2016) 163, <http://dx.doi.org/10.22323/1.2240163>, [arXiv:1501.06270 [gr-qc]].
- [49] Y.F. Cai, R. Brandenberger, X. Zhang, *J. Cosmol. Astropart. Phys.* 1103 (2011) 003, <http://dx.doi.org/10.1088/1475-7516/2011/03/003>, [arXiv:1101.0822 [hep-th]].
- [50] R. Brandenberger, *Phys. Rev. D* 80 (2009) 043516, <http://dx.doi.org/10.1103/PhysRevD.80.043516>, [arXiv:0904.2835 [hep-th]].
- [51] J. de Haro, J. Amorós, *J. Cosmol. Astropart. Phys.* 1408 (2014) 025, <http://dx.doi.org/10.1088/1475-7516/2014/08/025>, [arXiv:1403.6396 [gr-qc]].
- [52] S.D. Odintsov, V.K. Oikonomou, *Phys. Rev. D* 90 (12) (2014) 124083, <http://dx.doi.org/10.1103/PhysRevD.90.124083>, [arXiv:1410.8183 [gr-qc]].
- [53] T. Qiu, K.C. Yang, *J. Cosmol. Astropart. Phys.* 1011 (2010) 012, <http://dx.doi.org/10.1088/1475-7516/2010/11/012>, [arXiv:1007.2571 [astro-ph.CO]].
- [54] K. Bamba, J. de Haro, S.D. Odintsov, *J. Cosmol. Astropart. Phys.* 1302 (2013) 008, <http://dx.doi.org/10.1088/1475-7516/2013/02/008>, [arXiv:1211.2968 [gr-qc]].
- [55] J. de Haro, *J. Cosmol. Astropart. Phys.* 1211 (2012) 037, [arXiv:1207.3621 [gr-qc]].
- [56] S. Nojiri, S.D. Odintsov, V.K. Oikonomou, T. Paul, *Phys. Rev. D* 100 (8) (2019) 084056, <http://dx.doi.org/10.1103/PhysRevD.100.084056>, [arXiv:1910.03546 [gr-qc]].
- [57] E. Elizalde, S.D. Odintsov, T. Paul, *Eur. Phys. J. C* 80 (1) (2020) 10, <http://dx.doi.org/10.1140/epjc/s10052-019-7544-3>, [arXiv:1912.05138 [gr-qc]].
- [58] E. Elizalde, S.D. Odintsov, V.K. Oikonomou, T. Paul, *Nuclear Phys. B* 954 (2020) 114984, <http://dx.doi.org/10.1016/j.nuclphysb.2020.114984>, [arXiv:2003.04264 [gr-qc]].
- [59] E. Wilson-Ewing, *J. Cosmol. Astropart. Phys.* 1303 (2013) 026, <http://dx.doi.org/10.1088/1475-7516/2013/03/026>, [arXiv:1211.6269 [gr-qc]].
- [60] G.F.R. Ellis, J. Murugan, C.G. Tsagas, *Classical Quantum Gravity* 21 (1) (2004) 233–250, <http://dx.doi.org/10.1088/0264-9381/21/1/016>, [arXiv:gr-qc/0307112 [gr-qc]].
- [61] B.C. Paul, S.D. Maharaj, A. Beesham, [arXiv:2008.00169 [astro-ph.CO]].
- [62] S.L. Li, H. Lü, H. Wei, P. Wu, H. Yu, *Phys. Rev. D* 99 (10) (2019) 104057, <http://dx.doi.org/10.1103/PhysRevD.99.104057>, [arXiv:1903.03940 [gr-qc]].
- [63] Y. Akrami, et al., [Planck Collaboration], arXiv:1807.06211 [astro-ph.CO].
- [64] S. Nojiri, S.D. Odintsov, V.K. Oikonomou, *Phys. Rep.* 692 (2017) 1, <http://dx.doi.org/10.1016/j.physrep.2017.06.001>, [arXiv:1705.11098 [gr-qc]].
- [65] S. Nojiri, S.D. Odintsov, *Phys. Rep.* 505 (2011) 59, <http://dx.doi.org/10.1016/j.physrep.2011.04.001>, [arXiv:1011.0544 [gr-qc]].
- [66] E. Elizalde, S.D. Odintsov, T. Paul, D. Sáez-Chillón Gómez, *Phys. Rev. D* 99 (6) (2019) 063506, <http://dx.doi.org/10.1103/PhysRevD.99.063506>, [arXiv:1811.02960 [gr-qc]].
- [67] S. Perlmutter, et al., [Supernova Cosmology Project], *Astrophys. J.* 483 (1997) 565, <http://dx.doi.org/10.1086/304265>, [arXiv:astro-ph/9608192 [astro-ph]].
- [68] S. Perlmutter, et al., [Supernova Cosmology Project], *Astrophys. J.* 517 (1999) 565–586, <http://dx.doi.org/10.1086/307221>, [arXiv:astro-ph/9812133 [astro-ph]].

- [69] A.G. Riess, et al., [Supernova Search Team], *Astron. J.* 116 (1998) 1009–1038, <http://dx.doi.org/10.1086/300499>, [arXiv:astro-ph/9805201 [astro-ph]].
- [70] S. Capozziello, M. De Laurentis, *Phys. Rep.* 509 (2011) 167, <http://dx.doi.org/10.1016/j.physrep.2011.09.003>, [arXiv:1108.6266 [gr-qc]].
- [71] V. Faraoni, S. Capozziello, *Fundam. Theor. Phys.* 170 (2010) <http://dx.doi.org/10.1007/978-94-007-0165-6>.
- [72] M.B. Green, J.H. Schwarz, *Phys. Lett. B* 149 (1984) 117–122, [http://dx.doi.org/10.1016/0370-2693\(84\)91565-X](http://dx.doi.org/10.1016/0370-2693(84)91565-X).
- [73] I. Antoniadis, E. Gava, K.S. Narain, *Phys. Lett. B* 283 (1992) 209–212, [http://dx.doi.org/10.1016/0370-2693\(92\)90009-S](http://dx.doi.org/10.1016/0370-2693(92)90009-S), [arXiv:hep-th/9203071 [hep-th]].
- [74] J.c. Hwang, H. Noh, *Phys. Rev. D* 71 (2005) 063536, <http://dx.doi.org/10.1103/PhysRevD.71.063536>, [arXiv:gr-qc/0412126 [gr-qc]].
- [75] K. Choi, J.c. Hwang, K.W. Hwang, *Phys. Rev. D* 61 (2000) 084026, <http://dx.doi.org/10.1103/PhysRevD.61.084026>, [arXiv:hep-ph/9907244 [hep-ph]].
- [76] M. Satoh, S. Kanno, J. Soda, *Phys. Rev. D* 77 (2008) 023526, <http://dx.doi.org/10.1103/PhysRevD.77.023526>, [arXiv:0706.3585 [astro-ph]].
- [77] M. Satoh, J. Soda, J. Cosmol. Astropart. Phys. 09 (2008) 019, <http://dx.doi.org/10.1088/1475-7516/2008/09/019>, [arXiv:0806.4594 [astro-ph]].
- [78] R. Jackiw, S.Y. Pi, *Phys. Rev. D* 68 (2003) 104012, <http://dx.doi.org/10.1103/PhysRevD.68.104012>, [arXiv:gr-qc/0308071 [gr-qc]].
- [79] A. Lue, L.M. Wang, M. Kamionkowski, *Phys. Rev. Lett.* 83 (1999) 1506–1509, <http://dx.doi.org/10.1103/PhysRevLett.83.1506>, [arXiv:astro-ph/9812088 [astro-ph]].
- [80] Z. Haghani, T. Harko, S. Shahidi, *Eur. Phys. J. C* 77 (8) (2017) 514, <http://dx.doi.org/10.1140/epjc/s10052-017-5078-0>, [arXiv:1704.06539 [gr-qc]].
- [81] A. Nishizawa, T. Kobayashi, *Phys. Rev. D* 98 (12) (2018) 124018, <http://dx.doi.org/10.1103/PhysRevD.98.124018>, [arXiv:1809.00815 [gr-qc]].
- [82] K. Inomata, M. Kamionkowski, *Phys. Rev. Lett.* 123 (3) (2019) 031305, <http://dx.doi.org/10.1103/PhysRevLett.123.031305>, [arXiv:1811.04959 [astro-ph.CO]].
- [83] M. Kamionkowski, A. Kosowsky, *Phys. Rev. D* 57 (1998) 685–691, <http://dx.doi.org/10.1103/PhysRevD.57.685>, [arXiv:astro-ph/9705219 [astro-ph]].
- [84] P. Wagle, N. Yunes, D. Garfinkle, L. Bieri, *Classical Quantum Gravity* 36 (11) (2019) 115004, <http://dx.doi.org/10.1088/1361-6382/ab0eed>, [arXiv:1812.05646 [gr-qc]].
- [85] S.D. Odintsov, V.K. Oikonomou, *Phys. Rev. D* 99 (6) (2019) 064049, <http://dx.doi.org/10.1103/PhysRevD.99.064049>, [arXiv:1901.05363 [gr-qc]].
- [86] S.D. Odintsov, V.K. Oikonomou, *Phys. Rev. D* 94 (6) (2016) 064022, <http://dx.doi.org/10.1103/PhysRevD.94.064022>, [arXiv:1606.03689 [gr-qc]].
- [87] S. Nojiri, S.D. Odintsov, V.K. Oikonomou, *Phys. Dark Univ.* 29 (2020) 100602, <http://dx.doi.org/10.1016/j.dark.2020.100602>, [arXiv:1912.13128 [gr-qc]].
- [88] S. Nojiri, S.D. Odintsov, V.K. Oikonomou, T. Paul, *Phys. Rev. D* 102 (2) (2020) 023540, <http://dx.doi.org/10.1103/PhysRevD.102.023540>, [arXiv:2007.06829 [gr-qc]].
- [89] F. Bajardi, D. Vernieri, S. Capozziello, [arXiv:2106.07396 [gr-qc]].
- [90] S. Nojiri, S.D. Odintsov, S. Tsujikawa, *Phys. Rev. D* 71 (2005) 063004, <http://dx.doi.org/10.1103/PhysRevD.71.063004>, [arXiv:hep-th/0501025 [hep-th]].
- [91] N. Aghanim, et al., [Planck], *Astron. Astrophys.* 641 (2020) A6, <http://dx.doi.org/10.1051/0004-6361/201833910>, [arXiv:1807.06209 [astro-ph.CO]].
- [92] E. Elizalde, S. Nojiri, S.D. Odintsov, L. Sebastiani, S. Zerbini, *Phys. Rev. D* 83 (2011) 086006, <http://dx.doi.org/10.1103/PhysRevD.83.086006>, [arXiv:1012.2280 [hep-th]].
- [93] S. Nojiri, S.D. Odintsov, T. Paul, *Symmetry* 13 (6) (2021) 928, <http://dx.doi.org/10.3390/sym13060928>, [arXiv:2105.08438 [gr-qc]].
- [94] V.A. Belinskii, I.M. Khalatnikov, E.M. Lifshitz, *Adv. Phys.* 19 (1970) 525.
- [95] J.K. Erickson, D.H. Wesley, P.J. Steinhardt, N. Turok, *Phys. Rev. D* 69 (2004) 063514, <http://dx.doi.org/10.1103/PhysRevD.69.063514>, [hep-th/0312009].
- [96] D. Garfinkle, W.C. Lim, F. Pretorius, P.J. Steinhardt, *Phys. Rev. D* 78 (2008) 083537, <http://dx.doi.org/10.1103/PhysRevD.78.083537>, [arXiv:0808.0542 [hep-th]].
- [97] S.D. Odintsov, V.K. Oikonomou, *Phys. Rev. D* 101 (4) (2020) 044009, <http://dx.doi.org/10.1103/PhysRevD.101.044009>, [arXiv:2001.06830 [gr-qc]].
- [98] S. Nojiri, S.D. Odintsov, V.K. Oikonomou, A.A. Popov, *Phys. Dark Univ.* 28 (2020) 100514, <http://dx.doi.org/10.1016/j.dark.2020.100514>, [arXiv:2002.10402 [gr-qc]].
- [99] H. Noh, J.c. Hwang, *Phys. Lett. B* 515 (2001) 231, [http://dx.doi.org/10.1016/S0370-2693\(01\)00875-9](http://dx.doi.org/10.1016/S0370-2693(01)00875-9), [astro-ph/0107069].
- [100] J.c. Hwang, H. Noh, *Phys. Rev. D* 66 (2002) 084009, <http://dx.doi.org/10.1103/PhysRevD.66.084009>, [hep-th/0206100].
- [101] Y.F. Cai, W. Xue, R. Brandenberger, X. Zhang, *J. Cosmol. Astropart. Phys.* 05 (2009) 011, <http://dx.doi.org/10.1088/1475-7516/2009/05/011>, [arXiv:0903.0631 [astro-ph.CO]].
- [102] I. Agullo, D. Kranas, V. Sreenath, *Classical Quantum Gravity* 38 (6) (2021) 065010, <http://dx.doi.org/10.1088/1361-6382/abc521>, [arXiv:2006.09605 [astro-ph.CO]].
- [103] B.P. Abbott, et al., [LIGO Scientific and Virgo], *Phys. Rev. Lett.* 118 (12) (2017) 121101, <http://dx.doi.org/10.1103/PhysRevLett.118.121101>, [arXiv:1612.02029 [gr-qc]]. erratum; *Phys. Rev. Lett.* 119 (2) (2017) 029901.
- [104] G. Janssen, G. Hobbs, M. McLaughlin, C. Bassa, A.T. Deller, M. Kramer, K. Lee, C. Mingarelli, P. Rosado, S. Sanidas, et al., *PoS AASKA14* (2015) 037, <http://dx.doi.org/10.22323/1.215.0037>, [arXiv:1501.00127 [astro-ph.IM]].
- [105] P. Amaro-Seoane, S. Aoudia, S. Babak, P. Binetruy, E. Berti, A. Bohe, C. Caprini, M. Colpi, N.J. Cornish, K. Danzmann, et al., *GW Notes* 6 (2013) 4–110, [arXiv:1201.3621 [astro-ph.CO]].