UDC 517.956.4

ELEMENTS OF THE POTENTIAL THEORY FOR A DEGENERATE DIFFUSION EQUATION

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Potential theory for a degenerate-type heat equation of the following type

$$L_\alpha u = \frac{\partial u}{\partial t} - \alpha(t)u = f(x,t), \ \ x > 0, \ t > 0$$

was constructed in [1]. This equation has been investigated in [1,3,4]. The aim of this paper is to study potentials of a degenerate diffusion equation. We consider the following degenerate diffusion equation:

$$\Diamond_\alpha = \frac{\partial u}{\partial t} - \alpha(t)\Delta_x u = f(x,t), \quad x \in R^n, t > 0 \tag{1}$$

Here and throughout the coefficient $\alpha(t) \in L_1[0,T]$ is defined in $[0,T]$ and satisfies one of the following conditions I and II:

I. $\alpha(t)$ is non-negative and can be zero only at isolated points in $[0,T]$;

II. $\alpha_1(t)$ defined by the formula

$$\alpha_1(t) = \int_0^t \alpha(z)dz, \quad (\alpha_1(0) = 0)$$

is positive for all $t > 0$, which provides $\alpha(t)$ to be negative in an interval.

We denote

$$b(t,\tau) = \int_\tau^t \alpha(z)dz = \alpha_1(t) - \alpha_1(\tau), \quad (b(t,0) = \alpha_1(t)).$$

If $\alpha(t)$ satisfies the condition I, then $b(t,\tau)$ is positive for all $t > \tau > 0$.

Under the condition II the fundamental solution of (1) can be found by means of the Fourier transform F_x.

Lemma 1. The fundamental solution of the equation (1) can be represented as

$$\varepsilon_{n,a}(x,t) = \varepsilon_n(x,\alpha_1(t)) = \frac{\theta(t)}{\left(2\sqrt{\pi\alpha_1(t)}\right)^n} e^{-\frac{|x|^2}{4\alpha_1(t)}}, \tag{2}$$

where $\varepsilon_n(x,t) = \frac{\theta(t)}{\left(2\sqrt{\pi\alpha_1(t)}\right)^n} e^{-\frac{|x|^2}{4t}}$ is the fundamental solution of the heat conduction operator (see [2, p.148]), $\theta(t)$ is the Heaviside function.

The function (2) has the following properties

$$\int \varepsilon_{n,a}(x,t)dx = \frac{\theta(t)}{\left(2\sqrt{\pi\alpha_1(t)}\right)^n}\int_{R^n} e^{-\frac{|x|^2}{4\alpha_1(t)}}dx = 1,$$

$$\varepsilon_{n,a}(x,t) \to \delta(x) \text{ with } t \to 0+$$

that are similar to the fundamental solution of the heat conduction operator (see [2, p.148]).

Degenerate diffusion potential (under the condition I) with a density $f(x,t)$ can be defined by

$$V(x,t) = \varepsilon_{n,b} * f = \int_0^t \int_{R^n} \varepsilon_{n,b}(x-\xi, t-\tau)\cdot f(\xi,\tau)d\xi d\tau =$$

$$= \int_0^t \int_{R^n} \frac{e^{-\frac{|x|^2}{4b(t,\tau)}}}{\left(2\sqrt{\pi \cdot b(t,\tau)}\right)^n}\cdot f(\xi,\tau)d\xi d\tau,$$

Degenerate diffusion potential (under the condition II) with a density $\varphi(x)\cdot\delta(t)$ (or surface degenerate diffusion potential) can be defined by

$$V^{(0)}(x,t) = \varepsilon_{n,a} * \left(\varphi(x)\cdot\delta(t)\right) = \int_0^t \int_{R^n} \varepsilon_{n,a}(x-\xi, t)\cdot\varphi(\xi)d\xi dt =$$

$$= \frac{\theta(t)}{\left(2\sqrt{\pi\alpha_1(t)}\right)^n}\int_{R^n} e^{-\frac{|x|^2}{4\alpha_1(t)}}\varphi(\xi)d\xi,$$

where $\varepsilon_{n,b}(x-\xi,t-\tau) = \varepsilon_n(x-\xi, b(t,\tau))$

It is well known that $\varepsilon_{n,b} * f$ satisfies the equation (1). We shall distinguish a class of densites $f(x,t)$ for which the degenerate diffusion potential exists. Let M denote a class of functions which are bounded in the strip $[0,T]$ and which vanish at $t<0$.

Theorem 1. Let $\alpha(t)$ satisfy the condition I. Then the following statements hold:

1. $V(x,t)$ belongs to the class M for any $f(x,t)\in M$;
2. $V(x,t)$ satisfies the estimate

$$\left|V(x,t)\right| \le t\cdot \sup_{0\le\tau\le t,\ \xi\in R^n}\left|f(\xi,\tau)\right|, \qquad t>0$$

3. $V(x,t)$ is a distributional solution of (1) for all $x\in R^n$, $t\ge 0$ satisfying zero initial condition as $t\to 0+$;

4. If $f\in C^2$ for all $x\in R^n$ and $t\ge 0$ and its deriratives up to the second order belong to the class M and $\alpha(t)\in C(R_+)$, then $V\in C^2(t>0)\cap C^1(t\ge 0)$.

Theorem 2. Let $\alpha(t)$ satisfy the condition II. Then:

1. $V^{(0)}(x,t)$ belongs to the class M for any bounded function $\varphi(x)$;
2. $V^{(0)}(x,t)$ satisfies the estimate

$$\left|V^{(0)}(x,t)\right| \le \sup_{\xi\in R^n}\left|\varphi(\xi)\right|, \qquad t>0$$

3. If $\varphi(x)\in C(\overline{\Omega})$, then the surface potential $V^{(0)}(x,t)$ satisfies the initial condition $V^{(0)}(x,t)\big|_{t=0} = \varphi(x)$;

4. If $\varphi\in C^2$ for all $x\in R^n$ and $t\ge 0$ and its deriratives up to the second order belong to the class M and $\alpha(t)\in C(R_+)$, then $V^{(0)}\in C^2(t>0)\cap C(t\ge 0)$.

Literature

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