UDC 517.5

THE O'NEIL-TYPE INEQUALITIES FOR ANISOTROPIC LORENTZ SPACES

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R. O'Neill [1] investigated the boundedness of the convolution operator

$$Af(y)=\int_{\Re^n} K(y-x)f(x)dx$$

in Lorentz spaces.

In particular, the following inequality was obtained: for $1<p,r,q<\infty$, $0<h_1,h_2,h_3\le\infty$, $1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$, and $\frac{1}{h_1}=\frac{1}{h_2}+\frac{1}{h_3}$, one has

$$\|f*K\|_{L_{q,h_1}(\Re^n)}\le C\|f\|_{L_{p,h_2}(\Re^n)}\|K\|_{L_{r,h_3}(\Re^n)}.$$

This theme was further developed in the works of Hunt [2], Yap [3], Blozinski [4], [5], [6], E. Nursultanov and S. Tikhonov [7], and other authors.

The main goal of this work is to investigate the Young-O'Neil-type inequality in anisotropic Lorentz spaces.

Let $\bar{p}=(p_1,p_2)$, $\bar{q}=(q_1,q_2)$ be such that if $1\le p_i<\infty$, then $1\le q_i\le\infty$, if $p_i=\infty$, then $q_i=\infty$, where $i=1,2$.

$L_{\bar{p},\bar{q}}([0,1]^2)$ is the anisotropic Lorentz space (see [8]), which is defined as the set of measurable, 1-periodic functions $f(x_1,x_2)$ with respect to each variable with finite norm

$$\|f\|_{L_{\bar{p},\bar{q}}([0,1]^2)}=\left(\int_0^1\left(\int_0^1\left(t_1^{\frac{1}{p_1}}t_2^{\frac{1}{p_2}}f^{*_1,*_2}(t_1,t_2)\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}},$$

where $f^{*_1,*_2}(t_1,t_2)$ is the function obtained by applying a decreasing rearrangement of $f(x_1,x_2)$ sequentially with respect to variables and for a fixed other variable. Here the expression $\left(\int_0^1 (G(s))^q\frac{ds}{s}\right)^{\frac{1}{q}}$, for $q=\infty$ is understood as $\sup_{s>0}G(s)$.

Let f be a measurable function locally integrable in $[0,1]^2$. We define a function $f^{\star_1,\star_2}(t_1,t_2)$ in $[0,1]^2$ as follows

$$f^{\star_1,\star_2}(t_1,t_2)=\frac{1}{t_1t_2}\int_0^{t_2}\int_0^{t_1} f^{*_1,*_2}(s_1,s_2)ds_1ds_2 .$$

The following relations hold:

$$f^{\star_1,\star_2}(t_1,t_2)=\sup_{\substack{|e_2|=t_2\\ e_2\subset[0,1]}}\frac{1}{|e_2|}\int_{e_2}\sup_{\substack{|e_1|=t_1\\ e_1\subset[0,1]}}\frac{1}{|e_1|}\int_{e_1}|f(x_1,x_2)|dx_1dx_2 ,$$

here the suprema is taken over all compact sets $e_i \subset [0,1]$, whose measure $|e_i| = t_i$ ($i=1,2$);

$$f^{\star_1,\star_2}(t_1,t_2)\ge f^{*_1,*_2}(t_1,t_2);$$

$$\int_0^1\int_0^1 f(x_1,x_2)g(x_1,x_2)dx_1dx_2 \le \int_0^1\int_0^1 f^{*_1,*_2}(x_1,x_2)g^{*_1,*_2}(x_1,x_2)dx_1dx_2 .$$

Lemma 1. *(Two-dimensional analogue of Hardy's inequality). Let* $1<p_i<\infty$, $1\le q_i\le\infty$, $\frac{1}{p_i}+\frac{1}{p_i'}=1$ *(* $i=1,2$ *), and* f *be a nonnegative measurable function on* $[0,1]^2$*, then*

$$\left(\int_0^1\left(\int_0^1\left(\frac{t_1^{\frac{1}{p_1}}t_2^{\frac{1}{p_2}}}{t_1t_2}\int_0^{t_2}\int_0^{t_1} f(s_1,s_2)ds_1ds_2\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \le p_1'p_2'\left(\int_0^1\left(\int_0^1\left(t_1^{\frac{1}{p_1}}t_2^{\frac{1}{p_2}}f(t_1,t_2)\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}$$

and

$$\left(\int_0^1\left(\int_0^1\left(t_1^{1-\frac{1}{p_1}}t_2^{1-\frac{1}{p_2}}\int_{t_2}^1\int_{t_1}^1 f(s_1,s_2)\frac{ds_1}{s_1}\frac{ds_2}{s_2}\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \le$$

$$\le p_1'p_2'\left(\int_0^1\left(\int_0^1\left(t_1^{1-\frac{1}{p_1}}t_2^{1-\frac{1}{p_2}}f(t_1,t_2)\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} .$$

Lemma 2. *Let* $1<p_i<\infty$, $1\le q_i\le\infty$, $\frac{1}{p_i}+\frac{1}{p_i'}=1$ *(* $i=1,2$ *). Then*

$$\|f\|_{L_{\bar p,\bar q}([0,1]^2)} \approx \sup_{\|g\|_{L_{\bar p',\bar q'}([0,1]^2)}=1}\int_0^1\int_0^1 f^{*_1,*_2}(t_1,t_2)g(t_1,t_2)dt_1dt_2 .$$

Lemma 3. *Let* $1<p_i<\infty$, $1\le q_i\le\infty$ *(* $i=1,2$ *). Then*

$$\|f\|_{L_{\bar p,\bar q}([0,1]^2)} \approx \left\|f^{\star_1,\star_2}\right\|_{L_{\bar p,\bar q}([0,1]^2)} .$$

Let f and K be measurable on $[0,1]^2$ functions. Let for almost everywhere $(x_1,x_2)\in[0,1]^2$ exists an integral

$$\int_0^1\int_0^1 f(y_1,y_2)K(x_1-y_1,x_2-y_2)dy_1dy_2 ,$$

which is called the convolution of the functions f, K and is denoted by $K*f$.

Lemma 4. *Let* f*,* K *and* g *be measurable on* $[0,1]^2$ *functions. Then*

$$\int_0^1\int_0^1 g(t_1,t_2)(f*K)^{*_1*_2}(t_1,t_2)dt_1dt_2 \le$$

$$\le 4\int_0^1\int_0^1 g^{*_1*_2}(t_1,t_2)\int_0^1\int_0^1 f^{*_1*_2}(s_1,s_2)K^{\star_1,\star_2}(\max(s_1,t_1),\max(s_2,t_2))ds_1ds_2dt_1dt_2.$$

Theorem 1. *Let* $1<q_i<\infty$, $1\le p_i,r_i,h_i,\xi_i,\eta_i<\infty$, *and* $1+\frac{1}{q_i}=\frac{1}{p_i}+\frac{1}{r_i}$, $\frac{1}{h_i}=\frac{1}{\xi_i}+\frac{1}{\eta_i}$ *(* $i=1,2$ *). Suppose* f *and* K *are measurable on* $[0,1]^2$ *functions such that* $f^{\star_1,\star_2}\in L_{\bar p,\bar\xi}([0,1]^2)$ *and* $K^{\star_1,\star_2}\in L_{\bar r,\bar\eta}([0,1]^2)$. *Then* $f*K\in L_{\bar q,\bar h}([0,1]^2)$ *and*

$$\|f*K\|_{L_{\bar q,\bar h}([0,1]^2)}\le C\|f^{\star_1,\star_2}\|_{L_{\bar p,\bar\xi}([0,1]^2)}\|K^{\star_1,\star_2}\|_{L_{\bar r,\bar\eta}([0,1]^2)}. \quad (1)$$

This theorem also covers the limiting cases when at least one of the parameters p_i, r_i is equal to 1. In the case $p_i>1$, $r_i>1$, the functions $f^{\star_1,\star_2}$ and $K^{\star_1,\star_2}$ in (1) can be changed to the functions f and K, respectively.

Let give an example showing the sharpness of the result of Theorem 1, where in inequality (1) for $1<q_i=p_i<\infty$ the factor $\|K^{\star_1,\star_2}\|_{L_{\bar 1,\bar\eta}([0,1]^2)}$ could not be changed to $\|K\|_{L_{\bar 1,\bar\eta}([0,1]^2)}$. That is, in general, for $1<q_i=p_i<\infty$ and $\frac{1}{h_i}=\frac{1}{\xi_i}+\frac{1}{\eta_i}$ the inequality

$$\|f*K\|_{L_{\bar q,\bar h}([0,1]^2)}\le C\|f^{\star_1,\star_2}\|_{L_{\bar p,\bar\xi}([0,1]^2)}\|K\|_{L_{\bar 1,\bar\eta}([0,1]^2)}. \quad (2)$$

does not hold. Also in the case $1<q_i=r_i<\infty$ and $\frac{1}{h_i}=\frac{1}{\xi_i}+\frac{1}{\eta_i}$ the norm $\|f^{\star_1,\star_2}\|_{L_{\bar p,\bar\xi}([0,1]^2)}$ could not be changed to $\|f\|_{L_{\bar p,\bar\xi}([0,1]^2)}$, that is, the following inequality

$$\|f*K\|_{L_{\bar q,\bar h}([0,1]^2)}\le C\|f\|_{L_{\bar 1,\bar\xi}([0,1]^2)}\|K^{\star_1,\star_2}\|_{L_{\bar r,\bar\eta}([0,1]^2)}$$

does not hold.

Example 1. Let $1<q_i<\infty$ *(* $i=1,2$ *)*, $N_1,N_2\in\mathbf{N}$, $N_1<4(e^{q_1}+1)$, $N_2<4(e^{q_2}+1)$. We define $f(t_1,t_2)=\left(\min\left(N_1,\frac{1}{t_1}\right)\right)^{\frac{1}{q_1}}\left(\min\left(N_2,\frac{1}{t_2}\right)\right)^{\frac{1}{q_2}}$ and $K(t_1,t_2)=\min\left(N_1,\frac{1}{t_1}\right)\min\left(N_2,\frac{1}{t_2}\right)$. Then

$$\|f^{\star_1,\star_2}\|_{L_{\bar p,\bar\xi}([0,1]^2)}\|K\|_{L_{\bar r,\bar\eta}([0,1]^2)}\approx(1+\ln N_1)^{\frac{1}{h_1}}(1+\ln N_2)^{\frac{1}{h_2}},$$

$$\|f*K\|_{L_{\bar q,\bar h}([0,1]^2)}\approx(1+\ln N_1)^{1+\frac{1}{h_1}}(1+\ln N_2)^{1+\frac{1}{h_2}}.$$

Thus, (2) implies

$$(1+\ln N_1)(1+\ln N_2)\le C.$$

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