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Quantum gravitational corrections to the geometry of charged AdS black holes

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ABSTRACT

We consider a modified gravity by higher-derivative gravity coupled with non-local terms and Maxwell electrodynamics. By employing the effective field theory framework for quantum gravity, we compute quantum corrections to the entropy of charged AdS black holes, focusing on second-order curvature terms. By incorporating scale-dependent coefficients, we establish that the Wald entropy is renormalization group (RG) invariant, confirming the robustness of the framework. Additionally, quantum corrections to thermodynamic quantities-temperature, pressure, specific heat, and Helmholtz free energy-are derived, all satisfying the first-law of thermodynamics. Specific heat and Helmholtz free energy also exhibit RG invariance, demonstrating stability under scale transformations. These findings highlight the effectiveness of the approach in describing quantum modifications to charged AdS black hole thermodynamics, offering insights into the interplay between quantum gravity and black hole physics.

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1. Introduction

One of the most profound advancements in theoretical physics over the last half-century has been the discovery that black holes exhibit entropy [1], a concept that bridges the domains of thermodynamics, quantum mechanics, and general relativity. Within the framework of Einstein's general relativity, the leading-order contribution to black hole entropy is elegantly characterized as one-quarter of the area of the event horizon. Extensions or modifications to gravitational theories introduce additional terms, enriching the entropy's theoretical structure. Recently, the application of effective field theory methods [2] has enabled a rigorous computation of quantum gravitational corrections to the entropy of a Schwarzschild black hole, leveraging the Wald entropy formalism [3]. A significant point to consider is that quantum gravitational effects can induce modifications to the metric, potentially altering the location of the event horizon. This development marks a significant step toward a deeper understanding of black hole thermodynamics in the context of quantum gravity.

The renormalization group (RG) invariance of black hole entropy is a crucial feature that provides a fundamental consistency check of the quantum corrections. This invariance ensures that physical observables remain independent of the arbitrary energy scale introduced in the effective field theory calculations. When quantum corrections modify the classical Bekenstein-Hawking entropy, the RG invariance serves as a powerful constraint on these modifications, guaranteeing that the corrected entropy maintains its physical meaning across all energy scales. This is particularly important in the context of charged AdS black holes, where multiple energy scales are present due to the interplay between the gravitational, electromagnetic, and cosmological constant contributions.

It is worth highlighting that black hole entropy may also receive corrections from various additional sources. For example, quantum fluctuations of matter fields in the vicinity of black hole spacetimes, along with fluctuations in the spacetime geometry within the canonical quantum gravity framework, contribute to entropy modifications. Notably, these corrections often exhibit a logarithmic dependence, reflecting their distinct origin and nature [4,5].

The exploration of black hole thermodynamics provides a compelling framework for testing and refining our theories of gravity [1,6,7]. This potential arises from the inherent simplicity of thermal systems, which can be characterized by a limited set of macroscopic parameters. In the case of black holes, these parameters are distilled into fundamental quantities such as mass, entropy, charge, angular momentum, and, depending on the gravitational model, a few additional variables. The significance of black hole thermodynamics lies in its profound connection to the principles of quantum gravity–a field undergoing active development through promising approaches like string theory. Particularly, the quantum aspects of gravity become most apparent in the vicinity of the event horizon, where intense gravitational effects render classical thermodynamic treatments insufficient. These considerations underscore the unique role of black holes as both theoretical challenges and windows into the deeper structure of spacetime and quantum gravity.

In this study, we employ the effective field theory framework for quantum gravity to systematically compute quantum gravitational corrections, up to second order in curvature, to the entropy of a physically significant class of black holes-namely, charged AdS black hole. Here, inserting the explicit scale dependence of the coefficients we find that the Wald entropy is RG invariant. Furthermore, we derive the quantum gravitational corrections to relevant thermodynamic quantities satisfying first-law of thermodynamics, namely, temperature, pressure, specific heat and Helmholtz free energy. Here, we check that both specific heat and Helmholtz free energy are RG invariant. It is important to note that our analysis relies on certain well-motivated approximations to make the calculations tractable. Specifically, we work in the regime where both the electric charge Q and the cosmological constant Λ are small compared to the black hole mass ($Q^2 \ll GM^2$ and $|\Lambda|G^2M^2 \ll 1$). While these approximations restrict the generality of our results, they allow us to probe the essential features of quantum corrections in a controlled manner. The physical relevance of these approximations is supported by observational evidence suggesting that astrophysical black holes typically have small charge-to-mass ratios. These findings align with recent studies examining quantum effects in black hole systems through various approaches [8–11].

The paper is structured as follows to systematically explore the quantum gravitational corrections to charged AdS black hole properties: In this section 2, we delve into the modifications to the black hole metric arising from quantum gravitational corrections. We analyze how these corrections influence the structure of the black hole spacetime, particularly focusing on changes to the event horizon and other key geometric properties. The section 3 is dedicated to investigating the effects of quantum gravity on the entropy of black holes. Using the Wald entropy formalism, we compute the corrections to the classical entropy and examine their dependence on the underlying parameters of the theory, such as curvature and coupling constants. Here, in section 4, we explore the quantum gravitational corrections to various thermodynamic quantities of black holes, including temperature, pressure, specific heat, and Helmholtz free energy. We demonstrate how these corrections maintain consistency with the first law of thermodynamics and assess the implications for the stability and equilibrium properties of black holes. In the final section, we summarize the key findings of the paper, emphasizing the implications of quantum gravitational corrections for black hole physics. Additionally, we discuss potential avenues for future research, particularly in the context of testing and validating these theoretical predictions.

This structured approach ensures a comprehensive exploration of the subject, linking theoretical insights with broader implications for black hole thermodynamics and quantum gravity.

2. Quantum gravitational corrections to the classical metric in AdS space

We start by reviewing some known facts about the effective field theory approach to quantum gravity. For a good introduction to this topic, see for example [12,13]. A possible way to study quantum effects in gravity is to modify the pure Einstein-Hilbert action by including additional terms normally suppressed at low energies. The main possibility, at second order in curvature, is the local action

$$\Gamma_L = \int d^4x\,\sqrt{-g}\left(\frac{R+2\Lambda}{16\pi G_N} + c_1(\mu)R^2 + c_2(\mu)R_{\mu\nu}R^{\mu\nu} + c_3(\mu)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right), \tag{2.1}$$

where Λ is the cosmological constant and μ is an energy scale. The exact values of the constants c_1, c_2, c_3 are calculable provided that one assumes an ultra-violet complete theory of quantum gravity, see for example [14,15]. The scale μ enters our effective field theory as the reference point for quantum corrections. These corrections are organized in a derivative expansion, where terms with higher powers of curvature are suppressed by appropriate powers of μ. We terminate our expansion at second order in curvature because in the low-energy regime ($E \ll M_{\rm Pl}$), higher-order terms are suppressed by additional powers of $E/M_{\rm Pl}$. This perturbative treatment assumes that quantum corrections generate small modifications to the classical Einstein-Hilbert action. However, this assumption could break down near the Planck scale or in regions of extreme curvature, where the higher-order terms become comparable to the Einstein-Hilbert term. In such cases, the effective field theory approach would need to be modified or abandoned.

By integrating out fluctuations of the graviton and of any matter field irrelevant to the problem under consideration, one also gets a non-local (NL) effective action [16–22], which, at second order in curvature, is

$$\Gamma_{NL} = -\int d^4x\sqrt{-g}\left[\alpha R\ln\left(\frac{\Box}{\mu^2}\right)R + \beta R_{\mu\nu}\ln\left(\frac{\Box}{\mu^2}\right)R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}\ln\left(\frac{\Box}{\mu^2}\right)R^{\mu\nu\rho\sigma}\right], \tag{2.2}$$

where α, β and γ are constants which can be computed in a model-independent way [23]. The operator $\ln\left(\Box/\mu^2\right)$ has the integral representation [24]

$$\ln\left(\frac{\Box}{\mu^2}\right) = \int_0^{+\infty} ds\left(\frac{1}{\mu^2+s} - \frac{1}{\Box+s}\right). \tag{2.3}$$

Using both the local and non-local Gauss-Bonnet identities [25], it is possible to eliminate the Riemann tensor by redefining the coefficients as

$$\begin{aligned} &c_1 \to \bar{c}_1 = c_1 - c_3, \quad c_2 \to \bar{c}_2 = c_2 + 4c_3, \quad c_3 \to 0,\\ &\quad\alpha \to \bar{\alpha} = \alpha - \gamma, \quad \beta \to \bar{\beta} = \beta + 4\gamma, \quad \gamma \to 0. \end{aligned} \tag{2.4}$$

We can couple gravity to electromagnetism by adding the Maxwell action

$$\Gamma_M = -\frac{1}{4}\int d^4x\,\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu}, \tag{2.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor and A_μ is the electromagnetic potential. In this paper we then consider the full action as

$$\Gamma = \Gamma_L + \Gamma_{NL} + \Gamma_M \equiv \int d^4x\,\sqrt{-g}\,\mathcal{L}. \tag{2.6}$$

The Maxwell equations, obtained by varying (2.6) with respect to A_μ, are

$$g^{\mu\nu}\nabla_\mu F_{\nu\tau} = 0. \tag{2.7}$$

While quantum effects could in principle also modify the gauge field, we focus on gravitational corrections in this work. The quantum corrections to the electromagnetic field would enter at order α (the fine structure constant) in the effective action, while our gravitational corrections are of order G_N. Given that $\alpha \ll 1$, the gravitational corrections dominate over electromagnetic quantum effects in the regime we consider. The quantum corrected Einstein equations, obtained by varying (2.6) with respect to the metric, are

$$\frac{1}{8\pi G_N}\left(G_{\mu\nu} - \Lambda g_{\mu\nu}\right) + 2\left(H_{\mu\nu} + K_{\mu\nu}\right) = T_{\mu\nu}, \tag{2.8}$$

where $T_{\mu\nu}$ and $G_{\mu\nu}$ are respectively the energy-momentum tensor and Einstein tensor, which are given by

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{2.9}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{2.10}$$

However, $H_{\mu\nu}$ and $K_{\mu\nu}$ encode the quantum effects. The local contribution is

$$H_{\mu\nu} = \bar{c}_1\left(2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 - 2\nabla_\mu\nabla_\nu R + 2g_{\mu\nu}\Box R\right)$$
$$+ \bar{c}_2\left(-\frac{1}{2}g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} + 2R^{\rho\sigma}R_{\mu\rho\nu\sigma} - \nabla_\mu\nabla_\nu R + \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box R\right), \tag{2.11}$$

while the non-local contribution is

$$K_{\mu\nu} = -2\bar{\alpha}\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\ln\left(\frac{\Box}{\mu^2}\right)R - \bar{\beta}\left(\delta^\rho{}_\mu R_{\nu\sigma} + \delta^\rho{}_\nu R_{\mu\sigma}\right.$$
$$\left. - \frac{1}{2}g_{\mu\nu}R^\rho{}_\sigma + \delta^\rho{}_\mu g_{\nu\sigma}\Box + g_{\mu\nu}\nabla^\rho\nabla_\sigma - \delta^\rho{}_\mu\nabla_\sigma\nabla_\nu - \delta^\rho{}_\nu\nabla_\sigma\nabla_\mu\right)\ln\left(\frac{\Box}{\mu^2}\right)R^\sigma{}_\rho. \tag{2.12}$$

We now solve the quantum corrected Einstein and Maxwell equations using perturbation theory in the gravitational coupling G_N. Classically, a four-dimensional charged AdS black hole is described by its mass M, its charge Q and the AdS radius L, which is related to the cosmological constant via

$$L = \sqrt{\frac{3}{|\Lambda|}}. \tag{2.13}$$

In turn, the cosmological constant is proportional to the vacuum energy density ρ as $\Lambda = 8\pi G_N \rho$. For our future calculations we find convenient to keep the classical metric written in terms of ρ. We now consider a small perturbation of order $\mathcal{O}\left(G_N^2\right)$ around the classical solution:

$$g_{\mu\nu} = g^{AdS}_{\mu\nu} + g^q_{\mu\nu}, \tag{2.14}$$

where

$$ds^2_{AdS} = g^{AdS}_{\mu\nu}dx^\mu dx^\nu,$$
$$= -\left(1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{r^2} + \frac{1}{3}8\pi\rho G_N r^2\right)dt^2,$$
$$+\left(1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{r^2} + \frac{1}{3}8\pi\rho G_N r^2\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{2.15}$$

The quantum effects are encoded in $g^q_{\mu\nu}$. We set $g^q_{\theta\theta} = g^q_{\phi\phi} = 0$ and introduce two functions $\Sigma(r)$ and $\Omega(r)$ such that

$$ds^2_q = g^q_{\mu\nu}dx^\mu dx^\nu = G_N^2\Delta(r)dt^2 + G_N^2\Sigma(r)dr^2. \tag{2.16}$$

Our metric ansatz preserves the spherical symmetry and time-independence of the classical solution while allowing for quantum corrections to the radial and temporal components. This form is the most general static, spherically symmetric metric compatible with the symmetries of our problem. The functions $\Delta(r)$ and $\Sigma(r)$ capture all possible quantum corrections that preserve these symmetries. The full metric is then

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$
$$= \left[-\left(1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{r^2} + \frac{1}{3}8\pi\rho G_N r^2\right) + G_N^2\Delta(r)\right]dt^2$$
$$+\left[\left(1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{r^2} + \frac{1}{3}8\pi\rho G_N r^2\right)^{-1} + G_N^2\Sigma(r)\right]dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{2.17}$$

Quantum effects can also shift the classical value of the non-vanishing components of the electromagnetic tensor $F_{\mu\nu}$. Thus, we introduce a function $\Omega(r)$ such that

$$F_{tr} = -F_{rt} = \frac{Q}{r^2} + G_N^2\Omega(r). \tag{2.18}$$

Next, we plug the metric (2.17) and the components of the electromagnetic tensor (2.18) into the equations (2.7), (2.8), keeping only terms up to order $\mathcal{O}\left(G_N^2\right)$. Since $H_{\mu\nu}$ and $K_{\mu\nu}$ are quadratic in curvature, they just need to be evaluated on the classical charged AdS solution, i.e. (2.8) can be expressed schematically as

$$\frac{1}{8\pi G_N}\left(G_{\mu\nu} - \Lambda g_{\mu\nu}\right)\left[g^{AdS} + g^q\right] + 2\left(H_{\mu\nu}\left[g^{AdS}\right] + K_{\mu\nu}\left[g^{AdS}\right]\right) = T_{\mu\nu}\left[g^{AdS} + g^q\right]. \tag{2.19}$$

The results of the action of $\ln\left(\Box/\mu^2\right)$ on several radial functions were collected in [26]. However, unlike the pure Reissner-Nordström solution, the charged AdS spacetime has the non-vanishing Ricci curvature $R = -32G\pi\rho$. Hence, we need to additionally compute the action of $\ln\left(\Box/\mu^2\right)$ on a constant. Using the formula [27]

$$\ln\left(\frac{\Box}{\mu^2}\right) f(r) = \frac{1}{r}\int_0^{+\infty} dr'\, \frac{r'}{r+r'} f(r') - \lim_{\epsilon\to 0^+}\left\{ \frac{1}{r}\int_0^{r-\epsilon} dr'\, \frac{r'}{r-r'} f(r') + \frac{1}{r}\int_{r+\epsilon}^{+\infty} dr'\, \frac{r'}{r'-r} f(r') + 2f(r)\left[\gamma_E + \ln(\mu\epsilon)\right]\right\}, \tag{2.20}$$

the result is

$$\ln\left(\frac{\Box}{\mu^2}\right)\cdot 1 = -2\left(\ln(\mu r) + \gamma_E - 1\right), \tag{2.21}$$

where γ_E is the Euler-Mascheroni constant. The corresponding calculation is collected in Appendix A, where we show how to regularize the integrals.

While our work builds upon the methodology developed in Ref. [26] for Reissner-Nordström black holes, the present study extends those results in several important directions. The addition of a cosmological constant introduces new geometric structures and coupling terms that fundamentally alter the nature of quantum corrections. This modification requires novel computational techniques to handle the interplay between charge and cosmological constant contributions. Furthermore, our analysis of RG invariance in this more complex setting provides new insights into the universal features of quantum gravitational corrections to black hole thermodynamics.

The final solution of the quantum corrected equations of motion (2.19) and (2.7) is

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{2.22}$$

with

$$f(r) = 1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{r^2} + \frac{8\pi}{3}\rho G_N r^2 + 256\pi^2\rho G_N^2\left[4c_1 + c_2 + 2(4\alpha+\beta)\ln(\mu r)\right] - \frac{32\pi G_N^2 Q^2}{r^4}\left[c_2 + 4c_3 + (\beta+4\gamma)\left(2\ln(\mu r) + 2\gamma_E - 3\right)\right], \tag{2.23}$$

$$g(r) = 1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{r^2} + \frac{1}{3}8\pi\rho G_N r^2 + 512\pi^2\rho G_N^2(4\alpha+\beta) - \frac{64\pi G_N^2 Q^2}{r^4}\left[c_2 + 4c_3 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - 2\right)\right], \tag{2.24}$$

and

$$F_{tr} = -F_{rt} = \frac{Q}{r^2} + \frac{16\pi G_N^2 Q^3}{r^6}\left[c_2 + 4c_3 + (\beta+4\gamma)\left(2\ln(\mu r) + 2\gamma_E - 5\right)\right] + \frac{128\pi^2\rho G_N^2 Q}{r^2}\left[4c_1 + c_2 + (4\alpha+\beta)\ln(\mu r)\right]. \tag{2.25}$$

In the limit $\rho \to 0$ (or equivalently $L \to \infty$) one correctly recovers the solution obtained in [26] for the pure Reissner-Nordström black hole. Although the metric seems to depend on the arbitrary energy scale μ, the renormalized constants c_1, c_2, and c_3 also carry an explicit scale dependence [28]:

$$\begin{aligned} c_1(\mu) &= c_1(\mu_*) - \alpha\ln\left(\frac{\mu^2}{\mu_*^2}\right), \\ c_2(\mu) &= c_2(\mu_*) - \beta\ln\left(\frac{\mu^2}{\mu_*^2}\right), \\ c_3(\mu) &= c_3(\mu_*) - \gamma\ln\left(\frac{\mu^2}{\mu_*^2}\right), \end{aligned} \tag{2.26}$$

where μ_* is some fixed scale where the effective theory is matched onto the full theory. Taking into account (2.26), one sees that the metric ultimately does not depend on μ, as it must be.

The geometry described by the quantum-corrected metric (2.22) maintains the basic features of the classical Reissner-Nordström-AdS solution, with modifications controlled by powers of G_N. The horizon structure can be analyzed by finding the zeros of $f(r)$. To leading order in the quantum corrections, the outer horizon is shifted by terms of order G_N^2 relative to the classical position, while maintaining the basic causal structure. The extremality condition is similarly modified but preserved. The central singularity at $r = 0$ remains, as expected since our effective field theory approach cannot resolve such short-distance features. All corrections to geometric invariants remain small compared to their classical values away from $r = 0$, confirming the consistency of our perturbative treatment in the non-extremal regime. The extremality condition is particularly interesting as it represents the limit where the inner and outer horizons coincide. In the classical RN-AdS case, extremality occurs when $Q^2 = G_N M^2$ (neglecting AdS corrections). Including quantum corrections, this condition is modified to:

$$Q^2 = G_N M^2 + G_N^2 M^2[c_2 + 4c_3 + 2(\beta + 4\gamma)(\ln(2G_N M\mu) + \gamma_E - 2)] + O(G_N^3) \tag{2.27}$$

This shows that quantum effects increase the charge-to-mass ratio required for extremality, effectively making it harder for the black hole to become extremal. This is a physically significant modification as it suggests quantum effects tend to counteract the formation of extremal black holes. Our analysis relies on the approximations $Q^2 \ll GM^2$ and $|\Lambda|G^2M^2 \ll 1$, which constrain the validity of our results. Near the Planck scale, where quantum effects become dominant, higher-order curvature terms may contribute significantly, potentially modifying our conclusions. Similarly, for near-extremal black holes ($Q^2 \approx GM^2$) or strong AdS curvature ($|\Lambda|G^2M^2 \sim 1$), additional effects could emerge. In these regimes, non-perturbative approaches or the inclusion of higher-order terms in the effective action may be necessary. The perturbative nature of our framework means our results should be interpreted cautiously as one approaches these limits.

3. Corrected entropy

In this section we compute the quantum gravitational corrections to the entropy. Analogously to what done in [26], we consider a small electric charge. In addition, we assume the cosmological constant to be small, or, equivalently, the AdS radius to be large. We then compute the entropy up to orders $\mathcal{O}\left(Q^2\right), \mathcal{O}(\rho)$. The corrections to the metric imply a shift to the classical outer horizon radius as

$$\begin{aligned} r_h =& 2G_N M - \frac{Q^2}{2M} + \frac{8\pi Q^2}{G_N M^3}\Big[c_2 + 4c_3 + 2(\beta+4\gamma)\left(\ln\left(2G_N M\mu\right) + \gamma_E - 2\right)\Big] \\ &+ \frac{8192\pi^3 G_N Q^2 \rho}{M^3}(4\alpha+\beta)\Big[2c_2 + 8c_3 + (\beta+4\gamma)\left(4\ln(2GM\mu) + 4\gamma_E - 9\right)\Big] \\ &+ \frac{512\pi^2 G_N^2 Q^2 \rho}{3M}\Big[c_2 + 4c_3 + (\beta+4\gamma)\left(2\ln(2G_N\mu) + 2\gamma_E - 5\right)\Big]. \end{aligned} \tag{3.1}$$

Our treatment of both electric charge and cosmological constant as small parameters ($Q^2 \ll G_N M^2$ and $|\Lambda| \ll 1/L^2$) is motivated by both technical and physical considerations. From a technical standpoint, this approximation makes the calculations tractable by allowing us to organize our perturbative expansion clearly. Physically, for astrophysical black holes, the electric charge is typically small due to efficient neutralization by surrounding plasma, with estimates suggesting $Q/M \sim 10^{-18}$ for typical black holes. Similarly, the observed cosmological constant is very small in Planck units, justifying our treatment of AdS corrections as subdominant. These approximations do constrain the generality of our results. Our expressions for quantum corrections are strictly valid only in this regime, and additional corrections would appear if either Q or Λ were allowed to be large. However, this limitation is mitigated by the fact that we capture the leading-order behavior in the physically relevant regime for most astrophysical black holes.

We introduce the totally antisymmetric symbol $\epsilon_{\mu\nu}$

$$\epsilon_{\mu\nu} = \begin{cases} \sqrt{f(r)/g(r)} & \text{if } (\mu,\nu) = (t,r) \\ -\sqrt{f(r)/g(r)} & \text{if } (\mu,\nu) = (r,t) \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

so that the Wald formula for the entropy is [29]

$$\begin{aligned} S_{\text{Wald}} &= -2\pi \int_{r=r_h} d\Sigma\, \epsilon_{\mu\nu}\epsilon_{\rho\sigma} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}, \\ &= -8\pi\sqrt{\frac{f(r_h)}{g(r_h)}} \int_{r=r_h} d\Sigma\, \frac{\partial \mathcal{L}}{\partial R_{rtrt}} = -8\pi\sqrt{\frac{f(r_h)}{g(r_h)}} \frac{\partial \mathcal{L}}{\partial R_{rtrt}}\Big|_{r=r_h} 4\pi r_h^2. \end{aligned} \tag{3.3}$$

All the terms in the Lagrangian have to be considered without invoking the Gauss-Bonnet theorem. Furthermore, the following relations are useful:

$$\frac{\partial}{\partial R_{\mu\nu\rho\sigma}} R = \frac{1}{2}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right), \tag{3.4}$$

$$\frac{\partial}{\partial R_{\mu\nu\rho\sigma}} R_{\alpha\beta}R^{\alpha\beta} = \frac{1}{2}\left(g^{\mu\rho}R^{\nu\sigma} - g^{\nu\rho}R^{\mu\sigma} - g^{\mu\sigma}R^{\nu\rho} + g^{\nu\sigma}R^{\mu\rho}\right). \tag{3.5}$$

$$\frac{\partial}{\partial R_{\mu\nu\rho\sigma}} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 2R^{\mu\nu\rho\sigma}. \tag{3.6}$$

In applying the formula one also needs the results of the action of $\ln\left(\Box/\mu^2\right)$ on some radial functions. These were computed in [26]. However, the Riemann and Ricci tensors obtained from (2.22) contain new terms proportional to ρ. In particular, one needs to know the action of $\ln\left(\Box/\mu^2\right)$ on $1/r$ and $1/r^2$. The corresponding calculations are collected in Appendix A. The final result for the entropy of a charged AdS black hole is, up to order $\mathcal{O}\left(Q^4,\rho^2\right)$,

$$S_{\text{Wald}} = \frac{A}{4G} - 2\pi Q^2 + S_{Sch} + S_{RN} + S_{AdS} + \mathcal{O}\left(Q^4,\rho^2\right), \tag{3.7}$$

where $A = 16\pi G^2 M^2$ is the classical area of the event horizon of a Schwarzschild black hole, S_{Sch} is the quantum correction to the entropy of a Schwarzschild black hole, S_{RN} represents the corrections to the entropy of a Reissner-Nordström black hole and S_{AdS} are the new corrections containing the vacuum energy density ρ. The expression of S_{Sch} is [26]

$$S_{Sch} = 64\pi^2 c_3 + 64\pi^2\gamma\left[4\ln\left(2G_N M\mu\right) + 2\gamma_E - 2\right]. \tag{3.8}$$

The expression of S_{RN} is [26]

$$\begin{aligned}
S_{RN} = {} & \frac{4\pi^2 Q^2}{G_N M^2}\left[5(c_2+4c_3) + \beta\left(10\gamma_E - 21\right) + 8\gamma\left(5\gamma_E - 11\right) + 10(\beta+4\gamma)\ln\left(2G_N M\mu\right)\right] \\
& + \frac{64\pi^3 Q^2}{9G_N^2 M^4}\left\{54(\beta+4\gamma)\left[c_1 + 2\alpha\ln\left(2G_N M\mu\right)\right] + 12(12\gamma_E - 23)\beta\left[c_2 + 2\beta\ln\left(2G_N M\mu\right)\right]\right. \\
& + 48(48\gamma_E - 97)\gamma\left[c_3 + 2\gamma\ln\left(2G_N M\mu\right)\right] + 6\left[c_2\gamma(96\gamma_E - 185) + c_3\beta(96\gamma_E - 193)\right. \\
& \left. + \ 12\beta\gamma(32\gamma_E - 63)\ln\left(2G_N M\mu\right)\right] + 36\left[c_2^2 + 4c_2\beta\ln\left(2G_N M\mu\right) + 4\beta^2\ln^2\left(2G_N M\mu\right)\right] \\
& + 576\left[c_3^2 + 4c_3\gamma\ln\left(2G_N M\mu\right) + 4\gamma^2\ln^2\left(2G_N M\mu\right)\right] + 288\left[c_2 c_3 + 2c_2\gamma\ln\left(2G_N M\mu\right)\right. \\
& \left. + \ 2c_3\beta\ln\left(2G_N M\mu\right) + 4\beta\gamma\ln^2\left(2G_N M\mu\right)\right] + (\beta+4\gamma)\left[9\alpha(12\gamma_E - 25)\right. \\
& \left.\left. + \ 8\beta\left(3\gamma_E(6\gamma_E - 23) + 40 + 3\pi^2\right) + 4\gamma\left(6\gamma_E\left(24\gamma_E - 97\right) + 331 + 30\pi^2\right)\right]\right\}.
\end{aligned} \tag{3.9}$$

The expression of S_{AdS} is quite lengthy and it is relegated in Appendix B. Inserting the explicit scale dependence of the coefficients according to (2.26), one can check with any computer software that

$$\frac{\partial S_{\text{Wald}}}{\partial\mu} = 0, \tag{3.10}$$

i.e. the entropy is RG invariant. The μ-independent entropy is then effectively obtained with the substitutions $\mu \to \mu_*$ and $c_i \to c_i(\mu_*)$ everywhere.

The RG invariance of the Wald entropy, demonstrated by equation (3.10), emerges as a consequence of the consistency of our quantum corrected theory. While this invariance might appear automatic given that we are computing a physical observable, its explicit verification is non-trivial due to the complex interplay between the scale-dependent coefficients $c_i(\mu)$ and the logarithmic quantum corrections. The fact that all μ-dependent terms cancel exactly provides a robust check on our calculations and confirms that our quantum corrections preserve the fundamental thermodynamic properties of black holes. This invariance holds order by order in our perturbative expansion, suggesting it is a general feature of quantum corrections to black hole entropy rather than an artifact of our approximations.

4. Thermodynamics

In this section we compute the quantum gravitational corrections to relevant thermodynamics quantities, namely temperature, pressure, specific heat and Helmholtz free energy. In analyzing the thermodynamics of quantum-corrected black holes, we must carefully define the relevant quantities appearing in the first law. Our framework differs from the extended black hole thermodynamics (black hole chemistry [30,31]) approach where the cosmological constant is treated as a thermodynamic pressure. Instead, following [32], we derive the pressure P from the requirement that the first law holds in its standard form. The volume $V = \frac{4}{3}\pi r_h^3$ is the geometric volume enclosed by the horizon, though we note this is a naive geometric definition rather than a thermodynamic volume as would appear in extended thermodynamics. As in Section 3, our analysis here maintains the assumptions of small charge ($Q^2 \ll G_N M^2$) and small cosmological constant. These approximations allow us to obtain explicit expressions for the thermodynamic quantities but restrict the validity of our results to this regime.

The first law of thermodynamics for a black hole with mass M and charge Q is given by

$$dM = TdS - Qd\Phi + PdV, \tag{4.1}$$

where P, T, V, Φ are the pressure, the temperature, the volume and the electric potential, respectively. The electric potential is

$$\begin{aligned}
\Phi &= \int_{r_h}^{+\infty} dr'\, F_{tr} = \int_{r_h}^{+\infty} dr'\left(\frac{Q}{r'^2} + G_N^2\Omega(r')\right), \\
&= \frac{Q}{2G_N M} + \frac{64\pi^2\rho G Q}{M}\left[4c_1 + c_2 + (8\alpha + 2\beta)\left(1 + \ln(2G_N M\mu)\right)\right] + \mathcal{O}\left(Q^3,\rho^2\right).
\end{aligned} \tag{4.2}$$

The temperature is given by

$$T = \frac{1}{4\pi}\sqrt{\frac{df(r)}{dr}\frac{dg(r)}{dr}}\Big|_{r=r_h}, \\ = \frac{1}{8\pi G_N M} + \frac{8}{3}\rho M G_N^2 - \frac{2\rho G_N Q^2}{3M} + T_{RN} + T_{AdS} + \mathcal{O}\left(Q^4, \rho^2\right), \tag{4.3}$$

where T_{RN} represents the quantum gravitational corrections to a Reissner-Nordström black hole and T_{AdS} contains the new corrections proportional to ρ. The corresponding expressions are

$$T_{RN} = \frac{Q^2}{4G_N^3 M^5}\Big[2(c_2+4c_3) + (\beta+4\gamma)\left(4\gamma_E - 9 + 4\ln(2G_N M\mu)\right)\Big], \tag{4.4}$$

and

$$T_{AdS} = \frac{32\pi\rho G_N}{M}(4\alpha+\beta) - \frac{8\pi\rho Q^2}{3M^3}\Big[4(c_2+4c_3) - 12\alpha - (27-8\gamma_E)\beta + 32(-3+\gamma_E)\gamma \\ + 8(\beta+4\gamma)\ln(2G_N M\mu)\Big] - \frac{128\pi^2\rho Q^2}{G_N M^5}(4\alpha+\beta)\Big[17(c_2+4c_3) \\ + 2(\beta+4\gamma)\left(17\gamma_E - 38 + 17\ln(2G_N M\mu)\right)\Big]. \tag{4.5}$$

This relation utilizes the behavior of the temperature of a AdS Reissner-Nordström black hole as a function of M. Quantum effects also produce corrections to the pressure [26]. For a charged AdS_4 black hole, the pressure is

$$P = -\frac{T\frac{dS}{dM} - Q\frac{d\Phi}{dM} - 1}{\frac{dV}{dM}}, \\ = -\frac{Q^2}{64\pi G_N^4 M^4} - \frac{2}{3}\rho + P_{sch} + P_{RN} + P_{AdS} + \mathcal{O}\left(Q^4, \rho^2\right), \tag{4.6}$$

where $V = 4/3\pi r_h^3$ is the volume, $P_{sch} = -\gamma/(G_N^4 M^4)$ is the quantum correction for a Schwarzschild black hole, P_{RN} corresponds to the corrections for a Reissner-Nordström black hole and P_{AdS} represents the new contributions containing the AdS_4 radius. The expression of P_{RN} is [26]

$$P_{RN} = \frac{Q^2}{32G_N^5 M^6}\Big[c_2 + 4c_3 + 2\beta(\gamma_E-4) + 8\gamma(\gamma_E-5) + 2(\beta+4\gamma)\ln(2G_N M\mu)\Big] \\ + \frac{\pi Q^2}{9G_N^6 M^8}\Big\{54(\beta+4\gamma)\Big[c_1 + 2\alpha\ln\left(2G_N M\mu\right)\Big] + 24(6\gamma_E-13)\beta\Big[c_2 + 2\beta\ln\left(2G_N M\mu\right)\Big] \\ + 768(3\gamma_E-7)\gamma\Big[c_3 + 2\gamma\ln\left(2G_N M\mu\right)\Big] + 6\Big[c_2\gamma(96\gamma_E-215) + c_3\beta(96\gamma_E-217) \\ + 96\beta\gamma(4\gamma_E-9)\ln\left(2G_N M\mu\right)\Big] + 36\Big[c_2^2 + 4c_2\beta\ln\left(2G_N M\mu\right) + 4\beta^2\ln^2\left(2G_N M\mu\right)\Big] \\ + 576\Big[c_3^2 + 4c_3\gamma\ln\left(2G_N M\mu\right) + 4\gamma^2\ln^2\left(2G_N M\mu\right)\Big] + 288\Big[c_2c_3 + 2c_2\gamma\ln\left(2G_N M\mu\right) \\ + 2c_3\beta\ln\left(2G_N M\mu\right) + 4\beta\gamma\ln^2\left(2G_N M\mu\right)\Big] + (\beta+4\gamma)\Big[36\alpha(3\gamma_E-7) \\ + 2\beta\left(8\gamma_E(9\gamma_E-39) + 229 + 12\pi^2\right) + \gamma\left(192\gamma_E(3\gamma_E-14) + 2095 + 120\pi^2\right)\Big]\Big\}. \tag{4.7}$$

The expression of P_{AdS} is reported in Appendix B. The satisfaction of the first law by our quantum-corrected quantities follows from our construction of the pressure term, which is defined precisely to ensure the first law holds. While this might appear to make the verification trivial, it provides a useful consistency check on our calculations and ensures our quantum corrections maintain the basic thermodynamic structure of black hole physics. Plugging the explicit scale dependence (2.26), it is easy to verify with any computer software that the pressure is RG invariant,

$$\frac{\partial P}{\partial\mu} = 0. \tag{4.8}$$

Next, we compute the specific heat which is an important quantity related to the thermodynamics stability. The black hole is stable if $C \geq 0$. One obtains

$$C = \frac{T}{M}\frac{dS}{dT} = \frac{T}{M}\frac{1}{\frac{dT}{dM}}\frac{dS}{dM} = -8\pi GM + C_{Sch} + C_{RN} + C_{AdS} + \mathcal{O}\left(Q^4, \rho^2\right), \tag{4.9}$$

where C_{Sch} is the quantum contribution to a Schwarzschild black hole given as

$$C_{Sch} = -\frac{128\pi^2\gamma}{M}, \tag{4.10}$$

and C_{RN} is the quantum correction to a Reissner-Nordström black hole given as

$$\begin{aligned} C_{RN} &= \frac{8\pi^2 Q^2}{G_N M^3}\left[21(c_2+4c_3)+2\beta(21\gamma_E-53)+4\gamma(42\gamma_E-107)+42(\beta+4\gamma)\ln(2GM\mu)\right] \\ &+ \frac{512\pi^3 Q^2}{9G^2M^5}\Big\{18c_2^2+288c_3^2+3c_3\left[(-217+96\gamma_E)\beta+8(-103+48\gamma_E)\gamma\right] \\ &+3c_2\left[48c_3+4(-13+6\gamma_E)\beta+(-197+96\gamma_E)\gamma\right]+(\beta+4\gamma)\left[27c_1+229\beta+773\gamma\right. \\ &+6\left.\left[3(-7+3\gamma_E)\alpha+12\gamma_E^2(\beta+4\gamma)+2\pi^2(\beta+5\gamma)-2\gamma_E(26\beta+103\gamma)\right]\right] \\ &+6(\beta+4\gamma)\ln\left(2G_N M\mu\right)\left[12c_2+48c_3+9\alpha+(-52+24\gamma_E)\beta\right. \\ &+\left.(-206+96\gamma_E)\gamma+12(\beta+4\gamma)\ln\left(2G_N M\mu\right)\right]\Big\}. \end{aligned} \tag{4.11}$$

The purely AdS part is given in Appendix B.

Now let us calculate the Helmholtz free energy as

$$\begin{aligned} F &= E-TS = M+Q\Phi-TS \\ &= \frac{M}{2}+\frac{3Q^2}{4G_N M}+8\pi G_N^2 MQ^2\rho+F_{Sch}+F_{RN}+F_{AdS}+\mathcal{O}\left(Q^3,\rho^2\right), \end{aligned} \tag{4.12}$$

with

$$F_{Sch} = -\frac{8\pi}{G_N M}\left[c_3+2\gamma\left(\ln(2G_N M\mu)+\gamma_E-1\right)\right], \tag{4.13}$$

and

$$\begin{aligned} F_{RN} &= -\frac{\pi Q^2}{2G_N^2 M^3}\left[9(c_2+4c_3)+3\beta(6\gamma_E-13)+8\gamma(9\gamma_E-20)+18(\beta+4\gamma)\ln(2G_N M\mu)\right] \\ &-\frac{8\pi^2Q^2}{9G_N^3M^5}\Big\{36c_2^2+720c_3^2+6c_2[54c_3+(-46+24\gamma_E)\beta+(-197+108\gamma_E)\gamma] \\ &+24c_3\left[(-55+27\gamma_E)\beta+(-233+120\gamma_E)\gamma\right]+(\beta+4\gamma)\Big[54c_1+9(-25+12\gamma_E)\alpha \\ &+8\left[40+3\gamma_E(-23+6\gamma_E)+3\pi^2\right]\beta+4\left[412+3\gamma_E(-233+60\gamma_E)+30\pi^2\right]\gamma\Big] \\ &+12\ln\left(2G_N M\mu\right)\Big[(\beta+4\gamma)\left[9\alpha+(-46+24\gamma_E)\beta+(-233+120\gamma_E)\gamma\right] \\ &+6c_2(2\beta+9\gamma)+6c_3(9\beta+40\gamma)+12(\beta+4\gamma)(\beta+5\gamma)\ln\left(2G_N M\mu\right)\Big]\Big\}. \end{aligned} \tag{4.14}$$

The expression for F_{AdS} is reported in Appendix B. Again, one can verify that both specific heat and Helmholtz free energy are RG invariant as

$$\frac{\partial C}{\partial\mu}=0, \qquad \frac{\partial F}{\partial\mu}=0. \tag{4.15}$$

The RG invariance of the thermodynamic quantities, demonstrated by equations (4.15), extends the result found for the entropy in Section 3. While physical observables must be independent of the arbitrary scale μ, the explicit verification of this invariance provides a non-trivial check due to the intricate structure of quantum corrections. Just as with entropy, this independence emerges from the scale-dependence of the coefficients $c_i(\mu)$ precisely canceling the explicit logarithmic terms, reflecting the consistency of our effective field theory treatment.

The thermal stability of our quantum-corrected black hole can be analyzed through the sign of the specific heat C. From equation (4.9), we see that the leading term $-8\pi GM$ corresponds to the classical Schwarzschild instability. The quantum corrections, being of order $O(Q^2)$, and the AdS corrections, which we treat as small, provide modifications but do not overcome this leading instability in our perturbative regime. This is consistent with what we expect - small quantum corrections should not drastically alter the classical stability properties.

It is worth noting that beyond our approximations, charged AdS black holes can exhibit rich phase behavior, including Hawking-Page transitions when charge and cosmological constant are allowed to be larger [33]. However, such transitions lie outside our perturbative treatment where $Q^2 \ll G_N M^2$ and $|\Lambda| \ll 1/L^2$. A full analysis of phase structure would require going beyond these approximations.

5. Conclusions and outlook

In this study, we have employed the effective field theory framework for quantum gravity to rigorously compute quantum gravitational corrections to the entropy of charged AdS black holes. Our calculations have specifically accounted for contributions up to the second order in curvature, ensuring a systematic and comprehensive approach to capturing higher-order effects. By explicitly incorporating the scale dependence of the coefficients in our analysis, we have established that the Wald entropy, a key thermodynamic quantity, exhibits RG invariance. This invariance underscores the robustness of the entropy calculations under changes in the energy scale, providing a critical consistency check for the theoretical framework.

In addition to entropy, we have derived quantum gravitational corrections to several fundamental thermodynamic quantities associated with black holes, including temperature, pressure, specific heat, and Helmholtz free energy. These quantities have been shown to satisfy the first law of thermodynamics, thereby reinforcing the internal consistency of the corrections within the thermodynamic framework. Furthermore, our detailed examination has revealed that specific heat and Helmholtz free energy, in particular, maintain their RG invariance even after the inclusion of quantum gravitational effects. This finding is significant, as it confirms the stability of these thermodynamic properties under scale transformations and highlights their compatibility with the principles of effective field theory.

Our findings provide important insights into quantum corrections to black hole thermodynamics that can be compared with other approaches in the literature. The obtained corrections to thermodynamic quantities show interesting parallels with one-loop quantum corrections derived through different methods. For instance, recent work by Turiaci and Verlinde [34] investigating one-loop corrections to black hole entropy finds logarithmic corrections similar in structure to ours, though arising from different physical considerations. Their approach through JT gravity [35] provides complementary evidence for the robustness of quantum modifications to black hole thermodynamics.

The quantum corrections we find can also be compared with results from exact quantum matter backreaction studies. Work by Emparan, Grumiller and Tanabe [36] on quantum black holes incorporating full backreaction of quantum fields shows how quantum effects modify the classical geometry. Similarly, examining quantum effects through holographic methods reveal corrections to thermodynamic quantities [37–39] that share some qualitative features with our findings, particularly in how quantum effects modify the relationship between entropy and area. The consistency between these different approaches, despite their distinct starting points, suggests the robustness of quantum corrections to black hole thermodynamics.

Our results also connect with developments in extended black hole thermodynamics, where the cosmological constant is treated as a thermodynamic pressure. The seminal work by Kastor, Ray and Traschen [40], as well as extensive studies by Kubiznak, Mann and collaborators [41], has established a rich thermodynamic structure for AdS black holes. Our quantum corrections provide a natural extension to this framework, showing how quantum effects modify the classical extended thermodynamic relations. Particularly interesting is how our RG-invariant corrections maintain consistency with the first law of black hole thermodynamics while incorporating quantum effects.

These comparisons suggest that our results capture universal features of quantum corrections to black hole thermodynamics. While different approaches emphasize different aspects of the quantum effects, the emergence of consistent patterns across multiple methods strengthens confidence in the robustness of our findings. Future work could further explore these connections, particularly in regimes where different approaches overlap.

The physical relevance of our approximations is supported by observational evidence. Astrophysical black holes typically exhibit charge-to-mass ratios of $Q/M \sim 10^{-18}$, well within our assumption of $Q^2 \ll GM^2$. This small charge arises from efficient neutralization mechanisms in astrophysical environments. Similarly, the observed cosmological constant satisfies $|\Lambda|G^2M^2 \ll 1$ for known black hole masses. Our finding that quantum effects increase the charge-to-mass ratio required for extremality suggests a quantum mechanical mechanism opposing the formation of extremal black holes, potentially explaining their apparent absence in observational data. This alignment between our theoretical predictions and observational constraints reinforces the physical relevance of our effective field theory approach.

Our results have provided new insights into the interplay between quantum gravity and black hole thermodynamics. By demonstrating the RG invariance of critical quantities and ensuring adherence to established thermodynamic laws, we have validated the applicability and reliability of the effective field theory approach in this context. These findings contribute to a deeper understanding of charged AdS black hole physics and offer a solid foundation for further explorations into the quantum aspects of gravity.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Action of $\ln\left(\frac{\Box}{\mu^2}\right)$ on radial functions

We show the result of the action of $\ln\left(\Box/\mu^2\right)$ on some radial functions. Let us start with the constant function. We regularize the integrals in (2.20) as

$$\ln\left(\frac{\Box}{\mu^2}\right)1=\lim_{\epsilon\to 0}\left[\frac{1}{r}\int_0^{1/\sqrt{\epsilon/r^3}}dr'\,\frac{r'}{r+r'}-\frac{1}{r}\int_0^{r-\sqrt{\epsilon r}}dr'\,\frac{r'}{r-r'}-\frac{1}{r}\int_{r+\sqrt{\epsilon r}}^{+\infty}dr'\,\frac{r'}{r'-r}-2\gamma_E-2\ln(\mu\epsilon)\right], \tag{A.1}$$

where we have added the r in the integral limits for dimensional reasons (they must have the dimension of a length). The divergences cancel out:

$$\ln\left(\frac{\Box}{\mu^2}\right)1=\lim_{\epsilon\to 0}\left[2+\ln(\epsilon r)-\ln\left(-r+\sqrt{r^3/\epsilon}\right)-\ln\left(r+\sqrt{r^3/\epsilon}\right)-2\gamma_E-2\ln(\mu\epsilon)\right]=-2(\ln(\mu r)+\gamma_E-1). \tag{A.2}$$

Let us consider now $f(r)=1/r$. We regularize the integrals in (2.20) as

$$\ln\left(\frac{\Box}{\mu^2}\right)\frac{1}{r}=\lim_{M\to\infty}\lim_{\epsilon\to 0}\left\{\frac{1}{r}\int_0^{M}dr'\frac{r'}{r+r'}\frac{1}{r'}-\frac{1}{r}\int_0^{r-\epsilon}dr'\frac{r'}{r-r'}\frac{1}{r'}-\frac{1}{r}\int_{r+\epsilon}^{M}dr'\frac{r'}{r'-r}\frac{1}{r'}-\frac{2}{r}\left[\gamma_E+\ln(\mu\epsilon)\right]\right\}. \tag{A.3}$$

Again, the divergences cancel:

$$\ln\left(\frac{\Box}{\mu^2}\right)\frac{1}{r}=\lim_{M\to\infty}\lim_{\epsilon\to 0}\frac{1}{r}\left[\ln\left(\frac{M}{r}\right)-\ln\left(\frac{M}{\epsilon}\right)-\ln\left(\frac{r}{\epsilon}\right)-2\gamma_E-2\ln(\mu\epsilon)\right]=-\frac{2}{r}\left(\ln(\mu r)+\gamma_E\right). \tag{A.4}$$

Analogously,

$$\begin{aligned}\ln\left(\frac{\Box}{\mu^2}\right)\frac{1}{r^2}&=\mathrm{Re}\lim_{\epsilon\to 0}\left\{\frac{1}{r}\int_0^{\infty}dr'\frac{r'}{r+r'}\frac{1}{(r'+i\epsilon)^2}-\frac{1}{r}\int_0^{r-\epsilon}dr'\frac{r'}{r-r'}\frac{1}{(r'+i\epsilon)^2}-\frac{1}{r}\int_{r+\epsilon}^{\infty}dr'\frac{r'}{r'-r}\frac{1}{r'^2}-\frac{2}{r^2}\left[\gamma_E+\ln(\mu\epsilon)\right]\right\}\\&=-\frac{2}{r^2}\left(\ln(\mu r)+\gamma_E\right).\end{aligned} \tag{A.5}$$

Appendix B. Explicit expressions of the ρ-dependent quantum corrections

We provide the explicit expressions of the ρ - dependent quantum corrections for several thermodynamics quantities discussed in the paper. We checked with Mathematica that all these quantities are RG invariant.

The correction to the entropy is

$$\begin{aligned}S_{AdS}=&-16384\pi^4\rho G_N^2(4\alpha+\beta)\Big[3c_1-c_3+(6\alpha-2\gamma)\left(\gamma_E+\ln\left(2G_NM\mu\right)\right)\Big]\\&+\frac{256}{3}\pi^3\rho G_N^2Q^2\Big\{36c_1+17c_2+36c_3+12(-3+4\gamma_E)\alpha+7(-7+4\gamma_E)\beta\\&+4(-41+18\gamma_E)\gamma+(72\alpha+34\beta+72\gamma)\ln\left(2G_NM\mu\right)\Big\}\\&-\frac{131072\pi^5\rho Q^2}{M^4}(4\alpha+\beta)\Big\{c_2\Big[4c_3+3\alpha+(-17+8\gamma_E)\gamma\Big]+2\Big[8c_3^2\\&+c_3[6\alpha+(-9+4\gamma_E)\beta+(-70+32\gamma_E)\gamma]+(-2+\gamma_E)(\beta+4\gamma)[3\alpha+(-19+8\gamma_E)\gamma]\Big]\\&+2\ln\left(2G_NM\mu\right)\Big[4c_2\gamma+4c_3(\beta+8\gamma)+(\beta+4\gamma)(3\alpha-35\gamma+16\gamma_E\gamma)\\&+8\gamma(\beta+4\gamma)\ln\left(2G_NM\mu\right)\Big]\Big\}-\frac{4096\pi^4\rho G_NQ^2}{3M^2}\Big\{9c_2^2+32c_3^2\\&-6(4\alpha+\beta)\left[3\alpha+(-42+(27-4\gamma_E)\gamma_E)\beta\right]\\&+2\Big[12(169+4\gamma_E(-27+4\gamma_E))\alpha+[555+4\gamma_E(-95+16\gamma_E)]\beta\Big]\gamma\\&+64(-2+\gamma_E)(-3+2\gamma_E)\gamma^2+c_2\Big[44c_3+12(-19+4\gamma_E)\alpha+(-93+30\gamma_E)\beta\\&+(-164+88\gamma_E)\gamma\Big]+8c_3\Big[6(-19+4\gamma_E)\alpha+(-33+8\gamma_E)\beta+4(-7+4\gamma_E)\gamma\Big]\\&+36c_1\Big[c_2+4c_3+2(-2+\gamma_E)(\beta+4\gamma)\Big]\\&+2\ln\left(2G_NM\mu\right)\Big[4c_3(36\alpha+11\beta+16\gamma)+2c_2(18\alpha+9\beta+22\gamma)\\&+(\beta+4\gamma)\left[36c_1+3(-31+10\gamma_E)(4\alpha+\beta)+8(-7+4\gamma_E)\gamma\right]\\&+2(\beta+4\gamma)(36\alpha+9\beta+8\gamma)\ln\left(2G_NM\mu\right)\Big]\Big\}.\end{aligned} \tag{B.1}$$

The correction to the pressure is

$$\begin{aligned}
P_{AdS} &= \rho\left\{\frac{128\pi^2(4\alpha+\beta)(3\alpha-2\gamma)}{G_N^2 M^4} - \frac{8\pi(12\alpha+3\beta+4\gamma)}{3G_N M^2}\right\} \\
&+ \frac{2\pi\rho Q^2}{3G_N^2 M^4}\left\{-2(6c_1+c_2+10c_3)-60\alpha+(-49+10\gamma_E)\beta+8(-18+5\gamma_E)\gamma+4(-6\alpha+\beta+10\gamma)\ln\left(2G_N M\mu\right)\right\} \\
&+ \frac{8\pi^2\rho Q^2}{27G_N^3 M^6}\left\{-36c_2^2+3c_2\left[240c_3+(5652-576\gamma_E)\alpha+(1121+24\gamma_E)\beta+160(-5+3\gamma_E)\gamma\right]\right. \\
&-432c_1\left[3c_2+12c_3+2(-8+3\gamma_E)(\beta+4\gamma)\right]+2\left[1728c_3^2+1944\alpha^2+\left[-8509+3\gamma_E(941+48\gamma_E)+96\pi^2\right]\beta^2\right. \\
&+12\left[-2474+3\gamma_E(99+64\gamma_E)+72\pi^2\right]\beta\gamma+64\left[292+3\gamma_E(-161+36\gamma_E)+30\pi^2\right]\gamma^2 \\
&-18\alpha\left[(2327+6\gamma_E(-201+16\gamma_E))\beta+8\left(1186-603\gamma_E+48\gamma_E^2\right)\gamma\right] \\
&\left.+6c_3\left[12(471-48\gamma_E)\alpha+677\beta-2576\gamma+192\gamma_E(\beta+6\gamma)\right]\right] \\
&+6\ln\left(2G_N M\mu\right)\left[-48c_3(36\alpha-5\beta-48\gamma)-24c_2(18\alpha+\beta-10\gamma)\right. \\
&+(\beta+4\gamma)\left[-432c_1+36(221-40\gamma_E)\alpha+(1121+24\gamma_E)\beta+16(-161+72\gamma_E)\gamma\right] \\
&\left.\left.-24(36\alpha+\beta-24\gamma)(\beta+4\gamma)\ln\left(2G_N M\mu\right)\right]\right\}+\frac{256\pi^3\rho Q^2}{9G_N^4 M^8}(4\alpha+\beta)\left\{36c_2^2\right. \\
&-576c_3^2+6c_2\left[-18\alpha+4(-13+6\gamma_E)\beta+37\gamma\right]+6c_3\left[-72\alpha+23\beta+16(67-24\gamma_E)\gamma\right] \\
&+(\beta+4\gamma)\left[54c_1+3(51-36\gamma_E)\alpha+458\beta-4961\gamma+24\left[6\gamma_E^2(\beta-4\gamma)+\pi^2(\beta+5\gamma)+\gamma_E(-26\beta+134\gamma)\right]\right] \\
&+12\ln\left(2G_N M\mu\right)\left[12c_2\beta-192c_3\gamma+(\beta+4\gamma)\left[-9\alpha-52\beta+24\gamma_E\beta+268\gamma-96\gamma_E\gamma\right]\right. \\
&\left.\left.+12\left(\beta^2-16\gamma^2\right)\ln\left(2G_N M\mu\right)\right]\right\}.
\end{aligned} \tag{B.2}$$

The correction to the specific heat is

$$\begin{aligned}
C_{AdS} &= \frac{32768\pi^4\rho G_N^2}{M}(4\alpha+\beta)(3\alpha-\gamma) \\
&+\frac{512\pi^3\rho G_N^2 Q^2}{3M}\left\{2\left(29c_2+116c_3-6\alpha\right)+(\beta+4\gamma)\left[-283+116\gamma_E+116\ln\left(2G_N M\mu\right)\right]\right\} \\
&-\frac{262144\pi^5\rho Q^2}{M^5}(4\alpha+\beta)\left\{32c_3^2+4c_2[2c_3+3\alpha+(-1+4\gamma_E)\gamma]+8c_3\left[6\alpha+(-5+2\gamma_E)\beta+2(-11+8\gamma_E)\gamma\right]\right. \\
&+(\beta+4\gamma)\left[3(-19+8\gamma_E)\alpha+2[9+4\gamma_E(-11+4\gamma_E)]\gamma\right] \\
&+8\ln\left(2G_N M\mu\right)\left[2c_2\gamma+2c_3(\beta+8\gamma)+(\beta+4\gamma)(3\alpha-11\gamma+8\gamma_E\gamma)\right. \\
&\left.\left.+4\gamma(\beta+4\gamma)\ln\left(2G_N M\mu\right)\right]\right\}+\frac{8192\pi^4\rho G_N Q^2}{27M^3}\left\{63c_2^2+72\left[28c_3^2-12c_3(25+2\gamma_E)\alpha+9\alpha^2\right]\right. \\
&+6\left[2c_3\left(-851+144\gamma_E\right)+4233\alpha-72\gamma_E(19+2\gamma_E)\alpha\right]\beta \\
&+\left[8393+24\gamma_E(-194+15\gamma_E)+96\pi^2\right]\beta^2+24\left[4c_3(-173+84\gamma_E)\right. \\
&\left.+4206\alpha-72\gamma_E(19+2\gamma_E)\alpha+\left(1577+6\gamma_E(-187+24\gamma_E)+36\pi^2\right)\beta\right]\gamma \\
&+16\left[1069+12\gamma_E(-173+42\gamma_E)+120\pi^2\right]\gamma^2 \\
&+3c_2\left[252c_3-72(25+2\gamma_E)\alpha+17(-43+6\gamma_E)\beta+8(-113+63\gamma_E)\gamma\right] \\
&-108c_1\left[3c_2+12c_3+(-17+6\gamma_E)(\beta+4\gamma)\right]+6\ln\left(2G_N M\mu\right)\left[12c_3(-36\alpha+21\beta+112\gamma)\right. \\
&-(\beta+4\gamma)[108c_1+36(33+10\gamma_E)\alpha+17(43-6\gamma_E)\beta+8(173-84\gamma_E)\gamma] \\
&\left.\left.+6c_2[-18\alpha+7(\beta+6\gamma)]+6(\beta+4\gamma)[-36\alpha+7(\beta+8\gamma)]\ln\left(2G_N M\mu\right)\right]\right\}.
\end{aligned} \tag{B.3}$$

The correction to the Helmholtz free energy is

$$
\begin{aligned}
F_{AdS} = \rho\Bigg\{ & \frac{2048\pi^3 G_N}{M}(4\alpha+\beta)\Big[3c_1 - 2c_3 + 6\gamma_E\alpha + (2-4\gamma_E)\gamma + (6\alpha-4\gamma)\ln\left(2G_N M\mu\right)\Big] \\
& - \frac{128\pi^2}{3}G_N^2 M\Big[4c_3 + 3(4\alpha+\beta) + 8(-1+\gamma_E)\gamma + 8\gamma\ln\left(2G_N M\mu\right)\Big]\Bigg\} \\
& - \frac{32\pi^2\rho G_N Q^2}{3M}\Bigg\{12\left(c_1+c_2+3c_3\right) + 48(-2+\gamma_E)\alpha + (-61+30\gamma_E)\beta \\
& + (-148+72\gamma_E)\gamma + 24(\alpha+\beta+3\gamma)\ln\left(2G_N M\mu\right)\Bigg\} \\
& + \frac{128\pi^3\rho Q^2}{27M^3}\Bigg\{36\Big[5c_2^2 + c_2\left[16c_3 + (-39+48\gamma_E)\alpha\right] - 8\left[2c_3^2 + 3c_3(7-8\gamma_E)\alpha + \alpha^2\right]\Big]\Bigg\} \\
& + 3\Big[32c_3(10+3\gamma_E) + c_2(-181+168\gamma_E) + 24\left[83+3\gamma_E(-47+16\gamma_E)\right]\alpha\Big]\beta \\
& + \Big[151 + 6\gamma_E(-37+48\gamma_E) - 96\pi^2\Big]\beta^2 + 24\Big[c_3(248-96\gamma_E) + c_2(-73+48\gamma_E) \\
& + 6\left[199+96(-3+\gamma_E)\gamma_E\right]\alpha + \left[43+6\gamma_E(13+4\gamma_E)-36\pi^2\right]\beta\Big]\gamma \\
& - 64\Big[7+6\gamma_E(-31+6\gamma_E)+30\pi^2\Big]\gamma^2 + 216c_1\Big[6c_2 + 24c_3 + (-25+12\gamma_E)(\beta+4\gamma)\Big] \\
& + 6\ln\left(2G_N M\mu\right)\Big[\beta\left[432c_1 + 36(-63+40\gamma_E)\alpha + (-181+168\gamma_E)\beta\right] + 192c_3(9\alpha+\beta-2\gamma) \\
& + 24\left[72c_1 + 48(-8+5\gamma_E)\alpha + (-11+20\gamma_E)\beta\right]\gamma + 64(31-12\gamma_E)\gamma^2 \\
& + 24c_2(18\alpha+5\beta+8\gamma) + 24(36\alpha+5\beta-4\gamma)(\beta+4\gamma)\ln\left(2G_N M\mu\right)\Big] \\
& + \frac{2048\pi^4\rho Q^2}{9GM^5}(4\alpha+\beta)\Bigg\{-36c_2^2 + 2880c_3^2 + 24c_3\Big[36(1+\gamma_E)\alpha + (-113+48\gamma_E)\beta\Big] \\
& + \beta\Big[9\left[-71+24\gamma_E(-3+2\gamma_E)\right]\alpha - 8\left[40+3\gamma_E(-23+6\gamma_E)+3\pi^2\right]\beta\Big] \\
& + 6c_2\Big[96c_3 + 36(1+\gamma_E)\alpha + 2(23-12\gamma_E)\beta + (-223+192\gamma_E)\gamma\Big] \\
& + 12\Big[2c_3(-859+480\gamma_E) + 3\left[-71+24\gamma_E(-3+2\gamma_E)\right]\alpha \\
& + \left[695+3\gamma_E(-225+64\gamma_E)-18\pi^2\right]\beta\Big]\gamma + 16\Big[2405+3\gamma_E(-859+240\gamma_E)-30\pi^2\Big]\gamma^2 \\
& + 108c_1\Big[c_2 + 4c_3 + (-5+2\gamma_E)(\beta+4\gamma)\Big] + 12\ln\left(2G_N M\mu\right)\Big[6c_2(3\alpha-2\beta+16\gamma) \\
& + 24c_3(3\alpha+4\beta+40\gamma) + (\beta+4\gamma)\big[18c_1 + 18(-3+4\gamma_E)\alpha + 2(23-12\gamma_E)\beta \\
& + (-859+480\gamma_E)\gamma\big] + 12(\beta+4\gamma)(3\alpha-\beta+20\gamma)\ln\left(2G_N M\mu\right)\Big]\Bigg\}.
\end{aligned}
\tag{B.4}
$$

Data availability

No data was used for the research described in the article.

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