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Solitary wave solutions for the nonlinear Schrodinger equation with power law nonlinearity

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Abstract.
In this paper, the nonlinear Schrödinger equation with power law nonlinearity is considered. This equation admits Lax pair and is integrable. The Jacobi elliptic function approach is used to obtain solitary waves solutions. This method is applied to get the exact solutions for various nonlinear equations. 2D and 3D plots of the obtained solutions are presented. It is discovered that the solutions acquired are crucial for the explanation of a few physical issues.

Keywords: Jacobi elliptic function method, Schrodinger equation, solitary wave solution

1. Introduction

Darboux transformation [1-4], Hirota method [5-6], sine-cosine method [7], Kudryashov method [8-9], and Jacobi elliptic method [10-12] are only a few of the several techniques that have been developed in the last ten years for precisely solving nonlinear evolutionary equations. These equations are fundamental in mathematical physics because they are significant to phenomena in several domains, including fluid mechanics, plasma physics, optical fibers, and many others [13-14].

In this work, we study the nonlinear Schrödinger equation with power law nonlinearity [15-18]

$$iq_t + \beta_1 q_{xx} + \beta_2 q|q|^{2m} - i\beta_3 q_x + i\nu q_{xxx} - i\gamma(q|q|^{2m})_x - i\lambda q(|q|^{2m})_x = 0, \tag{1}$$

where $q(x,t)$ is a complex function of spatial coordinates x and time t. This equation is used to simulate electromagnetic pulses in optical waveguides, and provides the most complete description of an interacting process between two surface waves at large distances in the oceanic depth than Manakov system. It also includes femtosecond optical pulse propagation in dual birefringent or dual-mode nonlinear fiber.

2. Description Jacobi elliptic functions method

Let's consider the nonlinear wave equation according to the method.

$$N(u, \frac{du}{dt}, \frac{du}{dx}, \frac{d^2u}{dt^2}, \frac{d^2u}{dx^2}, ...) = 0, \tag{2}$$

We are looking for wave solutions with the following structure:

$$u = u(\xi), \quad \xi = k(x - ct), \tag{3}$$

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where c is the wave speed, k is the wave number. The Jacobi elliptic function is expressed through the following transformation [10-12]:

$$u(\xi) = \Sigma_{j=0}^{n} a_j sn^j(\xi). \tag{4}$$

After the transformation is performed, the highest degree of the functions is:

$$O(u(\xi)) = n.$$

And the derivative of the function in (4) is:

$$\frac{du}{d\xi} = \Sigma^n{}_{j=0} j a_j sn^{j-1}(\xi) cn(\xi) dn(\xi),$$

where $cn(\xi)$ and $dn(\xi)$ are the Jacobi elliptic unctions, and

$$cn^2\xi = 1 - sn^2\xi, \quad dn^2(\xi) = 1 - m^2 sn^2\xi,$$

where the modulus of the Jacobi elliptic function is $m(0 < m < 1)$.

$$\frac{d}{d\xi} cn\xi = -sn\xi dn\xi, \quad \frac{d}{d\xi} sn\xi = cn\xi dn\xi, \quad \frac{d}{d\xi} dn\xi = -m^2 sn\xi cn\xi,$$

The highest degree is expressed as the following expression: $\frac{d^p u}{d\xi^p}$

$$O(\frac{d^p u}{d\xi^p}) = n + p, \quad p = 1, 2, 3, ...,$$

$$O(u^q \frac{d^p u}{d\xi^p}) = (q+1)n + p, \quad q = 0, 1, 2, ...$$

We can therefore select n by equating the nonlinear term with the highest order derivative term. We know that if $m \to 1, \quad sn(\xi) \to tanh(\xi)$ is chosen this way, then (4) undergoes the following transformation:

$$u(\xi) = \Sigma_{j=0}^{n} a_j tanh^j(\xi). \tag{5}$$

The decomposition of Jacobi elliptic functions is more general that that of hyberbolic tangent functions. By using the Jacobi elliptic function method, we determine explicit solutions to the nonlinear Schrödinger equation with power law nonlinearity (1). This method is applicable to various nonlinear equations, such as the perturbed nonlinear Schrodinger equation, the Schrodinger-Hirota fractional nonlinear equation, and others.

3. Application Jacobi elliptic functions method

To transform the nonlinear Schrödinger equation with power law nonlinearity into a ordinary differential equation, we use the following transformation:

$$q(x,t) = e^{i(ax+bt)} Q(\xi), \quad \xi = x + ct, \tag{6}$$

where $Q(\xi)$ is a real function, a, b are constants. Equation (1) simplifies to the following real-and imaginary-parts system:

$$\frac{(-b - \beta_1 a^2 + \beta_3 a + a^3 \upsilon)}{\beta_1 - 3a\upsilon} Q + Q'' + \frac{\beta_2 - \gamma a}{\beta_1 - 3a\upsilon} Q^{2k+1} = 0, \tag{7}$$

$$\frac{c+2a\beta_1-\beta_3-3a^2\upsilon}{\upsilon}Q+Q''-\frac{\gamma+2k\gamma+2\lambda k}{(2k+1)\upsilon}Q^{2k+1}=0. \tag{8}$$

Equations (7) and (8) are similar, so the relationship between the coefficients is as follows:

$$b=a^3\upsilon+\beta_3 a-\beta_1 a^2-\frac{(c+2a\beta_1-\beta_3-3a^2\upsilon)(\beta_1-3a\upsilon)}{\upsilon}, \tag{9}$$

$$\beta_2=-\frac{\beta_1(\gamma+2k\gamma+2\lambda k)-4a\upsilon\gamma(2k+1)-6a\upsilon\lambda k}{2k\upsilon+\upsilon}. \tag{10}$$

Next, we will solve equation (8). In this case, $k=1$ it takes the form:

$$Q''+\frac{(c+2a\beta_1-\beta_3-3a^2\upsilon)}{\upsilon}Q-\frac{3\lambda+2\lambda}{3\upsilon}Q^3=0. \tag{11}$$

3.1. dn solutions

According to the method, we seek the solution of the equation in terms of Jacobi elliptic functions.

$$Q(\xi)=\sum_{j=0}^{n}a_j dn^j(\xi), \tag{12}$$

where $a_j(j=1,2,...n)$ are constants, and n can be obtained by a combination of balance between higher-order nonlinear terms and higher-order derivatives. Thus, we have $n=1$. This equation appears as:

$$Q(\xi)=a_0+a_1 dn\xi. \tag{13}$$

We will find the derivatives and degrees of the equation:

$$\frac{dQ}{d\xi}=-m^2 a_1 sn\xi cn\xi,$$

$$\frac{d^2Q}{d\xi^2}=(2-m^2)a_1 dn\xi-2a_1 dn^3\xi, \tag{14}$$

$$Q^3(\xi)=a_0^3+3a_0^2a_1 dn\xi+3a^0a_1^2dn^2\xi+a_1^3dn^3\xi, \tag{15}$$

where $m(0<m<1)$ implying to the modulus of Jacobi elliptic function. After substituting the defined equations (13)-(15) into (11), we have:

$$\left(\frac{c+2a\beta_1-\beta_3-3a^2\upsilon}{\upsilon}-\frac{3\lambda+2\lambda}{3\upsilon}a_0^2\right)a^0+ \tag{16}$$

$$+\left(\frac{c+2a\beta_1-\beta_3-3a^2\upsilon}{\upsilon}+(2-m^2)-\frac{3\lambda+2\lambda}{\upsilon}a_0^2\right)a_1 dn\xi-$$

$$-\frac{3\lambda+2\lambda}{\upsilon}a_0a_1^2dn^2\xi-\left(\frac{3\lambda+2\lambda}{3\upsilon}a_1^2+2\right)a_1dn^3\xi=0.$$

By collecting dn the terms of the same degree for the functions and combining the coefficients, we receive:

$$dn^0(\xi):\left(\frac{c+2a\beta_1-\beta_3-3a^2\upsilon}{\upsilon}-\frac{3\lambda+2\lambda}{3\upsilon}a_0^2\right)a^0=0, \tag{17}$$

$$dn^1(\xi):\left(\frac{c+2a\beta_1-\beta_3-3a^2\upsilon}{\upsilon}+(2-m^2)-\frac{3\gamma+2\lambda}{\upsilon}a_0^2\right)a_1=0. \tag{18}$$

$$dn^2(\xi):-\frac{3\lambda+2\lambda}{\upsilon}a_0a_1^2=0, \tag{19}$$

$$dn^3(\xi):-(\frac{3\lambda+2\lambda}{3\upsilon}a_1^2+2)a_1=0, \tag{20}$$

The solution to the system of equations above yields the following values:

$$a_0=0,\quad a_1=\pm\sqrt{-\frac{6\upsilon}{3\gamma+2\lambda}},\quad c=3a^2\upsilon+m^2\upsilon-2\alpha\beta_1+\beta_3-2\upsilon. \tag{21}$$

By substituting the determined values into equation (13) and then substituting the resulting expression into equation (6), we obtain dn solutions for the nonlinear Schrödinger equation with power law nonlinearity.

$$q_{31}(x,t)=\pm e^{i(ax+bt)}\sqrt{-\frac{6\upsilon}{3\gamma+2\lambda}}dn(x+ct),\quad \frac{6\upsilon}{3\gamma+2\lambda}<0, \tag{22}$$

where $c=3a^2\upsilon+m^2\upsilon-2a\beta_1+\beta_3-2\upsilon$, and

$$b=a^3\upsilon+\beta_3a-\beta_1a^2-\frac{(c+2a\beta_1-\beta_3-3a^{2\upsilon})(\beta_1-3a\upsilon)}{\upsilon}. \tag{23}$$

We observe that Jacobi elliptic functions transform into the next functions:

$$dn(\xi)\rightarrow sech(\xi), m\rightarrow 1. \tag{24}$$

By adopting $m=1$ in this form, we obtain solutions in terms of hyperbolic functions.

$$q_{32}(x,t)=\pm e^{i(ax+bt)}\sqrt{-\frac{6\upsilon}{3\gamma+2\lambda}}sech(x+ct),\quad \frac{6\upsilon}{3\gamma+2\lambda}<0, \tag{25}$$

where $c=3a^2\upsilon-2a\beta_1+\beta_3-\upsilon$, $b=a^3\upsilon+\beta_3a-\beta_1a^2-\frac{(c+2a\beta_1-\beta_3-3a^2\upsilon)(\beta_1-3a\upsilon)}{\upsilon}$. Figs. 1 and 2 displays the graphs of the solutions (22) and (25) that were obtained.

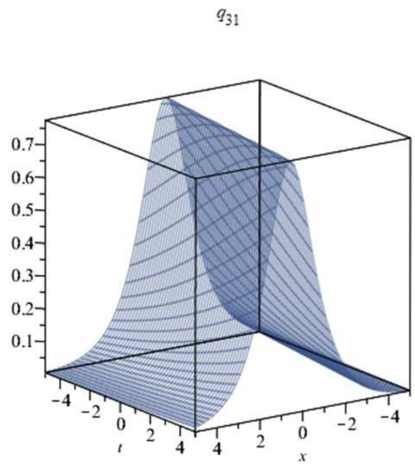

Figure 1: $q_{31}(x,t)$ the 3D graph of the solution is presented. The parameters are as follows: $a=\lambda=\beta_1=\beta_3=\gamma=1$, $\upsilon=0.5$, $m=1$, $\beta_2=0.5$

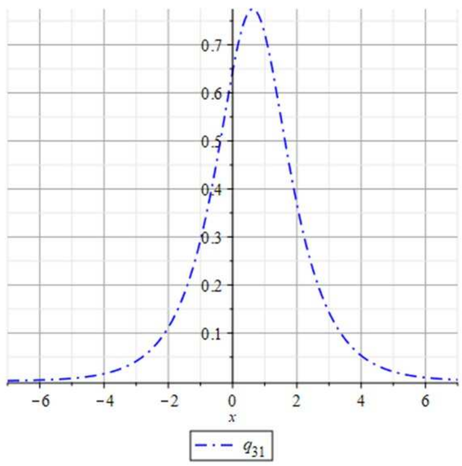

Figure 2: $q_{31}(x,t)$ the 2D graph of the solution is presented. The parameters are as follows: $a=\lambda=\beta_1=\beta_3=\gamma=1$, $\upsilon=0.5, m=1, \beta_2=0.5$

3.2. cs solutions

According to the method, we seek the solution of the equation cs in terms of Jacobi elliptic functions.

$$Q(\xi)=\sum_{j=0}^{n} a_j cs^j(\xi), \tag{26}$$

where $a_j(j=1,2,...,n)$ are consts, and n can be defined by considering the balance between higher-order nonlinear terms and higher-order derivatives, thus, we have $n=1$. The equation takes the following form:

$$Q(\xi) = a_0 + a_1 cs\xi, cs\xi = \frac{cn\xi}{sn\xi}. \tag{27}$$

We will find the derivatives and degrees of the equation:

$$\frac{dQ}{d\xi} = -a_1(1 + cs^2\xi)dn\xi,$$

$$\frac{d^2Q}{d\xi^2} = (2 - m^2)a_1 cs\xi + 2a_1 cs^3\xi, \tag{28}$$

$$Q^3(\xi) = a_0^3 + 3a_0^2 a_1 cs\xi + 3a_0 a_1^2 cs^2\xi + a_1^3 cs^3\xi, \tag{29}$$

where the modulus of the Jacobi elliptic function is $m(0 < m < 1)$. By use as a replacement the defined equations (27)-(29) into (11), we can express it in the following form:

$$\left(\frac{c + 2a\beta_1 - \beta_3 - 3a^2\upsilon}{\upsilon} - \frac{3\lambda + 2\lambda}{3\upsilon}a_0^2\right)a^0 + \tag{30}$$

$$\left(\frac{c + 2a\beta_1 - \beta_3 - 3a^2\upsilon}{\upsilon} + (2 - m^2) - \frac{3\lambda + 2\lambda}{\upsilon}a_0^2\right)a_1 cs\xi -$$

$$-\frac{3\lambda + 2\lambda}{\upsilon}a_0 a_1^2 cs^2\xi - \left(\frac{3\lambda + 2\lambda}{3\upsilon}a_1^2 + 2\right)a_1 cs^3\xi = 0.$$

By collecting the term cs of the same degree for the functions and combining the coefficients, we can get the system:

$$cs^0(\xi) : \left(\frac{c + 2a\beta_1 - \beta_3 - 3a^2\upsilon}{\upsilon} - \frac{3\gamma + 2\lambda}{3\upsilon}a_0^2\right)a_0 = 0, \tag{31}$$

$$cs^1(\xi) : \left(\frac{c + 2a\beta_1 - \beta_3 - 3a^2\upsilon}{\upsilon} + (2 - m^2) - \frac{3\gamma + 2\lambda}{3\upsilon}a_0^2\right)a_1 = 0, \tag{32}$$

$$cs^2(\xi) : -\left(\frac{3\gamma + 2\lambda}{\upsilon}a_0 a_1^2\right) = 0, \tag{33}$$

$$cs^3(\xi) : -\left(\frac{3\gamma + 2\lambda}{3\upsilon}a_1^2 - 2\right)a_1 = 0. \tag{34}$$

The solution to the system of equations above yields the following values:

$$a_0 = 0, \quad a_1 = \pm\sqrt{\frac{6\upsilon}{3\gamma + 2\lambda}}, \quad c = 3a^2\upsilon + m^2\upsilon - 2a\beta_1 + \beta_3 - 2\upsilon. \tag{35}$$

By substituting the determined values into equation (27) and then substituting the resulting expression into equation (6), we obtain cs solutions for nonlinear Schrödinger equation with power law nonlinearity.

$$q_{41}(x,t) = \pm e^{i(ax+bt)}\sqrt{\frac{6\upsilon}{3\gamma+2\lambda}}cs(x+ct), \frac{6\upsilon}{3\gamma+2\lambda} > 0, \tag{36}$$

where $c = 3a^2\upsilon + m^2\upsilon - 2a\beta_1 + \beta_3 - 2\upsilon$ and $b = a^3\upsilon + \beta_3 a - \beta_1 a^2 - \frac{(c+2a\beta_1-\beta_3-3a^2\upsilon)(\beta_1-3a\upsilon)}{\upsilon}$. We observe that the following functions are trasformed from Jacobian elliptic functions.

$$cs(\xi) \to csch(\xi), \quad m \to 1. \tag{37}$$

By accepting this, the solutions in the hyperbolic function is obtained

$$q_{42}(x,t) = \pm e^{i(ax+bt)}\sqrt{\frac{6\upsilon}{3\gamma+2\lambda}}csch(x+ct), \quad \frac{6\upsilon}{3\gamma+2\lambda} > 0, \tag{38}$$

where $c = 3a^2\upsilon - 2a\beta_1 + \beta_3 - \upsilon$ and $b = a^3\upsilon + \beta_3 a - \beta_1 a^2 - \frac{(c+2a\beta_1-\beta_3-3a_2\upsilon)(\beta_1-3a\upsilon)}{\upsilon}$.
Figs. 3 and 4 dispalys the graphical depiction of the received solutions (36) and (38).

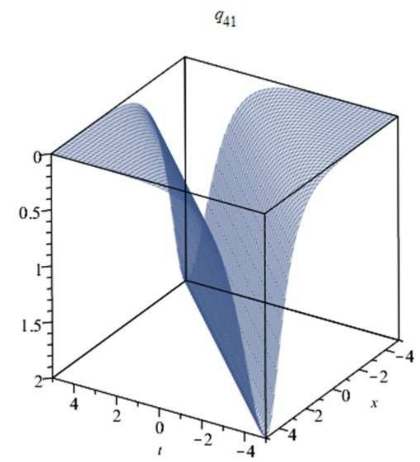

Figure 3: $q_{41}(x,t)$ the 3D graph of the solution is presented. The parameters are as follows: $a = \lambda = \beta_1 = \beta_3 = \gamma = 1$, $\upsilon = 0.5, m = 1, \beta_2 = 0.5$

4. Conclusions

The nonlinear Schrodinger equation with power law nonlinearity was investigated in this work. Different kinds of analytical solutions, including dark and bright solitons, were produced by using the Jacobi elliptic function approach to dn and cs solutions. Additionally, by establishing appropriate values for the related parameters, we plot 3D and 2D profiles for the physical representation of the results.

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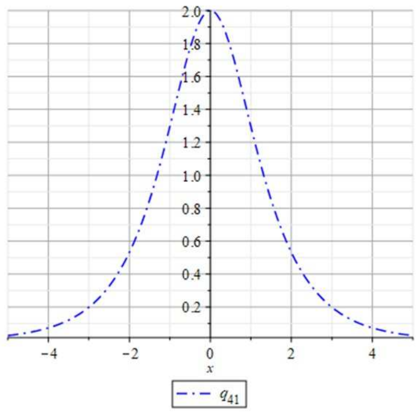

Figure 4: $q_{41}(x,t)$ the 2D graph of the solution is presented. The parameters are as follows: $a = \lambda = \beta_1 = \beta_3 = \gamma = 1,\ \upsilon = 0.5, m = 1, \beta_2 = 0.5$

References

[1] Yesmakhanova K., Bekova G., Myrzakulov R., Shaikhova G. 2017 *J. Phys.: Conf. Series* **804** 012004 (1-8).
[2] Yesmakhanova K., Bekova G., Shaikhova G., Myrzakulov R. 2016 *J. Phys.: Conf. Seriess* **738(1)** 012018.
[3] Shaikhova G., Serikbayev N., Yesmakhanova K., Myrzakulov R 2020 *Geom., Integrab. and Quant.* **21** 265–271.
[4] Bekova G., Yesmakhanova K., Ozat N., Shaikhova G 2018 *J. Phys.: Conf. Series* **965** 012035 (1-10).
[5] Hirota R 1971 *Phys. Rev. Lett.* 1192-1195.
[6] Burdik C, Shaikhova G, Rakhimzhanov B 2021 *European Physical Journal Plus* **136(10)** 1095.
[7] Ala V., Shaikhova G 2022 *Lobachevskii Journal of Mathematics* **43(11)** 3033-3038.
[8] Shaikhova G.N., Rakhimzhanov B.K., Zhanbosinova Zh.K. 2021 *J. Phys.: Conf. Series* **2090** 012062.
[9] Serikbayev N.S, Shaikhova G N, Yesmakhanova K R, Myrzakulov R 2019 *J. Phys.: Conf. Series* **1391(1)** 012166.
[10] Shaikhova G.N., Kutum B.B., Myrzakulov R 2022 *AIMS Mathematics* **7(10)** 18948-18970.
[11] Syzdykova A., Kudaibergenov G 2024 *Springer Proceedings in Mathematics and Statistics* **449** 179-187.
[12] Shaikhova G.N., Kutum B.B., Syzdykova A 2023 *Results in Physics* **55** 107173.
[13] SShaikhova G., Yesmakhanova K., Bekova G., Ybyraiymova S 2017 *J. Phys.: Conf. Seriess* **936(1)** 012098.
[14] Wazwaz A 2009 *Partial differential equations and solitary waves theory* (Springer) P.746.
[15] Nasreena N., Seadawy A. R., Lua D., Arshada M. 2019 *Results in Physics* **13** 102263.
[16] Potasek M J. 1987 *Opt Lett.* **12(11)** 921-3.
[17] Zayed E.ME, Al-Nowehy A.G., Elshater M.E. 2017 *Eur Phys J Plus* 132:259.