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RESEARCH ARTICLE

Shear-free anisotropic cosmological models in $f(R)$ gravity

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Abstract We study a class of shear-free, homogeneous but anisotropic cosmological models with imperfect matter sources in the context of $f(R)$ gravity. We show that the anisotropic stresses are related to the electric part of the Weyl tensor in such a way that they balance each other. We also show that within the class of orthogonal $f(R)$ models, small perturbations of shear are damped, and that the electric part of the Weyl tensor and the anisotropic stress tensor decay with the expansion as well as the heat flux of the curvature fluid. Specializing in locally rotationally symmetric spacetimes in orthonormal frames, we examine the late-time behaviour of the de Sitter universe in $f(R)$ gravity. For the Starobinsky model of $f(R)$, we study the evolutionary behavior of the Universe by numerically integrating the Friedmann equation, where the initial conditions for the expansion, acceleration and jerk parameters are taken from observational data.

Keywords Shear-free spacetimes · Homogeneity · Cosmic anisotropy · Modified gravity · Expansion · Inflation · Cosmological perturbations

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1 Introduction

As a result of the current understanding that the Universe is in a state of accelerated expansion, many modifications to general relativity (GR), the theory on which modern cosmology is based, have been proposed recently. One such modification consists of a class of higher-order gravity models that attempt to address the shortcomings of GR in the infrared (IR) and ultraviolet (UV) ranges [1–8]. These models are generally obtained by including higher-order curvature invariants in the Einstein–Hilbert action, by making the action nonlinear in the Ricci curvature R, or contain terms involving combinations of derivatives of R, in which case the models are known as $f(R)$ *theories of gravity*.

First proposed by Buchdal [9], $f(R)$ theories gained more popularity after further developments by Starobinsky [10] and later following the realization of the discrepancy between theory and observation [5,11–15].

The role of shear in general relativistic [16–26] and $f(R)$ cosmologies [27–34] has been the subject of intense study for some time now, with the studies focusing mostly on the special nature of shear-free cases. In particular, it was shown in [16] that in the orthogonally spatially homogeneous models with vanishing shear, the anisotropic stresses are related to the anisotropic curvature of the spatial hypersurface through the electric part of the Weyl tensor. It was also shown that within the class of orthogonal models, small perturbations of shear are damped, and that the electric part of the Weyl tensor and the anisotropic stress tensor decay with the expansion.

The main focus of this work is the analysis of anisotropic but homogeneous, shear-free models whose underlying theory of gravitational interaction is $f(R)$-gravity.

The rest of this paper is organised as follows: in Sect. 2 a covariant description of $f(R)$ field equations is presented. In Sect. 3 we specialise to orthogonal cosmological models with anisotropic matter sources and analyse the properties of such models in the case of shear-free imperfect fluids in Sect. 4. In Sect. 5, the analysis is taken further by considering subclasses of locally rotationally symmetric spacetimes with barotropic equations of state and a qualitative analysis of such models has been made. Finally in Sect. 6 we discuss the results and give conclusions.

Natural units ($\hbar = c = k_B = 8\pi G = 1$) will be used throughout this paper, and Latin indices run from 0 to 3. The symbols ∇, $\tilde{\nabla}$ and the overdot $\cdot$ represent the usual covariant derivative, the spatial covariant derivative, and differentiation with respect to cosmic time. We use the $(- + ++)$ spacetime signature and the Riemann tensor is defined by

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd}\Gamma^a_{ce} - \Gamma^f_{bc}\Gamma^a_{df},$$

where the Γ^a_{bd} are the Christoffel symbols (i.e., symmetric in the lower indices), defined by

$$\Gamma^a_{bd} = \frac{1}{2}g^{ae}\left(g_{be,d} + g_{ed,b} - g_{bd,e}\right).$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices of the Riemann tensor:

$$R_{ab} = g^{cd} R_{cadb}.$$

The completely anti-symmetric pseudotensor η^{abcd} is defined such that

$$\eta_{0123} = \sqrt{-g},$$

where $g = \det(g_{ab})$ is the determinant of the metric g_{ab}.

Unless otherwise stated, primes $'$ etc are shorthands for derivatives with respect to the Ricci scalar

$$R = R^a{}_a$$

and f is used as a shorthand for $f(R)$. Moreover the following standard notations are used:

(ab) : symmetrization over the indices a and b,
$[ab]$: anti-symmetrization over the indices a and b,
$\langle ab \rangle$: orthogonal, symmetric, trace-free projection over the indices a and b.

2 Covariant description of the field equations

In the standard $f(R)$-gravity formulation, one starts with the modified Einstein–Hilbert action

$$\mathcal{A} = \tfrac{1}{2} \int d^4x \sqrt{-g}\, [f(R) + 2\mathcal{L}_m]\,, \tag{1}$$

where $\mathcal{L}_m$ stands for the matter field contribution to the Lagrangian, and uses the variational principle of least action with respect to the metric g_{ab} to obtain the generalised Einstein field equations (EFEs)

$$G_{ab} = \tilde{T}^m_{ab} + T^R_{ab} \equiv T_{ab}. \tag{2}$$

Here we have defined

$$\tilde{T}^m_{ab} \equiv \frac{T^m_{ab}}{f'}, \quad T^R_{ab} \equiv \frac{1}{f'} \left[\tfrac{1}{2}(f - Rf')g_{ab} + \nabla_b \nabla_a f' - g_{ab} \nabla_c \nabla^c f'\right] \tag{3}$$

as the effective matter and curvature energy-momentum tensors (EMTs), respectively. The EMT of standard matter is given by

$$T^m_{ab} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} = \mu_m u_a u_b + p_m h_{ab} + q^m_a u_b + q^m_b u_a + \pi^m_{ab}, \tag{4}$$

where μ_m, p_m, q^m_a and π^m_{ab} are the associated energy density, isotropic pressure, heat flux and anisotropic pressure, respectively, and $u^a \equiv \frac{dx^a}{dt}$ is the normalized 4-velocity of fundamental observers comoving with the fluid. We use this vector to define the covariant time derivative for any tensor $S^{a..b}_{c..d}$ along an observer's worldlines:

$$\dot{S}^{a..b}_{c..d} = u^e \nabla_e S^{a..b}_{c..d}. \tag{5}$$

On the other hand, we use the projection tensor $h_{ab} \equiv g_{ab} + u_a u_b$ to define the fully orthogonally projected covariant derivative for any tensor $S^{a..b}_{c..d}$:

$$\tilde{\nabla}_e S^{a..b}_{c..d} = h^a_f h^p_c ... h^b_g h^q_d h^r_e \nabla_r S^{f..g}_{p..q}, \tag{6}$$

with total projection on all the free indices. We extract the orthogonally projected symmetric trace-free part of vectors and rank-2 tensors using

$$V^{\langle a \rangle} = h^a_b V^b, \quad S^{\langle ab \rangle} = \left[h^{(a}_c h^{b)}_d - \tfrac{1}{3} h^{ab} h_{cd} \right] S^{cd}, \tag{7}$$

and the volume element for the restspaces orthogonal to u^a is given by [35]

$$\varepsilon_{abc} = u^d \eta_{dabc} = -\sqrt{|g|}\delta^0_{[a}\,\delta^1_b \delta^2_c \delta^3_{d]} u^d \Rightarrow \varepsilon_{abc} = \varepsilon_{[abc]}, \ \varepsilon_{abc}u^c = 0, \tag{8}$$

where η_{abcd} is the 4-dimensional volume element satisfying the conditions

$$\eta_{abcd} = \eta_{[abcd]} = 2\varepsilon_{ab[c}u_{d]} - 2u_{[a}\varepsilon_{b]cd}. \tag{9}$$

The covariant spatial divergence and curl of vectors and rank-2 tensors are given as [36]

$$\mathrm{div} V = \tilde{\nabla}^a V_a \,, \qquad (\mathrm{div} S)_a = \tilde{\nabla}^b S_{ab} \,, \tag{10}$$

$$curl V_a = \varepsilon_{abc} \tilde{\nabla}^b V^c \,, \quad curl S_{ab} = \varepsilon_{cd(a} \tilde{\nabla}^c S_{b)}{}^d. \tag{11}$$

The 4-velocity vector field u^a can be split into its irreducible parts as follows

$$\nabla_a u_b = -A_a u_b + \tfrac{1}{3} h_{ab} \Theta + \sigma_{ab} + \varepsilon_{abc} \omega^c, \tag{12}$$

where $A_a \equiv \dot{u}_a$, $\Theta \equiv \tilde{\nabla}_a u^a$, $\sigma_{ab} \equiv \tilde{\nabla}_{\langle a} u_{b \rangle}$ and $\omega^a \equiv \varepsilon^{abc} \tilde{\nabla}_b u_c$.
We can also split the Weyl conformal curvature tensor [35,37]

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c} R^{b]}{}_{d]} + \frac{R}{3} g^{[a}{}_{[c} g^{b]}{}_{d]} \tag{13}$$

into its "gravito-electric" (GE) and "gravito-magnetic" (GM) parts, respectively, as

$$E_{ab} \equiv C_{agbh} u^g u^h, \qquad H_{ab} = \tfrac{1}{2} \eta_{ae}{}^{gh} C_{ghbd} u^e u^d. \tag{14}$$

The GE and GM components represent the free gravitational field [35] and they describe gravitational action at a distance—tidal forces and gravitational waves. They influence the motion of matter and radiation through the geodesic deviation for timelike and null-vector fields, respectively.

The total energy density, isotropic and anisotropic pressures and heat flux of the $f(R)$ universe are given, respectively, by [38]

$$\mu \equiv \frac{\mu_m}{f'} + \mu_R, \quad p \equiv \frac{p_m}{f'} + p_R, \quad \pi_{ab} \equiv \frac{\pi^m_{ab}}{f'} + \pi^R_{ab}, \quad q_a \equiv \frac{q^m_a}{f'} + q^R_a, \tag{15}$$

where the thermodynamic quantities for the *curvature fluid* component are defined as

$$\mu_R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right], \tag{16}$$

$$p_R = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R - f'''\tilde{\nabla}^a R\tilde{\nabla}_a R\right)\right], \tag{17}$$

$$q^R_a = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \frac{1}{3}f''\Theta\tilde{\nabla}_a R\right], \tag{18}$$

$$\pi^R_{ab} = \frac{1}{f'}\left[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R - \sigma_{ab}\dot{R}f''\right]. \tag{19}$$

In the $1+3$ covariant decomposition [39,40], a fundamental observer slices spacetime into time and space. The Bianchi and Ricci identities

$$\nabla_{[a}R_{bc]d}{}^{e} = 0, \quad (\nabla_a\nabla_b - \nabla_b\nabla_a)u_c = R_{abc}{}^{d}u_d \tag{20}$$

applied on the total fluid 4-velocity u^a result in evolution equations—which propagate consistent initial data on some initial ($t = t_0$) hypersurface S_0 uniquely along timelike congruences - and constraint equations—which restrict the initial data to be specified [41]. In $f(R)$ gravity, the evolution equations are given by [38]

$$\dot{\mu}_m = -(\mu_m + p_m)\Theta - \tilde{\nabla}^a q^m_a - 2A_a q^a_m - \sigma^a_b\pi^b_{a,m}, \tag{21}$$

$$\dot{\mu}_R = -(\mu_R + p_R)\Theta + \frac{\mu_m f''}{f'^2}\dot{R} - \tilde{\nabla}^a q^R_a - 2A_a q^a_R - \sigma^a_b\pi^b_{a,R}, \tag{22}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p) + \tilde{\nabla}_a A^a - A_a A^a - \sigma_{ab}\sigma^{ab} + 2\omega_a\omega^a, \tag{23}$$

$$\dot{q}^m_a = -\tfrac{4}{3}\Theta q^m_a - (\mu_m + p_m)A_a - \tilde{\nabla}_a p_m - \tilde{\nabla}^b\pi^m_{ab} - \sigma^b_a q^m_b - A^b\pi^m_{ab} - \varepsilon_{abc}\omega^b q^c_m, \tag{24}$$

$$\dot{q}^R_a = -\tfrac{4}{3}\Theta q^R_a + \frac{\mu_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b\pi^R_{ab} - \sigma^b_a q^R_b - (\mu_R + p_R)A_a - A^b\pi^R_{ab} - \varepsilon_{abc}\omega^b q^c_R, \tag{25}$$

$$\dot{\omega}_a = -\tfrac{2}{3}\Theta\omega_a - \tfrac{1}{2}\varepsilon_{abc}\tilde{\nabla}^b A^c + \sigma_a^b\omega_b, \tag{26}$$

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle} + A_{\langle a}A_{b\rangle} - \sigma^c_{\langle\, a}\sigma_{b\rangle c} - \omega_{\langle a}\omega_{b\rangle}, \tag{27}$$

$$\begin{aligned}\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} =& \varepsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \Theta\left(E_{ab} + \tfrac{1}{6}\pi_{ab}\right) - \tfrac{1}{2}(\mu + p)\,\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q_{b\rangle} \\ &+3\sigma_a^{\langle c}\left(E_{b\rangle c} - \tfrac{1}{6}\pi_{b\rangle c}\right) - A_{\langle a}q_{b\rangle} + \varepsilon_{cd\langle a}\left[2A^c H^d_{b\rangle} + \omega^c(E^d_{b\rangle} + \tfrac{1}{2}\pi^d_{b\rangle})\right],\end{aligned} \tag{28}$$

$$\begin{aligned}\dot{H}_{ab} =& -\Theta H_{ab} - \varepsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\varepsilon_{cd\langle a}\tilde{\nabla}^c\pi^d_{b\rangle} \\ &+ 3\sigma_a^{\langle c}H_{b\rangle c} + \tfrac{3}{2}\omega_{\langle a}q_{b\rangle} - \varepsilon_{cd\langle a}\left[2A^c E^d_{b\rangle} - \tfrac{1}{2}\sigma^c_{b\rangle}q^d - \omega^c H^d_{b\rangle}\right],\end{aligned} \tag{29}$$

whereas the constraints read

$$(C^1)_a := \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta + \varepsilon_{abc}\left(\tilde{\nabla}^b\omega^c + 2A^b\omega^c\right) + q_a = 0, \tag{30}$$

$$(C^2)_{ab} := \varepsilon_{cd(a}\tilde{\nabla}^c\sigma_{b)}{}^d + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} - 2A_{\langle a}\omega_{b\rangle} = 0, \tag{31}$$

$$\begin{aligned}(C^3)_a :=& \tilde{\nabla}^b H_{ab} + (\mu + p)\omega_a + \varepsilon_{abc}\left[\tfrac{1}{2}\tilde{\nabla}^b q^c + \sigma_{bd}\left(E^d{}_c + \tfrac{1}{2}\pi^d{}_c\right)\right] \\ &+3\omega_b\left(E^{ab} - \tfrac{1}{6}\pi^{ab}\right) = 0,\end{aligned} \tag{32}$$

$$\begin{aligned}(C^4)_a :=& \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b\pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\mu + \tfrac{1}{3}\Theta q_a \tfrac{1}{2}\sigma_a^b q_b \\ &-3\omega^b H_{ab} - \varepsilon_{abc}[\sigma^{bd}H^c_d - \tfrac{3}{2}\omega^b q^c] = 0,\end{aligned} \tag{33}$$

$$(C^5) := \tilde{\nabla}^a\omega_a - A_a\omega^a = 0. \tag{34}$$

The Gauß–Codazzi equations are given by

$$\tilde{R}_{ab} + \dot{\sigma}_{\langle ab\rangle} + \Theta\sigma_{ab} - \tilde{\nabla}_{\langle a}A_{b\rangle} - A_{\langle a}A_{b\rangle} - \pi_{ab} - \frac{1}{3}\left(2\mu - \frac{2}{3}\Theta^2\right)h_{ab} = 0, \tag{35}$$

where $\tilde{R}_{ab}$ is the Ricci tensor on 3-D spatial hypersurfaces with $\tilde{R} = 2\mu - \frac{2}{3}\Theta^2 + 2\sigma^2$ as its corresponding (3-curvature) Ricci scalar.

3 Orthogonal models

Following [16], the orthogonal models are characterised by the matter EMT representing an anisotropic fluid without heat fluxes

$$T^m_{ab} = \mu_m u_a u_b + p_m h_{ab} + \pi^m_{ab}, \tag{36}$$

the matter energy density and isotropic pressure measured by an observer moving with the velocity u^a. In this setting, we have an irrotational and non-accelerated flow of the vector field u^a and therefore $\omega_a = 0 = A_a$. Thus the corresponding evolution and constraint equations are given by

$$\dot{\mu}_m = -(\mu_m + p_m)\Theta - \sigma^a_b\pi^b_{a,m}, \tag{37}$$

$$
\begin{aligned}
\dot{\mu}_R &= -(\mu_R + p_R)\Theta + \frac{\mu_m f''}{f'^2}\dot{R} - \tilde{\nabla}^a q_a^R - \sigma^a_b \pi^b_{a,R}, &(38)\\
\dot{\Theta} &= -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p) - \sigma_{ab}\sigma^{ab}, &(39)\\
\dot{q}_a^R &= -\tfrac{4}{3}\Theta q_a^R + \frac{\mu_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b \pi^R_{ab} - \sigma^b_a q^R_b, &(40)\\
\dot{\sigma}_{ab} &= -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi_{ab} - \sigma^c_{\langle a}\sigma_{b\rangle c}, &(41)\\
\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} &= \varepsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \Theta\left(E_{ab} + \tfrac{1}{6}\pi_{ab}\right) - \tfrac{1}{2}(\mu + p)\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a} q^R_{b\rangle}\\
&\quad + 3\sigma_a^{\langle c}\left(E_{b\rangle c} - \tfrac{1}{6}\pi_{b\rangle c}\right), &(42)\\
\dot{H}_{ab} &= -\Theta H_{ab} - \varepsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\varepsilon_{cd\langle a}\tilde{\nabla}^c \pi^d_{b\rangle} + 3\sigma_a^{\langle c} H_{b\rangle c}\\
&\quad + \tfrac{1}{2}\varepsilon_{cd\langle a}\sigma^c_{b\rangle} q^d_R, &(43)\\
(C^{*1})_a &:= \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta + q_a^R = 0, &(44)\\
(C^{*2})_{ab} &:= \varepsilon_{cd(a}\tilde{\nabla}^c\sigma_{b)}{}^d - H_{ab} = 0, &(45)\\
(C^{*3})_a &:= \tilde{\nabla}^b H_{ab} + \varepsilon_{abc}\left[\tfrac{1}{2}\tilde{\nabla}^b q^c_R + \sigma_{bd}\left(E^d{}_c + \tfrac{1}{2}\pi^d{}_c\right)\right] = 0, &(46)\\
(C^{*4})_a &:= \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b\pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\mu + \tfrac{1}{3}\Theta q_a^R - \tfrac{1}{2}\sigma^b_a q_b\\
&\quad - \varepsilon_{abc}\sigma^{bd}H^c_d = 0, &(47)
\end{aligned}
$$

where the new equations corresponding to Eqs. (26) and (34) become trivial, and with Eq. (24) resulting in the constraint

$$(C^{*5})_a := \tilde{\nabla}_a p_m + \tilde{\nabla}^b \pi^m_{ab} = 0. \tag{48}$$

4 Shear-free anisotropic models with an imperfect fluid

From causal relativistic thermodynamical relationships for imperfect fluids, the anisotropic pressure is known to evolve according to [42–45]

$$\tau\dot{\pi}_{ab} + \pi_{ab} = -\lambda\sigma_{ab}, \tag{49}$$

where τ and λ are relaxation and viscosity parameters. If we consider cases where τ is negligible and λ is a positive constant, and use the fairly popular ansatz (valid near thermal equilibrium, such as in the very early stages of the Universe) for the equation of state [16,25,46]

$$\pi_{ab} = -\lambda\sigma_{ab}, \tag{50}$$

then Eqs. (15) and (19) imply that we can rewrite (50) as

$$\pi^m_{ab} + f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R = \sigma_{ab}\left(\dot{R}f'' - \lambda f'\right). \tag{51}$$

For a general case of vanishing shear tensor during the entire cosmic evolution, one can see from Eq. (51) that

$$\pi^m_{ab} = -f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R - f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R. \tag{52}$$

Moreover, the Gauß–Codazzi equations (35) reduce to

$$\tilde{R}_{ab} - \tfrac{1}{3}\tilde{R}h_{ab} = \pi_{ab} = \frac{1}{f'}\left(\pi^m_{ab} + f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R\right), \tag{53}$$

thus showing that even if the matter anisotropic stress vanishes, no constant-curvature geometries are guaranteed and hence no necessarily FLRW universes. It is also worth noticing that, unlike in GR, if we allow the matter anisotropic pressure to be nonzero despite a vanishing shear, constant-curvature models are allowed provided

$$f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R = 0. \tag{54}$$

The converse also holds, i.e., it is possible, unlike in GR, to have a vanishing matter anisotropic pressure π^m_{ab} for a non-constant curvature geometry.

One can see the tidal effect on the anisotropic stresses by dropping the shear terms of Eq. (41), obtaining the equation

$$\pi_{ab} = 2E_{ab}, \tag{55}$$

which shows that, in this case as in GR [16], the anisotropic stresses are related to the electric part of the Weyl tensor in such a way that they balance each other, a necessary and sufficient condition for the shear to remain zero if initially vanishing.

If the shear is nonzero, but with very small second-order contributions, then one can show that Eq. (41) can be approximated by

$$\dot{\sigma}_{ab} \approx -\tfrac{2}{3}\Theta\sigma_{ab}. \tag{56}$$

Rewriting Eq. (56) as

$$\left(\sigma^2\right)^{\cdot} \approx -\tfrac{4}{3}\Theta\sigma^2 \tag{57}$$

shows that the shear decays with expansion. One can, therefore, conclude that *within the class of orthogonal f(R) models, small perturbations of shear are damped*, i.e. that these models are stable if expanding, a result similar to that obtained in [16] for models whose underlying theory is GR.

For shear-free orthogonal models satisfying Eq. (55), we see that Eq. (45) implies a purely electric Weyl tensor, i.e., $H_{ab} = 0$, and hence Eq. (43) reduces to an identity:

$$\varepsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} = \tfrac{1}{2}\varepsilon_{cd\langle a}\tilde{\nabla}^c\pi^d_{b\rangle}. \tag{58}$$

Moreover, it is straighforward to show using Eqs. (42) and (47) that

$$\dot{E}_{ab} = -\tfrac{2}{3}\Theta E_{ab} - \tfrac{1}{4}\tilde{\nabla}_{\langle a} q^R_{b\rangle}, \tag{59}$$

$$\tilde{\nabla}^b E_{ab} = \tfrac{1}{6}\left(\tilde{\nabla}_a \mu - \tfrac{1}{3}\Theta q^R_a\right). \tag{60}$$

Defining $E^2 \equiv E_{ab}E^{ab}$, we can rewrite Eq. (59) as

$$\left(E^2\right)^{\cdot} = -\tfrac{4}{3}\Theta E^2 - \frac{1}{8}\left(\tilde{\nabla}_{\langle a} q^R_{b\rangle} E^{ab} + \tilde{\nabla}^{\langle a} q^{b\rangle}_R E_{ab}\right), \tag{61}$$

thus showing the decay of the electric part of the Weyl tensor and the anisotropic stress tensor with the expansion. This equation also implies decay with the heat flux of the curvature fluid if the bracketed terms in the r.h.s are overall positive.

Let us now consider the generalized Friedman equation

$$\Theta^2 = 3\left(\mu - \tfrac{1}{2}\tilde{R}\right). \tag{62}$$

Since the total energy density μ is not always guaranteed to be positive for generic $f(R)$ models, it is not straightforward to comment on the asymptotic isotropization of expanding shear-free anisotropic models for the different values of the spatial curvature. This is in contrast to the GR result where, for example, expanding shear-free models which exhibit negative spatial curvature asymptotically approach isotropy [16].

5 Anisotropic LRS models

Let us consider the locally rotationally symmetric (LRS) metric given by

$$ds^2 = -dt^2 + a^2(t)dr^2 + b^2(t)\left[d\theta^2 + f^2(\theta)d\phi^2\right], \tag{63}$$

where

$$f(\theta) = \begin{cases} \sin(\theta) & \text{for } \tilde{R} > 0 \text{ (Kantowski-Sachs)}, \\ \theta & \text{for } \tilde{R} = 0 \text{ (Bianchi I)}, \\ \sinh(\theta) & \text{for } \tilde{R} < 0 \text{ (Bianchi III)}. \end{cases}$$

Here $\tilde{R} = 2k/b^2$ for $k = \pm 1, 0$. The non-vanishing kinematic quantities for these models are the expansion and shear, respectively given as

$$\Theta = \frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}, \tag{64}$$

$$2\sigma^2 = \sigma_{ab}\sigma^{ab} = \frac{1}{\sqrt{3}}\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right). \tag{65}$$

Consider the EMT of the imperfect fluid matter source to be of the form

$$T_{ab} = \mu u_a u_b + \bar{p} h_{ab} - \bar{\pi}(e_1)_a (e_1)_b \tag{66}$$

where, because of the rotational symmetry, $e_1 = \frac{1}{a}\frac{\partial}{\partial r}$ is defined as the unit vector along the axis of symmetry. Whereas μ represents the total energy density measured by a comoving observer, the pressure measured by the same observer is

$$p = \bar{p} - \bar{\pi}. \tag{67}$$

Here the anisotropic stress tensor in the orthonormal tetrad bases

$$e_0 = \frac{\partial}{\partial t}, \quad e_1 = \frac{1}{a}\frac{\partial}{\partial r}, \quad e_2 = \frac{1}{b}\frac{\partial}{\partial \theta}, \quad e_3 = \frac{1}{b\sin\theta}\frac{\partial}{\partial \phi} \tag{68}$$

is given by

$$\pi_{ab} = diag\left(0, -\frac{2}{3}\bar{\pi}, \frac{1}{3}\bar{\pi}, \frac{1}{3}\bar{\pi}\right). \tag{69}$$

This way we can write the modified EFEs as

$$2\frac{\dot{a}\dot{b}}{ab} + \frac{k + \dot{b}^2}{b^2} = \mu, \tag{70}$$

$$2\frac{\ddot{b}}{b} + \frac{k + \dot{b}^2}{b^2} = -\bar{p} + \bar{\pi}, \tag{71}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} = -\bar{p}, \tag{72}$$

whereas the conservation Eqs. (37), (38), (40) and (48) are rewritten as

$$\dot{\mu}_m = -\left(\mu_m + \bar{p}_m - \tfrac{1}{3}\bar{\pi}_m\right)\Theta, \tag{73}$$

$$\dot{\mu}_R = -\left(\mu_R + \bar{p}_R - \tfrac{1}{3}\bar{\pi}_R\right)\Theta + \frac{\mu_m f''}{f'^2}\dot{R} - \tilde{\nabla}^a q_a^R, \tag{74}$$

$$\dot{q}_a^R = -\tfrac{4}{3}\Theta q_a^R + \frac{\mu_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a \bar{p}_R - \tilde{\nabla}_a \bar{\pi}_R, \tag{75}$$

$$\tilde{\nabla}_a \bar{p}_m (e_1)^a = \tilde{\nabla}_a \bar{\pi}_m (e_1)^a. \tag{76}$$

As a result of the homogeneity assumption, $\bar{\pi} = \bar{\pi}_m(t)$ and therefore Eq. (76) is trivially satisfied.

We notice from Eq. (65) that for the case of vanishing shear, $a(t) = b(t)$ and thus the modified EFES (70)–(72) reduce to

$$3\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \mu, \tag{77}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -\bar{p} + \bar{\pi}, \tag{78}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -\bar{p}. \tag{79}$$

Subtracting Eq. (79) from (78) yields

$$\bar{\pi} = \frac{k}{a^2}, \tag{80}$$

and therefore

$$E_{ab} = diag\,(0, -2E, E, E)\,, \tag{81}$$

where $E = \frac{\bar{\pi}}{6}$.

We adopt the barotropic EoS , $p_m = (\gamma_m - 1)\mu_m$, where $p_m = \bar{p}_m - \bar{\pi}_m/3$, from the continuity Eq. (73) for p_m, we obtain $\mu_m = \mu_m^0 a^{-3\gamma_m}$. To integrate (77) we need to know μ_R. Indeed, it is a hard job to integrate (74) although we are working in the homogeneous case. But we can rewrite (77) in the following form:

$$3\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \mu_m^0 a^{-3\gamma_m} + \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \frac{3\dot{a}}{a}f''\dot{R}\right]. \tag{82}$$

Here μ_m^0 is the matter density at the time $t = t_0$ and γ_m is the EoS parameter for the matter content. As we see, Eq. (82) is model dependent. To specify solutions we must choose a specific model of $f(R)$ gravity. Otherwise, we cannot integrate it explicitly. Let us have a brief qualitative analysis of (82). If we are looking for the late-time behavior of the solutions for (82) and if we suppose that the space is flat $k = 0$, and without matter, the evolution is defined by the de Sitter (dS) solution, in which we put $R = 6H_0^2$, where H_0 is the time scale of the dS universe. In this simple case, we can solve Eq. (82) to obtain:

$$H_0^2 = \frac{1}{6f'}(Rf' - f). \tag{83}$$

But this is not the only case we can solve (77). Suppose that we choose a model of $f(R)$, so (77) reduces generally to a fourth-order ODE, which can be solved in terms of quadratures. For example, in the so-called Starobinsky model, $f(R) = R + \alpha R^2$, which is motivated for the inflationary universe scenario [10], Eq. (82) reduces to the following differential equation:

$$3\frac{\dot{a}^2}{a^2}=\mu_m^0 a^{-3\gamma_m}+\frac{\alpha}{2}\frac{R^2-12H\dot{R}}{1+2\alpha R}, \tag{84}$$

where $R=6\left(\frac{\dot{a}^2}{a^2}+\frac{\ddot{a}}{a}\right)$. Eq. (84) is a third oder ODE for $a(t)$. So we need to specify initial condition(s) (ICs), as well as integrability condition(s). The cosmological ICs are fitted using the Hubble H, deceleration q, jerk j, and snap s parameters evaluated at the present time $t=t_0$. We can adjust the first derivatives of the scale factor as $a(0)=a_0=1, \dot{a}(0)=H_0 a_0, \ddot{a}(0)=-H_0^2 a_0 q_0, \dddot{a}(0)=H_0^3 j_0 a_0^{-1}$ where q_0 is the deceleration parameter at the initial time (present time), j_0 is the jerk parameter at the instant $t=t_0$, etc [47]. Fortunately, these data have been measured with high precisions.

A series solution for $a(t)$ in Eq. (84) has been developed using these cosmographic parameters which are all evaluated at $t=t_0$:

$$\begin{aligned}
a(t) &= 1+H_0(t-t_0)-1/2\,{H_0}^2 q_0 (t-t_0)^2\\
&-\frac{1}{216}\frac{\left(-3\,{H_0}^2+54\,{H_0}^4\alpha+\mu_m+12\,\alpha\,\mu_m {H_0}^2-12\,\alpha\,\mu_m^0 {H_0}^2 q_0+18\,\alpha\,{H_0}^4{q_0}^2+36\,{H_0}^4\alpha\,q_0\right)}{\alpha\,H_0}\\
&\times(t-t_0)^3\\
&+\frac{1}{2592}\frac{(t-t_0)^4}{\alpha\,{H_0}^2}\times\left(9\,{H_0}^4+162\,{H_0}^6\alpha-12\,\mu_m^0{H_0}^2+18\,{H_0}^4\alpha\,\mu_m^0\right.\\
&+108\,{H_0}^4\alpha\,\mu_m^0 q_0-54\,{H_0}^6\alpha\,{q_0}^2\\
&+324\,{H_0}^6\alpha\,q_0-108\,\alpha\,{H_0}^6{q_0}^3-6\,\mu_m^0{H_0}^2 q_0+9\,\mu_m^0\gamma_m{H_0}^2+{\mu_m}^2\\
&+90\,\alpha\,\mu_m^0{H_0}^4{q_0}^2-12\,\alpha\,{\mu_m^0}^2{H_0}^2 q_0\\
&\left.+12\,\alpha\,{\mu_m}^2{H_0}^2+108\,\mu_m^0\gamma_m{H_0}^4\alpha-108\,\mu_m^0\gamma_m{H_0}^4\alpha\,q_0\right)+O\left[(t-t_0)^5\right].
\end{aligned}\tag{85}$$

The above solution can be used to check observational constraints. As an alternative, we can also solve Eq. (84) numerically. A numerical solution for the Hubble parameter is developed in Fig. 1 where we put $a_0=H_0=1, q_0=-0.7$.

We see in Fig. 1 that H is an oscillatory function, it reaches maxima and minima several times. It defines an oscillatory solution but it is not in the form of Type IV future singularity [48–52]. But it can be identified in the late-time as the ΛCDM era.

We can classify the future singularities as follows:

- Type I: ("Big Rip"): $t\to t_s$, $a\to\infty$, $\mu\to\infty$ and $|p|\to\infty$.
- Type II: ("sudden"): $t\to t_s$,$a\to a_s$, $\mu\to\mu_s$ and $|p|\to\infty$.
- Type III : $t\to t_s$, $a\to a_s$, $\mu\to\infty$ and $|p|\to\infty$
- Type IV : $t\to t_s$, $a\to a_s$, $\mu\to 0$ and $|p|\to 0$ and higher derivatives of H diverge. Here t_s, a_s and μ_s are constants with $a_s\neq 0$.

For our case, the factor given in Fig. 1, the Hubble parameter and first, second and third derivatives of H are plotted in Fig. 2. No higher derivatives of H diverges.

A phase portrait for Starobinsky model is plotted in Fig. 3. Here we solved the ODE with parameters $\Omega_m^0\equiv\frac{\mu_m^0}{3H_0^2}=0.3$, $\gamma_m=1$.

The phase portrait shows that the scale factor $a(t)$ is a monotonic increasing function of time. It is always increasing, and never decreasing.

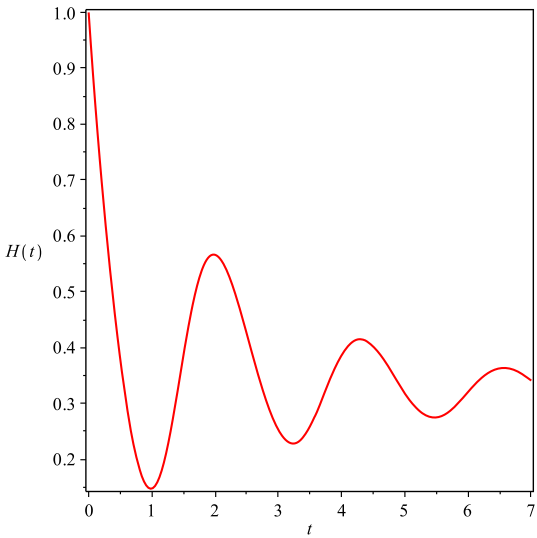

Fig. 1 Numerical solution for $H(t)$. The model of $f(R)$ is the one proposed by Starobinsky, with $\alpha = 0.02$. The cosmological data are fitted with observational data for extended cosmological parameters

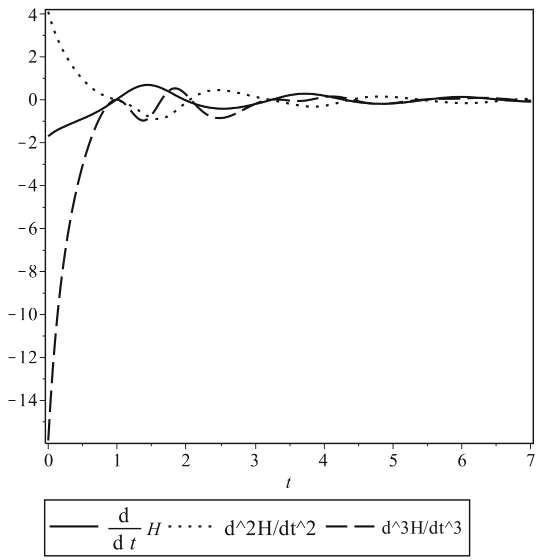

Fig. 2 Numerical solution for $\dot{H}, \ddot{H}, \dddot{H}$

For curiosity we are interested to know if the system has attractors or not. The late-time or asymptotic attractors are a class of solutions which have a generic form independent of the initial conditions. We examine our model for such types of solutions and solve the equations of motion for some initial conditions. The model is well established as an attractor in the following Fig. 4.

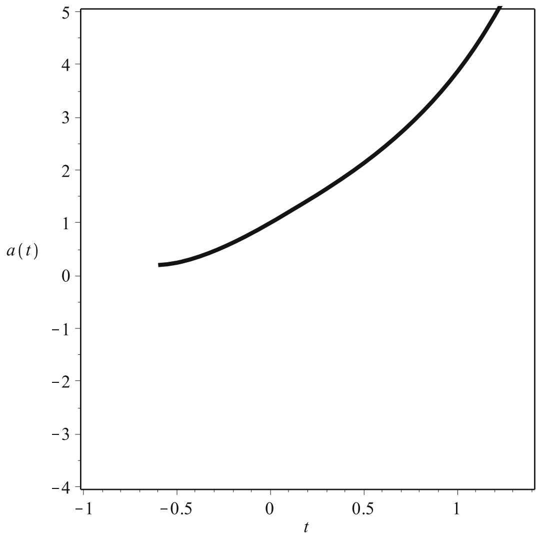

Fig. 3 Phase portrait for Starobinsky's model

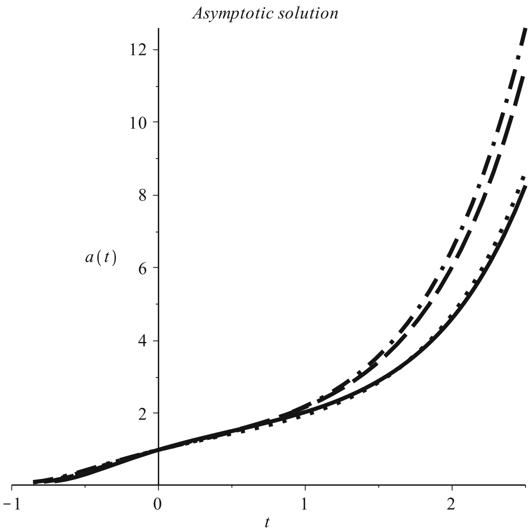

Fig. 4 Attractors for Starobinsky's model

6 Discussions and conclusion

In this work we looked at classes of shear-free anisotropic cosmological spacetimes in $f(R)$ gravity. Focusing on orthogonal models with irrotational and non-accelerated fluid flows without heat fluxes, we showed that the anisotropic stresses are related to the electric part of the Weyl tensor in such a way that they balance each other. This is considered necessary and sufficient condition for the shear to be vanishing forever if vanishing initially. This turned out to be a generalization of a previous result [16] for models whose underlying theory is GR. We also showed that within the class of orthogonal $f(R)$ models, small perturbations of shear are damped, i.e,. that these models are stable if expanding, and that the electric part of the Weyl tensor and the anisotropic stress tensor decay with the expansion as well as the heat flux of the curvature fluid.

As an application, we considered a subclass of locally rotationally symmetric space-times with barotropic equations of state and studied the evolutionary dynamics of the Universe. In particular, we showed that the late-time behaviour of the dS universe in $f(R)$ gravity should satisfy Eq. (83). For the Starobinsky model of $f(R)$, we provided a power-series solution for $a(t)$ and we studied the behavior of the expansion parameter $H(t)$ by numerically integrating the Friedmann Eq. (84), where the intial conditions for H_0, q_0 and j_0 are taken from observational data. The result is the oscillatory solution presented in Fig. 1 and describes the late-time universe in the ΛCDM era. The first three derivatives of H have also been calculated as shown in Fig. 2; none of these derivatives diverges. A phase-portrait anaysis for this model with $\Omega_m^0 = 0.3$, $\gamma_m = 1$, given in Fig. 3, shows that the scale factor is a monotonically increasing function of time. Finally, we examined our model for late-time or asymptotic attractors, with well established solutions depicted in Fig. 4.

References

1. Capozziello, S., De Laurentis, M.: Extended theories of gravity. Phys. Rep. **509**, 167–321 (2011)
2. Modesto, L.: Super-renormalizable quantum gravity. Phys. Rev. D **86**, 044005 (2012)
3. Modesto, L., Rachwal, L.: Super-renormalizable and finite gravitational theories. Nucl. Phys. B **889**, 228–248 (2014)
4. Biswas, T., Gerwick, E., Koivisto, T., Mazumdar, A.: Towards singularity and ghost free theories of gravity. Phys. Rev. Lett. **108**, 031101 (2012)
5. Clifton, T., Ferreira, P.G., Padilla, A., Skordis, C.: Modified gravity and cosmology. Phys. Rep. **513**, 1–189 (2012)
6. De Felice, A., Tsujikawa, S.: $f(R)$ theories. Living Rev. Rel. **13**, 1002–4928 (2010)
7. Nojiri, S., Odintsov, S.D.: Introduction to modified gravity and gravitational alternative for dark energy. Int. J. Geom. Methods Mod. Phys. **4**, 115–145 (2007)
8. Nojiri, S., Odintsov, S.D.: Unified cosmic history in modified gravity: from f (r) theory to lorentz non-invariant models. Phys. Rep. **505**, 59–144 (2011)
9. Buchdahl, H.A.: Non-linear Lagrangians and cosmological theory. Mon. Not. R. Astron. Soc. **150**, 1 (1970)
10. Starobinsky, A.A.: A new type of isotropic cosmological models without singularity. Phys. Lett. B **91**, 99–102 (1980)
11. Carroll, S., Duvvuri, V., Turner, M., Trodden, M.: Is cosmic speed-up due to new gravitational physics? Phys. Rev. D **70**, 043528 (2004)
12. Nojiri, S., Odintsov, S.D.: Modified gravity with negative and positive powers of curvature: unification of inflation and cosmic acceleration. Phys. Rev. D **68**, 123512 (2003)
13. Sotiriou, T.P., Liberati, S.: Metric-affine $f(R)$ theories of gravity. Ann. Phys. **322**, 935–966 (2007)
14. Nojiri, S., Odintsov, S.D.: Modified $f(R)$ gravity consistent with realistic cosmology: from a matter dominated epoch to a dark energy universe. Phys. Rev. D **74**, 086005 (2006)
15. Starobinsky, A.A.: Disappearing cosmological constant in $f(R)$ gravity. JETP Lett. **86**, 157–163 (2007)
16. Mimoso, J.P., Crawford, P.: Shear-free anisotropic cosmological models. Class. Quantum Gravity **10**, 315 (1993)
17. Ellis, G.: Dynamics of pressure-free matter in general relativity. J. Math. Phys. **8**, 1171 (1967)
18. G ö del, K.: Rotating universes in general relativity theory . In: Graves, L.M. et al. Proceedings of the International Congress of Mathematicians, vol. 1, p. 175. Cambridge, Mass (1952)

19. Goldberg, J., Sachs, R.: A theorem on petrov type (field equations for proving theorem identifying geometrical properties of null congruence with existence of algebraically special riemann tensor). 1966. 13–23 (1962)
20. Robinson, I., Schild, A.: Generalization of a theorem by goldberg and sachs. J. Math. Phys. **4**, 484 (1963)
21. Ellis, G., MacCallum, M.A.: A class of homogeneous cosmological models. Commun. Math. Phys. **12**, 108–141 (1969)
22. MacCallum, M.: A class of homogeneous cosmological models iii: asymptotic behaviour. Commun. Math. Phys. **20**, 57–84 (1971)
23. Collins, C.: Shear-free fluids in general relativity. Can. J. Phys. **64**, 191–199 (1986)
24. Barrow, J.D., Matzner, R.A.: The homogeneity and isotropy of the universe. Mon. Not. R. Astron. Soc. **181**, 719–727 (1977)
25. MacCallum, M., Stewart, J., Schmidt, B.: Anisotropic stresses in homogeneous cosmologies. Commun. Math. Phys. **17**, 343–347 (1970)
26. Koivisto, T.S., Mota, D.F., Quartin, M., Zlosnik, T.G.: Possibility of anisotropic curvature in cosmology. Phys. Rev. D **83**, 023509 (2011)
27. Barrow, J.D., Ottewill, A.C.: The stability of general relativistic cosmological theory. J. Phys. A Math. Gen. **16**, 2757 (1983)
28. Barrow, J.D., Clifton, T.: Exact cosmological solutions of scale-invariant gravity theories. Class. Quant. Gravity **23**, L1 (2006)
29. Clifton, T., Barrow, J.D.: Further exact cosmological solutions to higher-order gravity theories. Class. Quant. Gravity **23**, 2951 (2006)
30. Middleton, J.: On the existence of anisotropic cosmological models in higher order theories of gravity. Class. Quant. Gravity **27**, 225013 (2010)
31. Abebe, A., Goswami, R., Dunsby, P.K.S.: Shear-free perturbations of $f(R)$ gravity. Phys. Rev. D **84**, 124027 (2011)
32. Abebe, A.: Beyond concordance cosmology. Ph.D. thesis, UCT (University of Cape Town) (2013)
33. Abebe, A.: Anti-newtonian cosmologies in $f(R)$ gravity. Class. Quant. Gravity **31**, 115011 (2014)
34. Abebe, A., Elmardi, M.: Irrotational-fluid cosmologies in fourth-order gravity. arXiv preprint arXiv:1411.6394 (2014)
35. Ellis, G.F.R., Van Elst, H.: Cosmological models. In: Lachiéze-Rey, M. (ed.) Theoretical and Observational Cosmology, pp. 1–116. Springer, The Netherlands (1999)
36. Maartens, R., Triginer, J.: Density perturbations with relativistic thermodynamics. Phys. Rev. D **56**, 4640 (1997)
37. Betschart, G.: General relativistic electrodynamics with applicantions in cosmology and astrophysics. Ph.D. thesis, University of Cape Town (2005)
38. Carloni, S., Dunsby, P., Troisi, A.: Evolution of density perturbations in $f(R)$ gravity. Phys. Rev. D **77**, 024024 (2008)
39. Ellis, G., Bruni, M.: Covariant and gauge-invariant approach to cosmological density fluctuations. Phys. Rev. D **40**, 1804 (1989)
40. Ellis, G., Maartens, R., MacCallum, M.A.: Relativistic Cosmology. Cambridge University Press, Cambridge (2012)
41. Maartens, R.: Covariant velocity and density perturbations in quasi-newtonian cosmologies. Phys. Rev. D **58**, 124006 (1998)
42. Israel, W.: Nonstationary irreversible thermodynamics: a causal relativistic theory. Ann. Phys. **100** (1976). http://gen.lib.rus.ec/scimag/index.php?s=10.1016/0003-4916(76)90064-6
43. Novella, M.: Cosmology and Gravitation Two, vol. 2, Atlantica S é guier Fronti è res, (1996)
44. Maartens, R. Causal thermodynamics in relativity. arXiv preprint astro-ph/9609119 (1996)
45. Rezzolla, L., Zanotti, O.: Relativistic Hydrodynamics. Oxford University Press, Oxford (2013)
46. Coley, A.A., Van den Hoogen, R.: Qualitative analysis of viscous fluid cosmological models satisfying the israel-stewart theory of irreversible thermodynamics. Class. Quant. Gravity **12**, 1977 (1995)
47. Visser, M.: Jerk, snap and the cosmological equation of state. Class. Quant. Gravity **21**, 2603 (2004)
48. Nojiri, S., Odintsov, S, & Tsujikawa, S.: Properties of singularities in (phantom) dark energy universe. Phys. Rev. d71 063004 (2005). arXiv preprint hep-th/0501025
49. Brandenberger, R. H.: The matter bounce alternative to inflationary cosmology. arXiv preprint arXiv:1206.4196 (2012)

50. Cai, Y.-F.: Exploring bouncing cosmologies with cosmological surveys. Sci. China Phys. Mech. Astron. **57**, 1414–1430 (2014)
51. Bamba, K., Nojiri, S., Odintsov, S.D.: The future of the universe in modified gravitational theories: approaching a finite-time future singularity. J. Cosmol. Astropart. Phys. **2008**, 045 (2008)
52. Barrow, J.D.: More general sudden singularities. Class. Quant. Gravity **21**, 5619 (2004)