Journal of Physics: Conference Series

PAPER • OPEN ACCESS

Integrable hierarchies of Heisenberg ferromagnet equation

To cite this article: G. Nugmanova and A. Azimkhanova 2016 *J. Phys.: Conf. Ser.* **738** 012132

Integrable hierarchies of Heisenberg ferromagnet equation

G. Nugmanova, A. Azimkhanova

L.N. Gumilyov Eurasian National University, Astana, Kazakhstan

E-mail: nugmanovagn@gmail.com

Abstract.
In this paper we consider the coupled Kadomtsev-Petviashvili system. From compatibility conditions we obtain the form of matrix operators. After using a gauge transformation, obtained a new type of Lax representation for the hierarchy of Heisenberg ferromagnet equation, which is equivalent to the gauge coupled Kadomtsev-Petviashvili system.

1. Introduction

Korteweg-de Vries equation (KdV) is one of the most important equation in the theory of integrable systems , which have multi-soliton solution, an infinite number of conservation laws and many scientific applications [1].

In this paper we consider the expansion of the KdV equation - coupled Kadomtsev-Petviashvili equation (KP) with three potentials:

$$q_t = \frac{1}{4}(q_{xxx} - 6qq_x + 3\int q_{yy}dx + 6(pr)_x), \tag{1}$$

$$p_t = \frac{1}{2}(-p_{xxx} - 6qp_x + 3qp_x + 3p\int q_y dx - 3p_{xy}), \tag{2}$$

$$r_t = \frac{1}{2}(-r_{xxx} + 3qr_x - 3r\int q_y dx + 3r_{xy}), \tag{3}$$

where indexes represent partial derivatives. This system is also integrable and can assume multi-soliton solutions [2].

By using certain transformations [3], the system (1)-(3) can be reduced to a coupled system of the KdV equation [4], and the standard KP equation [5].

The main goal of this paper is to construct the equivalent spin system to the coupled KP system through gauge transformation.

2. Searching unknown elements of matrix operators for the Lax representation

In [6] presented decomposed form of this system by the first two terms into (1+1) dimensional hierarchy of Ablowitz-Kaup-Newell-Sigur:

$$u_y = -u_{xx} + 2u^2 v, \tag{4}$$

$$v_y = v_{xx} - 2uv^2, \tag{5}$$

$$u_t = u_{xxx} - 6uvu_x, \tag{6}$$

$$v_t = v_{xxx} - 6uvv_x \tag{7}$$

and offers the following assumption: if (u, v) is the solution of system (4)-(7), then the functions (p, q, r), defined as $q = 4uv, p = -2u^2, r = -2v^2$ is a solution of (1)-(3) [7].

The Lax representation of the system (4)-(7) is defined as

$$\Psi_x = U\Psi, \tag{8}$$

$$\Psi_y = V\Psi, \tag{9}$$

$$\Psi_t = W\Psi, \tag{10}$$

where U, V, W - matrix operators. Let U looks like:

$$U = -\frac{\lambda}{2}\sigma_3 + Q, \tag{11}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$. Matrix operator V given in the form

$$V = \lambda^2 V_2 + \lambda V_1 + V_0, \tag{12}$$

where $V_n = \begin{pmatrix} k_n & m_n \\ l_n & -k_n \end{pmatrix}$, $n = 0, 1, 2$. And operator W given by:

$$W = \lambda^3 W_3 + \lambda^2 W_2 + \lambda W_1 + W_0, \tag{13}$$

where $W_m = \begin{pmatrix} a_m & b_m \\ c_m & -a_m \end{pmatrix}$, $m = 0, 1, 2, 3$. Let's find the unknown elements of given matrices V and W. For this purpose, we will use the compatibility conditions of the system (4)-(7) and after this we obtain that

$$U_y - V_x + [U, V] = 0, \tag{14}$$

$$U_t - W_x + [U, W] = 0, \tag{15}$$

$$V_t - W_y + [V, W] = 0. \tag{16}$$

From (14) after substitution matrix (11) and (12) we have

$$Q_y - (\lambda^2 V_2 + \lambda V_1 + V_0)_x - \frac{1}{2}[\sigma_3, V] + [Q, V] = 0, \tag{17}$$

from which we obtain next system of equations in powers of λ:

$$\lambda^3 \quad : \quad -\frac{1}{2}[\sigma_3, V] = 0, \tag{18}$$

$$\lambda^2 \quad : \quad -V_{2x} - \frac{1}{2}[\sigma_3, V_1] + [Q, V_2] = 0, \tag{19}$$

$$\lambda^1 \quad : \quad -V_{1x} - \frac{1}{2}[\sigma_3, V_0] + [Q, V_1] = 0, \tag{20}$$

$$\lambda^0 \quad : \quad Q_y - V_{0x} + [Q, V_0] = 0. \tag{21}$$

Consistently solving the equations (18) - (21) we determine the unknown elements and form of the matrix operator V as following

$$V = \begin{pmatrix} -\frac{\lambda^2}{2} + uv & \lambda u - u_x \\ \lambda v + v_x & \frac{\lambda^2}{2} - uv \end{pmatrix}. \tag{22}$$

From (15) in terms of matrices U, V, W we have

$$Q_t - (\lambda^3 W_3 + \lambda^2 W_2 + \lambda W_1 + W_0)_x - \frac{\lambda}{2}[\sigma_3, W] + [Q, W] = 0. \tag{23}$$

Expanding in powers of λ, we obtain the following equation:

$$\lambda^4 \quad : \quad -\frac{1}{2}[\sigma_3, W_3] = 0, \tag{24}$$

$$\lambda^3 \quad : \quad -W_{3x} - \frac{1}{2}[\sigma_3, W_2] + [Q, W_3] = 0, \tag{25}$$

$$\lambda^2 \quad : \quad -W_{2x} - \frac{1}{2}[\sigma_3, W_1] + [Q, W_2] = 0, \tag{26}$$

$$\lambda^1 \quad : \quad -W_{1x} - \frac{1}{2}[\sigma_3, W_0] + [Q, W_1] = 0, \tag{27}$$

$$\lambda^0 \quad : \quad Q_t - W_{0x} + [Q, W_0] = 0. \tag{28}$$

From the system of equations (24)-(28) we obtain the unknown elements and form of the matrix W as

$$W = \begin{pmatrix} -\frac{\lambda^3}{2} + uv\lambda - vu_x + uv_x & u\lambda^2 - u_x\lambda - 2u^2v + u_{xx} \\ v\lambda^2 + v_x\lambda - 2uv^2 + v_{xx} & \frac{\lambda^3}{2} - uv\lambda + vu_x - uv_x \end{pmatrix}. \tag{29}$$

Thus, we found all unknown elements of the matrix operators V, W.

3. The new Lax representation form and coupled KP equivalent spin system

Now we are ready to obtain the Lax representation of a desired spin system by using a gauge transformation in form

$$\Phi = g^{-1}\Psi. \tag{30}$$

By definition, gauge equivalence is $g = \Psi|_{\lambda=0}$ and spin matrix is $S = g^{-1}\sigma_3 g$.

After using a gauge transformation, we get a new Lax representation for the hierarchy of Heisenberg ferromagnet equation, which is the gauge equivalence of the coupled KP system (1)-(3).

New Lax represenatation looks like:

$$\Phi_x = -\frac{\lambda}{2}S\Phi, \tag{31}$$

$$\Phi_y = -\frac{\lambda^2}{2}S\Phi + \frac{\lambda}{2}SS_x\Phi, \tag{32}$$

$$\Phi_t = -\frac{\lambda^3}{2}S\Phi + \frac{\lambda^2}{2}SS_x\Phi - \frac{3\lambda}{8}tr(S_x^2)S\Phi - \frac{\lambda}{2}S_{xx}\Phi. \tag{33}$$

Using the conditions of compatibility of the Lax representation we obtain a new spin system. From first condition $\Phi_{xy} = \Phi_{yx}$ after expanding by powers of λ we get the following equations:

$$S_x + \frac{1}{2}[SS_x, S] = 0, \tag{34}$$

$$S_y + \frac{1}{2}[S, S_{xx}] = 0. \tag{35}$$

From condition $\Phi_{xt} - \Phi_{tx} = 0$, expanding by powers of λ we have:

$$S_x + \frac{1}{2}[SS_x, S] = 0, \tag{36}$$

$$S_t - S_{xxx} - \frac{3}{4}tr(S_x^2)S_x - \frac{3}{4}[tr(S_x^2)]_x S = 0. \tag{37}$$

From the third conditions $\Phi_{yt} - \Phi_{ty} = 0$, expanding by powers of λ leads to the equations:

$$S_y + \frac{1}{2}[S, S_{xx}] = 0, \tag{38}$$

$$S_t + (SS_x)_y - \frac{3}{4}tr(S_x^2)S_x - \frac{1}{2}[S_{xx}, SS_x] = 0, \tag{39}$$

$$(SS_x)_t + S_{xxy} + \frac{3}{4}[tr(S_x^2)]_y S + \frac{3}{4}tr(S_x^2)S_y = 0. \tag{40}$$

Let us transform last two equations. After we consider (40)

$$S_t + (SS_x)_y - \frac{3}{4}tr(S_x^2)S_x - \frac{1}{2}[S_{xx}, SS_x] = 0. \tag{41}$$

In order to facilitate computations we divide this equation into the following two parts:

$$1) S_t - \frac{3}{4}tr(S_x^2)S_x = A, \tag{42}$$

$$2) (SS_x)_y - \frac{1}{2}[S_{xx}, SS_x] = B. \tag{43}$$

We consider the first term $(SS_x)_y$. Transform it considering the (35) and its derivative with respect to x $(S_{xy} = S_{yx})$:

$$(SS_x)_y = S_y S_x + SS_{xy} = \frac{1}{2}[S_{xx}, SS_x] - \frac{1}{2}S_{xxx} + \frac{1}{2}SS_{xxx}S. \tag{44}$$

Substituting this in (43) we find that

$$-\frac{1}{2}S_{xxx} + \frac{1}{2}SS_{xxx}S = B. \tag{45}$$

The second term $SS_{xxx}S$ we will express through the third derivative of the expression $S \cdot S = I$, multiplying it to S:

$$SS_{xxx}S = [S_{xx}S, S_x] - S_{xxx} - 2(S_x^2)_x S. \tag{46}$$

Substituting (46) into (45), we obtain

$$\frac{1}{2}[S_{xx}S, S_x] - S_{xxx} - (S_x^2)_x S = B. \tag{47}$$

Now we transform the first term of (47) $\frac{1}{2}[S_{xx}S, S_x]$:

$$\frac{1}{2}[S_{xx}S, S_x] = \frac{1}{2}(S_{xx}SS_x - S_x S_{xx}S) =$$

$$= \frac{1}{2}\big[(4uvg^{-1}\sigma_3 g + 2g^{-1}\begin{pmatrix} 0 & u_x \\ -v_x & 0 \end{pmatrix} g)g^{-1}\sigma_3 g \cdot 2g^{-1}\begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix} g -$$
$$-2g^{-1}\begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix} g \cdot (4uvg^{-1}\sigma_3 g + 2g^{-1}\begin{pmatrix} 0 & u_x \\ -v_x & 0 \end{pmatrix} g)g^{-1}\sigma_3 g\big]. \tag{48}$$

As a result, we find that

$$\frac{1}{2}[S_{xx}S, S_x] = -\frac{1}{4}[tr(S_x^2)]_x S. \tag{49}$$

Substituting this expression in (47)

$$-S_{xxx} - \frac{1}{4}[tr(S_x^2)]_x S - (S_x^2)_x S = B. \tag{50}$$

Considering the third term as the $(S_x^2)_x S = -(4uv)_x S = \frac{1}{2}[tr(S_x^2)]_x S$, we get in result B:

$$-S_{xxx} - \frac{1}{4}[tr(S_x^2)]_x S - \frac{1}{2}[tr(S_x^2)]_x S = -S_{xxx} - \frac{3}{4}[tr(S_x^2)]_x S = B. \tag{51}$$

Add the A and B, according to (41) we obtain

$$S_t - S_{xxx} - \frac{3}{4}tr(S_x^2)S_x - \frac{3}{4}[tr(S_x^2)]_x S = 0, \tag{52}$$

which corresponds to the equation (37).

Now let's consider the equation (40)

$$(SS_x)_t + S_{xxy} + \frac{3}{4}[tr(S_x^2)]_y S + \frac{3}{4}tr(S_x^2)S_y = 0. \tag{53}$$

We will transform each term. Let's start with the first, taking into account the expressions for S_x and its derivative with respect to t

$$(SS_x)_t = \frac{1}{2}S_t S_x + \frac{1}{2}SS_{xt} - \frac{1}{2}S_{xt}S - \frac{1}{2}S_x S_t, \tag{54}$$

$$\frac{1}{2}S_t S_x = \frac{1}{2}S_{xxx}S_x + \frac{3}{8}[tr(S_x^2)]_x SS_x + \frac{3}{8}tr(S_x^2)S_x^2,$$
$$\frac{1}{2}S_x S_t = \frac{1}{2}S_x S_{xxx} + \frac{3}{8}[tr(S_x^2)]_x S_x S + \frac{3}{8}tr(S_x^2)S_x^2. \tag{55}$$

Due to the validity of the expression $S_{xt} = S_{tx}$:

$$\frac{1}{2}SS_{xt} = \frac{1}{2}SS_{xxxx} + \frac{3}{4}[tr(S_x^2)]_x SS_x + \frac{3}{8}[tr(S_x^2)]_{xx} + \frac{3}{8}tr(S_x^2)SS_{xx}, \tag{56}$$

$$\frac{1}{2}S_{xt}S = \frac{1}{2}S_{xxxx}S + \frac{3}{4}[tr(S_x^2)]_x S_x S + \frac{3}{8}[tr(S_x^2)]_{xx} + \frac{3}{8}tr(S_x^2)S_{xx}S. \tag{57}$$

Combining the above, we find that the first term $(SS_x)_t$

$$(SS_x)_t = \frac{1}{2}[S_{xxx}, S_x] + \frac{1}{2}[S_x, S_{xxx}]\frac{9}{4}[tr(S_x^2)]_x SS_x + \frac{3}{8}tr(S_x^2)[S, S_{xxx}]. \tag{58}$$

Let's find the second $S_{xxy} = S_{yxx}$:

$$S_{yxx} = \frac{1}{2}[S_{xxxx}, S] + [S_{xxx}, S_x]. \tag{59}$$

By substituting the first two terms in the original equation (53), we obtain

$$\frac{3}{2}[S_{xxx}, S_x] + \frac{9}{4}[tr(S_x^2)]_x S S_x + \frac{3}{4}[tr(S_x^2)]_y S = 0. \tag{60}$$

We extend further transformation until we get in result the equation (37), or 0.

After the transformation of the first term, we have

$$\frac{3}{2}[S_{xxx}, S_x] = -\frac{9}{4}[tr(S_x^2)]_x S S_x + 6g^{-1}\begin{pmatrix} -vu_{xx} + uv_{xx} & 0 \\ 0 & -v_{xx}u + vu_{xx} \end{pmatrix} g. \tag{61}$$

Substituting the latest into (60) and obtain

$$\frac{3}{4}[tr(S_x^2)]_y S + 6g^{-1}\begin{pmatrix} -vu_{xx} + uv_{xx} & 0 \\ 0 & -v_{xx}u + vu_{xx} \end{pmatrix} g. \tag{62}$$

Transforming the first term, we find that

$$\frac{3}{4}[tr(S_x^2)]_y S = \frac{3}{2}(S_x S_x)_y S, \tag{63}$$

$$(S_x S_x)_y = S_{xy} S_x + S_x S_{xy} = -4g^{-1}\begin{pmatrix} -vu_{xx} + uv_{xx} & 0 \\ 0 & -v_{xx}u + vu_{xx} \end{pmatrix} g, \tag{64}$$

substituting the last expression into (63), and finally we find the following:

$$\frac{3}{4}[tr(S_x^2)]_y S = -\frac{3}{2}4g^{-1}\begin{pmatrix} -vu_{xx} + uv_{xx} & 0 \\ 0 & -v_{xx}u + vu_{xx} \end{pmatrix} g = -6g^{-1}\begin{pmatrix} -vu_{xx} + uv_{xx} & 0 \\ 0 & -v_{xx}u + vu_{xx} \end{pmatrix} g.$$

And finally substituting the last expression into (62), we obtain

$$-6g^{-1}\begin{pmatrix} -vu_{xx} + uv_{xx} & 0 \\ 0 & -v_{xx}u + vu_{xx} \end{pmatrix} g + 6g^{-1}\begin{pmatrix} -vu_{xx} + uv_{xx} & 0 \\ 0 & -v_{xx}u + vu_{xx} \end{pmatrix} g = 0. \tag{65}$$

Thus, in this paper was obtained integrable hierarchy of Heisenberg ferromagnet equation

$$S_t - S_{xxx} - \frac{3}{4}tr(S_x^2)S_x - \frac{3}{4}[tr(S_x^2)]_x S = 0, \tag{66}$$

which is the equivalent to a coupled KP system (1)-(3). Much more detailed study of the spin system (66) is the subject of our further research.

Acknowledgments

We express gratitude to Professor R. Myrzakulov for useful discussions and advices. The work is performed under the financial support of the grant of MES RK GPh4-2016.

4. References

[1] Wazwaz A 2011 *Pramana - journal of physics* **77** 233
[2] Isojima S, Willox R and Satsuma J 2002 *Journal of Physics A: Mathematical and General* **35** 6893
[3] Hirota R and Ohta Y 1991 *Journal of the Physical Society of Japan* **60** 798
[4] David D, Levi D and Winternitz P 1987 *Studies in Applied Mathematics* **76** 133
[5] Kakei S 2000 *Physics Letters A* **264** 449
[6] Geng X G 2003 *Journal of Physics A: Mathematical and General* **36** 2289
[7] Qi F-H, Tiana Bo, Liua W-J, Guoa R, Xua T and Zhanga H-Q 2011 *Verlag der Zeitschrift fur Naturforschung* Tubingen