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DYNAMICS OF THE EXPANDING UNIVERSE IN THE HORNDESKI GRAVITY

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Evidence that our Universe had accelerated expansion after the Big Bang and the presence of dark energy may indicate that a modified theory of gravity lies behind our Univers, which differs from the General Theory of Relativity.

In 1971, Horndeski found a general class of scalar-tensor gravity models, the equation of motion of which are second order. The aim of this article is to find some new spherically symmetric solutions in the Horndeski gravity.

We use units of $\kappa_B = c = h = 1$ and set Planck Mass as $\frac{8\pi}{M_{PI}^2} = 1$ [1].

The most common scalar-tensor gravitational models, in which the equations of motion are of the second order, as in GR, belong to the class of theories of gravitation by Hornedeski, whose action is determined by,

$$I = \int d^4x\sqrt{-g}\left[\frac{R}{2} + \mathrm{L}_H\right],\ \mathrm{L}_H = \sum_{i=1}^{5}\mathrm{L}_i, \tag{1}$$

with

$$\begin{aligned}
\mathrm{L}_2 &= P(\varphi, X),\\
\mathrm{L}_3 &= -G_3(\varphi, X)W\varphi,\\
\mathrm{L}_4 &= G_4(\varphi, X)R + G_{4,X}\left[(W\varphi)^2 - (\nabla_\mu\nabla_\nu\varphi)(\nabla^\mu\nabla^\nu\varphi)\right],\\
\mathrm{L}_5 &= \ G_5(\varphi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\varphi) - \frac{1}{6}G_{5,X}[(W\varphi)^3 -\\
&\quad 3(W\varphi)(\nabla_\mu\nabla_\nu\varphi)(\nabla^\mu\nabla^\nu\varphi) + 2(\nabla^\mu\nabla_{\beta\varphi})(\nabla^\beta\nabla_{\mu\varphi})].
\end{aligned} \tag{2}$$

In our article we will consider the case $G_4(\varphi, X) \to \sqrt{|X|}$, namely

$$I=\int dx^4\sqrt{-g}\left[\frac{R}{2}+P(\varphi,X)+\alpha\sqrt{|X|}R+\frac{\alpha}{2\sqrt{|X|}}\left(\frac{|X|}{X}\right)\left[(W\varphi)^2-\left(\nabla_\mu\nabla_\nu\varphi\right)\left(\nabla^\mu\nabla^\nu\varphi\right)\right]\right], \quad (3)$$

which corresponds to

$$G_4(\varphi,X)=\alpha\sqrt{|X|},\quad G_3(\varphi,X)=G_5(\varphi,X)=0, \quad (4)$$

where α is a constant.

A general spherically symmetric static solution is described by the metric

$$ds^2=-A(r)dt^2+\frac{dr^2}{B(r)}+r^2(d\theta^2+\sin^2\theta d\varphi^2), \quad (5)$$

where $A(r)$ and $B(r)$ are functions of the radial coordinate r. As a consequence, $\varphi\equiv\varphi(r)$ and $X\equiv X(r)$ such that

$$X=-B(r)\frac{\varphi'^2}{2}, \quad (6)$$

where the prime index denotes the derivative with respect to r and the field is real, namely $0<\varphi'^2$ [2].

We can write the type of the covariant components of the metric tensor $g_{\mu\nu}$ as follows

$$g_{\mu\nu}=\begin{pmatrix}-A(r) & 0 & 0 & 0\\ 0 & \frac{1}{B(r)} & 0 & 0\\ 0 & 0 & r^2 & 0\\ 0 & 0 & 0 & r^2\sin^2\theta\end{pmatrix} \quad (7)$$

Covariant components of the metric tensor $g_{\mu\nu}$ correspond to countervariant components by the formula

$$g^{\mu\nu}=\frac{\Delta_{\mu\nu}}{g}, \quad (8)$$

where $g=|g_{\mu\nu}P|$ and $\Delta_{\mu\nu}$ – algebraic complement of the corresponding matrix element.

The contravariant metric tensor $g^{\mu\nu}$ is

$$g^{\mu\nu}=\begin{pmatrix}-A^{-1}(r) & 0 & 0 & 0\\ 0 & B(r) & 0 & 0\\ 0 & 0 & r^{-2} & 0\\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta\end{pmatrix} \quad (9)$$

Sixty-four components of the Christoffel symbols $\Gamma^{\lambda}_{\mu м}$ are calculated by the following formula

$$\Gamma^{\lambda}_{\mu м} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right). \tag{10}$$

Nonzero Christoffel symbols

$$\Gamma^{1}_{11} = -\frac{B'}{2B'}, \quad \Gamma^{1}_{00} = -\frac{AB}{2}, \quad \Gamma^{1}_{22} = -Br,$$
$$\Gamma^{1}_{33} = -Br\sin^2\theta, \quad \Gamma^{2}_{33} = -\sin\theta\cos\theta, \quad \Gamma^{0}_{01} = \frac{A'}{2A}, \tag{11}$$
$$\Gamma^{0}_{10} = \frac{A'}{2A}, \quad \Gamma^{2}_{12} = \frac{1}{r}, \quad \Gamma^{2}_{21} = \frac{1}{r},$$
$$\Gamma^{3}_{13} = \frac{1}{r}, \quad \Gamma^{3}_{31} = \frac{1}{r}, \quad \Gamma^{3}_{23} = \theta, \; \Gamma^{3}_{32} = \theta.$$

Then using the (11) and this formula

$$R_{ik} = \frac{\partial \Gamma^{l}_{ik}}{\partial x^{l}} - \frac{\partial \Gamma^{l}_{il}}{\partial x^{k}} + \Gamma^{l}_{lm}\Gamma^{m}_{lm} - \Gamma^{m}_{il}\Gamma^{l}_{km} \tag{12}$$

we find the components of the Ricci tensor, also taking into account the symmetry

$$R_{ik} = R_{ki}. \tag{13}$$

As a result, we obtain the following components of the Ricci tensor

$$\begin{aligned} R_{00} &= \frac{B'A'}{4} + \frac{A''B}{2} + \frac{A'B}{r} - \frac{A'^2 B}{4A}, \\ R_{11} &= -\frac{A''}{2A} + \frac{1}{4}\left(\frac{A'}{A}\right)^2 - \frac{A'B'}{4AB} - \frac{B'}{Br}, \\ R_{22} &= -\frac{B'}{2r} - B + 1 - \frac{A'Br}{2B}, \\ R_{33} &= \left(-\frac{B'}{2r} - B + 1 - \frac{A'Br}{2A} \right)\sin^2\theta. \end{aligned} \tag{14}$$

Using

$$R = g^{ik} R_{ik} \tag{15}$$

we calculate the scalar curvature of space. Finally, we get the following expression

$$R = -\frac{A'B'}{2A} - \frac{A''B}{A} - \frac{2A'B}{Ar} + \frac{A'^2 B}{2A^2} - \frac{2B'}{r} - \frac{2B}{r^2} + \frac{2}{r^2}. \tag{16}$$

The Equations of motion (EOMs) are derived from the action, whose on-shell form for the metric (5) is given by

$$
\begin{aligned}
I = \int dx^4 [& \frac{1}{4A(r)^2}\sqrt{\frac{A(r)}{B(r)}}(r^2 B(r) A'(r)^2 \left(\sqrt{2}\alpha\sqrt{B(r)}\phi'(r)+1\right) - \\
& -rA(r)(2\sqrt{2}\alpha r\sqrt{B(r)}A'(r)B'(r)\phi'(r) + rA'(r)B'(r) + \\
& +2\sqrt{2}\alpha B(r)^{\frac{3}{2}}\left(rA''(r)\phi'(r) + A'(r)\left(r\phi''(r)+4\phi'(r)\right)\right) + \\
& +2B(r)\left(rA''(r)+2A'(r)\right)) - 4A(r)^2(\sqrt{2}\alpha\sqrt{B(r)}(2rB'(r) - \\
& -1)\phi'(r) + rB'(r) + 2\sqrt{2}\alpha B(r)^{\frac{3}{2}}\left(r\phi''(r)+\phi'(r)\right) + B(r) - \\
& -P(\phi, X)r^2 - 1))].
\end{aligned} \tag{17}
$$

After integration by parts, we are able to recast the Lagrangian in a standard form where only the first derivatives of the metric appear, namely,

$$
I_B = -\sqrt{\frac{A(r)}{B(r)}}\left(\frac{1}{2} + \frac{\alpha\sqrt{B(r)\phi'(r)^2}}{\sqrt{2}}\right)\left(4B(r)r + \frac{A'(r)B(r)r^2}{A(r)}\right) \tag{18}
$$

Here, the boundary term [3] is given by

$$
I_B = -\sqrt{\frac{A(r)}{B(r)}}\left(\frac{1}{2} + \frac{\alpha\sqrt{B(r)\phi'(r)^2}}{\sqrt{2}}\right)\left(4B(r)r + \frac{A'(r)B(r)r^2}{A(r)}\right) \tag{19}
$$

The variations with respect to the metric functions $A(r)$ [4], [5]

$$
1 - B(r) - B'(r)r + \sqrt{2}\alpha\sqrt{B(r)\phi'^2} = -r^2 P(\phi, X). \tag{20}
$$

We also got a variation with respect to the metric function $B(r)$

$$
-1 + B(r)\left(1 + r\frac{A'(r)}{A(r)}\right) = r^2 P(\phi, X) + r^2 P_X(\phi, X)\phi'^2 B(r). \tag{21}
$$

Moreover, the variation with respect to the field reads

$$
\begin{aligned}
& P_\phi(\phi, X) + \frac{2B(r)}{r}\phi' P_X(\phi, X) + \frac{B'(r)\phi'}{2} P_X(\phi, X) + B(r)\phi'' P_X(\phi, X) \\
& + \frac{A'(r)}{2A(r)\phi'}\left[-\frac{\alpha\sqrt{2B(r)\phi'^2}}{r^2} + B(r)\phi'^2 P_X(\phi, X)\right] + B(r)\phi'^2 P_{X\phi}(\phi, X) \\
& -\frac{1}{2}B(r)B'(r)\phi'^3 P_{XX} - B^2(r)\phi'^2\phi'' P_{XX}(\phi, X) = 0.
\end{aligned} \tag{22}
$$

We will consider the case

$$P(\phi, X) \Rightarrow P(\phi). \quad (23)$$

And we get the following variations

$$1 - B(r) - B'(r)r + \sqrt{2}\alpha\sqrt{B(r)\phi'^2} = -r^2 P(\phi). \quad (24)$$

$$-1 + B(r)\left(1 + r\frac{A'(r)}{A(r)}\right) = r^2 P(\phi). \quad (25)$$

and

$$P_\phi(\phi) - \frac{A'(r)\alpha\sqrt{2B(r)\phi'^2}}{2A(r)\phi' r^2} = 0. \quad (26)$$

We will consider a special case, where

$$\begin{aligned} P(\phi) &= \phi, \\ B(r) &= r^2. \end{aligned} \quad (27)$$

Thus, Equations (24), (25) and (26) reads

$$1 - 3r^2 + \sqrt{2}\alpha r\phi' = -r^2\phi. \quad (28)$$

$$-\frac{1}{r^2} + 1 + \frac{A'(r)}{A(r)} = \phi. \quad (29)$$

$$\frac{A'(r)\beta}{\sqrt{2}A(r)r} = 1. \quad (30)$$

From Eq. (30) we have

$$A(r) = \exp\left[\frac{\sqrt{2}}{\alpha}\left(\frac{r^2}{2} + A_0\right)\right]. \quad (31)$$

And also from Eq. (29) we get the following solution for ϕ

$$\phi = 1 + \frac{1}{r^2} + \frac{\sqrt{2}r}{\alpha}. \quad (32)$$

In this paper, based on the general interest in the Horndeski theory of gravity, we tried to analyze static spherically symmetric solutions for a subclass of the Horndeski models.

We considered the action in vacuum in Horndeski theory of gravity and found the scalar curvature of space. Using a variation, we found the equations of motion, then found the solution for the special case.

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