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Integrable Kuralay Equations: Geometry, Solutions and Generalizations

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Abstract: In this paper, we study the Kuralay equations, namely the Kuralay-I equation (K-IE) and the Kuralay-II equation (K-IIE). The integrable motion of space curves induced by these equations is investigated. The gauge equivalence between these two equations is established. With the help of the Hirota bilinear method, the simplest soliton solutions are also presented. The nonlocal and dispersionless versions of the Kuralay equations are considered. Some integrable generalizations and other related nonlinear differential equations are presented.

Keywords: geometry; soliton solution; integrable generalizations; gauge equivalence; nonlocal and dispersionless equations

1. Introduction

In this paper, we continue our programme of construction of integrable spin systems (or Heisenberg ferromagnet-type equations) in 1 + 1 and 2 + 1 dimensions (see, e.g., [1–4] and references therein). We present some such type-integrable systems and their gauge-equivalent counterparts. Generally speaking, soliton equations (integrable equations) are the most important class of nonlinear differential equations (NDE) in mathematics and physics. They have remarkable applications to many physical systems, such as hydrodynamics, nonlinear optics, plasma physics, field theories and so on. The nonlinear integrable system is characterized by several features: solitons, Lax pairs, Painleve tests and so on. Searching for such integrable NDE is an extremely important task in modern mathematical physics and its applications. Another very important actual problem is the construction of the exact solutions of such integrable systems. At present, to find exact solutions of integrable nonlinear equations, there exist several powerful mathematical tools, such as the inverse scattering transform, the Hirota bilinear method, the Wronskian and pfaffian technique, the Bell polynomial approach, the Darboux and Bäcklund transformations, Painleve analysis, etc. Among these methods for the construction of exact solutions, the Hirota bilinear method is most efficient for the construction of exact solutions and multiple collisions of solitons. Note that soliton solutions have a wide range of applications in nonlinear physics and other branches of science. For example, such nonlinear solutions arise in different areas, such as fluid mechanics, nonlinear optics, atomic physics, biophysics, biology, field theory, in plasma physics and Bose–Einstein condensates and so on. The main subject of this work is the following Kuralay-II equation (K-IIE) [5–9]

$$iq_t + q_{xt} - vq = 0, \tag{1}$$
$$v_x - 2\epsilon(|q|^2)_t = 0, \tag{2}$$

where $q(x,t)$ is a complex function, $\bar{q}$ is the complex conjugate of q, $v(x,t)$ is a real function (potential), $\epsilon = \pm 1$, x and t are independent real variables. A subscript denotes a partial derivative with respect to x and t. In the soliton theory, the gauge equivalence and geometrical equivalence between integrable equations play an important role [10,11]. In this paper, we prove that the gauge and geometrical equivalent counterpart of the K-IIE (1) and (2) is the following Kuralay-I equation (K-IE) [5–9]

$$\mathbf{S}_t - \mathbf{S} \wedge \mathbf{S}_{xt} - u\mathbf{S}_x = 0, \tag{3}$$

$$u_x + \frac{1}{2}(\mathbf{S}_x^2)_t = 0, \tag{4}$$

where $\mathbf{S} = (S_1, S_2, S_3)$ is the unit spin vector, $\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2 = 1$, $\mathbf{S}_x^2 = S_{1x}^2 + S_{2x}^2 + S_{3x}^2$ and u is the real scalar function (potential). This K-IE is one of the examples of integrable spin systems (see, e.g., [1–4] and references therein).

The paper is organized as follows. In Section 2, we consider the Kuralay-II equation. The traveling wave solutions and the simplest soliton solution of the K-IIE are considered in Section 3. The integrable motion of the space curves induced by the K-IIE is presented in Section 4. In the following Section 5, the gauge equivalence between the K-IE and the K-IIE is established. The Hirota bilinear form and soliton solutions of the K-IE are considered in Section 6. The nonlocal and dispersionless versions of the Kuralay equations are presented in Sections 7 and 8, respectively. In Sections 9 and 10, we present some generalizations of the K-IE and K-IIE, respectively. We conclude in Section 11.

2. The Kuralay-II Equation

In this paper, we will study the Kuralay equations (KE). For example, there are two forms, which are the two versions of the Kuralay-II equation (K-IIE). They are the Kuralay-IIA equation (K-IIAE) and the Kuralay-IIB equation (K-IIBE). In this section, we study these two forms of the K-IIE.

2.1. Kuralay-IIA Equation (K-IIAE)

In this paper, we study the following form of the Kuralay-II equation (K-IIE) [5–9]

$$iq_t - q_{xt} - vq = 0, \tag{5}$$
$$ir_t + r_{xt} + vr = 0, \tag{6}$$
$$v_x + 2d^2(rq)_t = 0, \tag{7}$$

which we call the K-IIAE. It is integrable by the inverse scattering transform (IST) method. It is well known that the Lax pair plays a key role in the theory of integrable systems. Below, we write that the nonlinear equation has the Lax representation if it can be written as the compatibility condition of two linear equations. Thus, in this paper, the Lax pair and the Lax representation are equivalent synonyms. In our case, for the K-IIAE, the corresponding Lax representation has the form

$$\Phi_x = U_2\Phi, \tag{8}$$

$$\Phi_t = V_2\Phi, \tag{9}$$

with

$$U_2 = id\lambda\sigma_3 + dQ, \tag{10}$$

$$V_2 = \frac{1}{1-2d\lambda}B. \tag{11}$$

Here,

$$B = -0.5iv\sigma_3 - di\sigma_3 Q_t \tag{12}$$

and

$$Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

The compatibility condition

$$U_{2t} - V_{2x} + [U_2, V_2] = 0 \tag{14}$$

is equivalent to the q-form of the KE (qKE) [5–9], i.e., to the Kuralay-IIA equation (K-IIAE) (5)–(7), where $U_{jt} = \frac{\partial U_j}{\partial t}$, $V_{jx} = \frac{\partial V_j}{\partial x}$ and so on. As $r = \epsilon \bar{q}$, $d = 1$, from these equations, we obtain the K-IIAE of the form (1) and (1).

2.2. Kuralay-IIB Equation (K-IIBE)

The second form of the K-IIE is given by

$$iq_x + q_{xt} - vq = 0, \tag{15}$$
$$ir_x - r_{xt} + vr = 0, \tag{16}$$
$$v_t - 2(rq)_x = 0, \tag{17}$$

which we call the K-IIBE. It is the second form of the K-IIE. It is natural that this K-IIBE is also integrable in the sense that it admits the Lax representation of the form

$$\Phi_t = U_3 \Phi, \tag{18}$$
$$\Phi_x = V_3 \Phi, \tag{19}$$

where

$$U_3 = -i\lambda\sigma_3 + Q, \quad V_3 = \frac{1}{1 - 2\lambda} B, \quad B = -0.5iv\sigma_3 - i\sigma_3 Q_x. \tag{20}$$

3. Soliton Solutions

Let us demonstrate that the K-IIE admits some exact solutions, e.g., the simplest traveling wave solutions, including the 1-soliton solution. As an example, here, we consider the K-IIAE. Let $d = 1$, $r = \epsilon \bar{q}$. Then, the K-IIAE takes the form

$$iq_t - q_{xt} - vq = 0, \tag{21}$$
$$v_x - 2\epsilon(|q|^2)_t = 0. \tag{22}$$

3.1. Traveling Wave Solutions

Let us assume that $q(x, t)$ has the form

$$q = \chi(x, t)e^{i(ax + bt + \delta)}, \tag{23}$$

where $\chi(x, t)$ is a real function and a, b, δ are some real constants. Then, the K-IIAE takes the form

$$i(\chi_t + ib\chi) - [\chi_{xt} + ia\chi_t + ib\chi_x - ab\chi] - v\chi = 0, \tag{24}$$
$$v_x - 2\epsilon(\chi^2)_t = 0. \tag{25}$$

Hence, we obtain

$$\chi_t - a\chi_t - b\chi_x = 0, \tag{26}$$
$$-b\chi - \chi_{xt} + ab\chi - v\chi = 0, \tag{27}$$
$$v_x - 2\epsilon(\chi^2)_t = 0, \tag{28}$$

or

$$\chi_t - a\chi_t - b\chi_x = 0, \tag{29}$$

$$\chi_{xt} - b(a-1)\chi + v\chi = 0, \tag{30}$$

$$v_x - 2\epsilon(\chi^2)_t = 0. \tag{31}$$

Let us now introduce the new independent variable $\xi = mx + ct$, where m, c are some real constants. Then, we have

$$(c - ac - bm)\chi_\xi = 0, \tag{32}$$

$$cm\chi_{\xi\xi} - [b(a-1) - c_1]\chi + 2cm^{-1}\chi^3 = 0, \tag{33}$$

$$mv - 2c\chi^2 - mc_1 = 0. \tag{34}$$

Hence, we obtain

$$m = \frac{c(1-a)}{b}, \tag{35}$$

$$\chi_{\xi\xi} = \frac{b(a-1) - n}{cm}\chi - 2m^{-2}\chi^3, \tag{36}$$

$$v = 2m^{-1}c\chi^2 + c_1. \tag{37}$$

It is well known that the solutions of Equation (36) are provided by the Jacobi elliptic functions *cn* and *dn*. It is well known from the literature that these functions (*cn* and *dn*) satisfy the following equations [12]:

$$\chi_{\xi\xi} + (1 - 2k^2)\chi + 2k^2\chi^3 = 0, \tag{38}$$

$$\chi_{\xi\xi} - (2 - k^2)\chi + 2\chi^3 = 0, \tag{39}$$

respectively. The corresponding two solutions of the K-IIE are given by

$$q_1 = cn(\xi|k)e^{i(ax+bt+\delta)}, \tag{40}$$

$$v_1 = 2cd^{-1}cn^2(\xi|k) + c_1, \tag{41}$$

and

$$q_2 = dn(\xi|k)e^{i(ax+bt+\delta)}, \tag{42}$$

$$v_2 = 2cm^{-1}dn^2(\xi, k) + c_1, \tag{43}$$

respectively. If $k = 1$, from these solutions, we obtain the following 1-soliton solution of the K-IIE

$$q = \frac{\alpha}{\cosh\xi}e^{i(ax+bt+\delta)} \tag{44}$$

$$v = \frac{2c\epsilon}{m\cosh^2\xi} + c_1, \tag{45}$$

where

$$\alpha = \pm\frac{m}{\sqrt{\epsilon}}, \quad c_1 = -cm^{-1}[(a-1)^2 + m^2], \quad b = cm^{-1}(1-a). \tag{46}$$

This 1-soliton solution represents a wave traveling that propagates with constant speed and shape [13].

3.2. Hirota Bilinear Form

One of the most powerful methods of the construction of exact and explicit solutions of the nonlinear differential equations is the famous Hirota bilinear method [14]. In this subsection, we apply it to solve the K-IIAE.

3.2.1. K-IIAE

To construct the N-soliton solution, we can use the Hirota bilinear form of the K-IIAE. It can be obtained by using the following transformation

$$q = \frac{h}{\phi}, \quad v = 2(\ln \phi)_{xt}, \tag{47}$$

where h is a complex function and ϕ is a real function. Then, we obtain the following Hirota bilinear equations

$$[iD_t + D_x D_t](h \circ \phi) = 0, \tag{48}$$
$$D_x^2(\phi \circ \phi) - 2\epsilon \bar{h} h = 0, \tag{49}$$

where the Hirota D-operators are defined as

$$D_x^n f(x) \circ g(x) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n f(x)g(x')|_{x=x'}. \tag{50}$$

The 1-soliton solution that we look for is

$$h = e^{\chi}, \quad \phi = 1 + \phi_2 = 1 + \frac{e^{(\chi + \bar{\chi})}}{2b}, \tag{51}$$

where $\chi = i(ax + bt + \delta)$, ($a = const, b = const, \delta = const$). Finally, we obtain the 1-soliton solution of the form (44)–(45). Proceeding in the standard way, we can construct the N-soliton solutions of the K-IIAE.

3.2.2. K-IIBE

Similarly, we can construct the soliton solutions of the K-IIBE via the Hirota bilinear method. The corresponding bilinear equations read as

$$[iD_x + D_x D_t](h \circ \phi) = 0, \tag{52}$$
$$D_t^2(\phi \circ \phi) - 2\epsilon \bar{h} h = 0. \tag{53}$$

4. Integrable Motion of Space Curves Induced by the K-IIE

It is well known that in $1 + 1$ and $2 + 1$ dimensions, there exists geometrical equivalence between spin systems and nonlinear Schrödinger-type equations [1–4,10–37], which we call the Lakshmanan equivalence or, in short, the L-equivalence. In this section, we find the L-equivalent counterpart of the K-IIAE (5)–(7). For this purpose, in this section, we want to study the integrable motion of space curves induced by the K-IIAE (5)–(7). For this purpose, consider a moving space curve in R^3 parametrized by the arclength x. It is well known that such a space curve is governed by the following spatial and temporal Serret–Frenet equations (SFE)

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x = C \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_t = D \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{54}$$

where

$$C = \begin{pmatrix} 0 & \kappa & \sigma \\ -\kappa & 0 & \tau \\ -\sigma & -\tau & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & \omega_3 & \omega_2 \\ -\omega_3 & 0 & \omega_1 \\ -\omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{55}$$

Here, κ and σ are the geodesic and normal curvatures of the space curve, τ is its torsion, and ω_j $(j = 1, 2, 3)$ are some real functions describing the motion of space curves. The latter functions must be expressed in terms of κ, σ, τ and their derivatives. Note that the SFE can be rewritten as

$$\mathbf{e}_{ix} = \mathbf{C} \wedge \mathbf{e}_i, \quad \mathbf{e}_{it} = \mathbf{D} \wedge \mathbf{e}_i, \tag{56}$$

where

$$\mathbf{C} = \tau \mathbf{e}_1 + \sigma \mathbf{e}_2 + \kappa \mathbf{e}_3, \quad \mathbf{D} = (\omega_1, \omega_2, \omega_3) \tag{57}$$

and $\mathbf{e}_i$s, $i = 1, 2, 3$, form the orthogonal trihedral. The compatibility condition of the linear Equation (54) reads as

$$C_t - D_x + [C, D] = 0 \tag{58}$$

or

$$\kappa_t = \omega_{3x} - \tau \omega_2 + \sigma \omega_1, \tag{59}$$

$$\sigma_t = \omega_{2x} - \kappa \omega_1 + \tau \omega_3, \tag{60}$$

$$\tau_t = \omega_{1x} - \sigma \omega_3 + \kappa \omega_2. \tag{61}$$

Let us now assume that functions τ, σ, κ have the following forms

$$\tau = -id(r + q), \quad \sigma = d(r - q), \quad \kappa = 2d\lambda, \tag{62}$$

$$\omega_1 = \frac{d}{1 - 2d\lambda}(r_t - q_t), \quad \omega_2 = \frac{di}{1 - 2d\lambda}(r_t + q_t), \quad \omega_3 = -v, \tag{63}$$

where r, q are some complex functions, v is a real function and $d = const$. Substituting these expressions into the set (59)–(61), we obtain the following equations for the functions q, r, v:

$$iq_t - q_{xt} - vq = 0, \tag{64}$$

$$ir_t + r_{xt} + vr = 0, \tag{65}$$

$$v_x + 2d^2(rq)_t = 0. \tag{66}$$

It is nothing but the K-IIAE (5)–(7). Therefore, we have constructed the integrable motion of the space curves induced by the K-IIAE. In this case, it is not difficult to verify that the unit vector $\mathbf{e}_3$ satisfies the following set of equations

$$\mathbf{e}_{3t} - \mathbf{e}_3 \wedge \mathbf{e}_{3xt} - u\mathbf{e}_{3x} = 0, \tag{67}$$

$$u_x + \frac{1}{2}(\mathbf{e}_{3x}^2)_t = 0. \tag{68}$$

This set of equations is the geometrical or Lakshmanan equivalent counterpart of the K-IIAE (5)–(7). Note that after the identification $\mathbf{e}_3 \equiv \mathbf{S}$, Equations (67) and (68) take the form of the K-IAE (3) and (4). Thus, this result proves that the K-IAE and the K-IIAE are geometrically equivalent to each other.

5. Gauge Equivalent Counterpart of the K-IIE

In the previous section, we obtained the geometrical equivalent of the K-IIAE, which has the form (67) and (68). In this section, our aim is to find the gauge-equivalent counterpart of the K-IIAE.

5.1. Derivation of the K-IAE

Thus, in this section, we want to find the gauge-equivalent partner of the K-IIAE. To do this, we consider the following gauge transformation

$$\Psi = g^{-1}\Phi, \tag{69}$$

where Φ is the solution of Equations (8) and (9) and $g(x,t) = \Phi|_{\lambda=0}$. After some standard algebra [11], we obtain the following equations for the new function Ψ:

$$\Psi_x = U_1\Psi, \tag{70}$$
$$\Psi_t = V_1\Psi, \tag{71}$$

where

$$U_1 = -i\lambda S, \quad V_1 = \frac{2\lambda}{1-2\lambda}Z, \quad Z = 0.25([S,S_t] + 2iuS). \tag{72}$$

Here,

$$S = g^{-1}\sigma_3 g. \tag{73}$$

The compatibility condition

$$U_{1t} - V_{1x} + [U_1, V_1] = 0 \tag{74}$$

is equivalent to the following Kuralay-IA equation (K-IAE):

$$iS_t = \frac{1}{2}[S, S_{xt}] + iuS_x, \tag{75}$$

$$u_x = \frac{i}{4}tr(S \cdot [S_x, S_t]), \tag{76}$$

or

$$iS_t = \frac{1}{2}[S, S_{xt}] + iuS_x, \tag{77}$$

$$u_x = -\frac{1}{4}tr\left((S_x^2)_t\right), \tag{78}$$

where

$$S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad S^2 = I, \quad S^{\pm} = S_1 \pm iS_2. \tag{79}$$

This K-IAE is one of examples of an integrable spin system (see, e.g., [1–4] and references therein). The solutions of the K-IE and the K-IIE are related by the following formulas:

$$tr(S_x^2) = 8|q|^2 = 2\mathbf{S}_x^2. \tag{80}$$

and

$$-2i\mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_{xx}) = tr(SS_xS_{xx}) = 4(\bar{q}q_x - \bar{q}_xq). \tag{81}$$

The K-IAE can be written in the vector form as [5]

$$\mathbf{S}_t - \mathbf{S} \wedge \mathbf{S}_{xt} - u\mathbf{S}_x = 0, \tag{82}$$

$$u_x + \frac{1}{2}(\mathbf{S}_x^2)_t = 0, \tag{83}$$

where $\mathbf{S} = (S_1, S_2, S_3)$ is the unit spin vector, $\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2 = 1$, $\mathbf{S}_x^2 = S_{1x}^2 + S_{2x}^2 + S_{3x}^2$ and u is the real scalar function (potential). Using the stereographic projection, one can obtain the following new form of the K-IAE:

$$iw_t + \omega_{xt} - uw_x - \frac{2\bar{w}w_x w_t}{1 + |w|^2} = 0, \tag{84}$$

$$u_x + \frac{2i(w_x\bar{w}_t - \bar{w}_x w_t)}{(1 + |w|^2)^2} = 0. \tag{85}$$

Here,

$$S^+ = S_1 + iS_2 = \frac{2w}{1 + |w|^2}, \quad S_3 = \frac{1 - |w|^2}{1 + |w|^2}, \tag{86}$$

and

$$w = \frac{S^+}{1 + S_3}. \tag{87}$$

5.2. Derivation of the K-IBE

Analogically, we can derive the K-IBE. It has the form

$$iS_x = \frac{1}{2}[S, S_{xt}] + iuS_t, \tag{88}$$

$$u_t = -\frac{1}{4}tr\left((S_t^2)_x\right), \tag{89}$$

or

$$iw_x + \omega_{xt} - uw_t - \frac{2\bar{w}w_x w_t}{1 + |w|^2} = 0, \tag{90}$$

$$u_t + \frac{2i(w_t\bar{w}_x - \bar{w}_t w_x)}{(1 + |w|^2)^2} = 0. \tag{91}$$

6. Soliton Solutions of the K-IE

There are two methods for the construction the exact solutions of a spin system such as the K-IE. One is to find solutions of the spin system using the corresponding solutions of its gauge-equivalent counterpart. Another method is finding the solutions of the spin system directly.

6.1. Solutions from Gauge Equivalence

The gauge equivalence between two equations allows us to construct the solutions of one equation using the solutions of the other gauge-equivalent equation. Here, we use this approach to find solutions of the K-IAE. Let the seed solution of the K-IIAE have the form $r = q = 0, v = 2c$. Then, the associated linear system (8) and (9) takes the form

$$\Phi_{0x} = id\lambda\sigma_3\Phi_0, \tag{92}$$

$$\Phi_{0t} = -\frac{ic}{1 - 2d\lambda}\sigma_3\Phi_0, \tag{93}$$

where

$$\Phi_0 = \begin{pmatrix} \phi_{01} & -\bar{\phi}_{02} \\ \phi_{02} & \bar{\phi}_{01} \end{pmatrix}, \quad \Phi_0^{-1} = \frac{1}{\det\Phi_0}\begin{pmatrix} \bar{\phi}_{01} & \bar{\phi}_{02} \\ -\phi_{02} & \phi_{01} \end{pmatrix}, \quad \det\Phi_0 = |\phi_{01}|^2 + |\phi_{02}|^2. \tag{94}$$

The corresponding solution of the linear Equations (92) and (93) has the form

$$\phi_{01} = c_1 e^{-\chi}, \quad \phi_{02} = c_2 e^{\chi + i\delta_{21}}, \tag{95}$$

where c_j are complex constants, $\chi = \chi_1 + i\chi_2 = i(d\lambda - \frac{c}{1-2d\lambda}t + \delta_1), \delta_{21} = \delta_2 - \delta_1, \lambda = \alpha + i\beta$ and δ_j, α, β are real constants. For the spin matrix S, we have

$$S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix} = \Phi_0^{-1}\sigma_3\Phi_0 = \begin{pmatrix} |\phi_{01}|^2 - |\phi_{02}|^2 & -2\bar{\phi}_{01}\bar{\phi}_{02} \\ -2\phi_{01}\phi_{02} & |\phi_{02}|^2 - |\phi_{01}|^2 \end{pmatrix}. \tag{96}$$

For the components of the spin matrix S, we obtain the following expressions

$$S_3 = \frac{|\phi_{01}|^2 - |\phi_{02}|^2}{\det \Phi_0}, \quad S^+ = -\frac{2\phi_{01}\phi_{02}}{\det \Phi_0}. \tag{97}$$

Substituting the expressions for the functions ϕ_{oj} into the formula (97), we obtain the following 1-soliton solution of the K-IE as

$$S_3 = \frac{|c_1|^2 e^{-2\chi_1} - |c_2|^2 e^{2\chi_1}}{|c_1|^2 e^{-2\chi_1} + |c_2|^2 e^{2\chi_1}}, \quad S^+ = -\frac{2c_1c_2 e^{i\delta_{21}}}{|c_1|^2 e^{-2\chi_1} + |c_2|^2 e^{2\chi_1}}. \tag{98}$$

or

$$S_3 = -\tanh(2\chi_1) = 1 - \frac{e^{2\chi_1}}{|c_1|\cosh(2\chi_1)}, \quad S^+ = -\frac{e^{i(\delta_{21}+\epsilon_1+\epsilon_2)}}{\cosh(2\chi_1)}, \quad S^- = \bar{S}^+, \tag{99}$$

where $c_j = |c_j|e^{i\epsilon_j}$. Thus, using the gauge equivalence between two Kuralay equations, we have constructed the 1-soliton solution of the K-IE. Finally, we note that the functions ϕ_{oj} are the analogies of eigenvectors. As it is well known, these eigenvectors are closely related to the soliton problems (see, e.g., [38–40] and references therein).

6.2. Hirota Bilinear Form of the K-IE

To construct the N-soliton solution of the K-IE, we can use the Hirota bilinear method. For this purpose, we consider the w-form of the K-IAE. Consider the transformation

$$\omega = \frac{g}{f}, \tag{100}$$

where f and g are some complex valued functions. Substituting this expression into the Kuralay-I equation, after some algebra, we obtain the following bilinear form

$$(iD_t - D_xD_t)(\bar{f} \circ g) = 0, \tag{101}$$
$$(iD_t - D_xD_t)(\bar{f} \circ f - \bar{g} \circ g) = 0, \tag{102}$$
$$D_x(\bar{f} \circ f + \bar{g} \circ g) = 0, \tag{103}$$

and

$$u = -\frac{iD_t(\bar{f} \circ f + \bar{g} \circ g)}{\bar{f} \circ f + \bar{g} \circ g}. \tag{104}$$

Here, D_x is the Hirota bilinear operator, defined by

$$D_x^k D_t^n (f \circ g) = (\partial_x - \partial_{x'})^k (\partial_t - \partial_{t'})^n f(x,t)g(x,t)\|_{x=x',t=t'}. \tag{105}$$

Note that from the definition of the D-operator, it follows that

$$u_x = -2i[D_t(f \circ g)D_x(\bar{f} \circ \bar{g}) - c \cdot c]. \tag{106}$$

On the other hand, the spin field takes the form

$$S^+ = \frac{2\bar{f}g}{|f|^2 + |g|^2}, \quad S_3 = \frac{|f|^2 - |g|^2}{|f|^2 + |g|^2}. \tag{107}$$

The bilinear form of the K-IE represents the starting point to obtain interesting classes of its solutions. The construction of the solutions is standard. One expands the functions g and f as a series

$$g = \epsilon g_1 + \epsilon^3 g_3 + \epsilon^5 g_5 + \cdots \cdots \tag{108}$$
$$f = 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \epsilon^6 f_6 + \cdots \cdots \tag{109}$$

Substituting these expansions into (101)–(103) and equating the coefficients of ϵ, one obtains the following system of equations from (101):

$$\epsilon^1 : ig_{1t} + g_{1xt} = 0, \tag{110}$$
$$\epsilon^3 : [i\partial_t + \partial_x\partial_t]g_3 = [iD_t - D_xD_t](\bar{f}_2 \cdot g_1), \tag{111}$$
$$\cdots \tag{112}$$
$$\cdots \tag{113}$$
$$\cdots \tag{114}$$
$$\epsilon^{2n+1} : [i\partial_t + \partial_x\partial_t]g_{2n+1} = \sum_{k+m=n} [iD_t - D_xD_t](\bar{f}_{2k} \cdot g_{2m+1}), \tag{115}$$

and from (102):

$$\epsilon^2 : i\partial_t(\bar{f}_2 - f_2) - \partial_x\partial_t(\bar{f}_2 + f_2) = [iD_t - D_xD_t](\bar{g}_1 \cdot g_1), \tag{116}$$
$$\epsilon^4 : i\partial_t(\bar{f}_4 - f_4) - \partial_x\partial_t(\bar{f}_4 + f_4) = [iD_t - D_xD_t](\bar{g}_1 \cdot g_3 + \bar{g}_3 \cdot g_1 - \bar{f}_2 \cdot f_2), \tag{117}$$
$$\vdots \tag{118}$$
$$\epsilon^{2n} : i\partial_t(\bar{f}_{2n} - f_{2n}) - \partial_x\partial_t(\bar{f}_{2n} + f_{2n}) = \tag{119}$$
$$(iD_t - D_xD_t)\left(\sum_{n_1+n_2=n-1} \bar{g}_{2n_1+1} \cdot g_{2n_2+1}\right) - (iD_t - D_xD_t)\left(\sum_{m_1+m_2=n} \bar{f}_{2m_1} \cdot f_{2m_2}\right). \tag{120}$$

Further from (103), we have the following:

$$\epsilon^2 : \partial_x(\bar{f}_2 - f_2) = -D_x(\bar{g}_1 \cdot g_1), \tag{121}$$
$$\epsilon^4 : \partial_x(\bar{f}_4 - f_4) = -D_x(\bar{g}_1 \cdot g_3 + \bar{g}_3 \cdot g_1 + \bar{f}_2 \cdot f_2), \tag{122}$$
$$\cdots \tag{123}$$
$$\cdots \tag{124}$$
$$\cdots \tag{125}$$
$$\epsilon^{2n} : \partial_x(\bar{f}_{2n} - f_{2n}) = -D_x\left[\sum_{n_1+n_2=n-1} (\bar{g}_{2n_1+1} \cdot g_{2n(2+1)}) + \sum_{n_1+n_2=n} \bar{f}_{2n_1} \cdot f_{2n_2}\right]. \tag{126}$$

Solving recursively the above equations, we obtain many interesting classes of solutions to the K-IE. At last, we note that the functions g, f are the analogies of eigenvectors. As it is well known, these eigenvectors are closely related to the soliton problems (see, e.g., [38–40] and references therein). However, in this paper, to construct the exact and explicit solutions of the Kuralay equations, we use a more modern and effective tool, namely the Hirota bilinear method.

7. Nonlocal KE

Recently, there has been significant interest in studying the nonlocal integrable NDE [41–43]. In the previous sections, we have considered the local Kuralay equations. In this section, let us we present some main results about the nonlocal Kuralay equations. In particular, the nonlocal K-IIE has the form

$$iq_t - q_{xt} - vq = 0, \tag{127}$$

$$ir_t + r_{xx} + vr = 0, \tag{128}$$

$$v_x - 2\epsilon(rq)_t = 0, \tag{129}$$

where

$$r = k\bar{q}(\epsilon_1 x, \epsilon_2 t), \quad r = kq(\epsilon_1 x, \epsilon_2 t), \quad k = \pm 1, \quad \epsilon_j^2 = 1 \tag{130}$$

or

$$r = k\bar{q}(-x, t), \quad r = k\bar{q}(x, -t), \quad r = k\bar{q}(-x, -t), \tag{131}$$

$$r = kq(-x, t), \quad r = kq(x, -t), \quad r = kq(-x, -t). \tag{132}$$

The gauge equivalent spin system corresponding to the K-IIE is given by (3) and (4). However, here, we must note that in contrast to the local case, in our nonlocal case, in the Serret–Frenet Equation (54), the curvatures $\kappa(t, x)$ and $\sigma(t, x)$, the torsion $\tau(t, x)$ and $\omega_j(t, x)$ are complex-valued functions. As a result, in the nonlocal case, the spin matrix S is not Hermitian and has PT symmetry $S(t, x) = \sigma_3 S^+(t, -x)\sigma_3$. The corresponding spin vector $\mathbf{S}(t, x) = (S_1(t, x), S_2(t, x), S_3(t, x))$ is a complex-valued vector. As mentioned above, in the nonlocal case, the spin matrix $S(t, x)$ is not Hermitian. However, we can decompose it as the sum of a Hermitian matrix and a skew-Hermitian matrix as [44]

$$S = M + iL, \tag{133}$$

where

$$M = \frac{1}{2}(S^+ + S), \quad L = \frac{i}{2}(S^+ - S). \tag{134}$$

Next, we use the standard Pauli matrix representations of these matrices: $M = \mathbf{m} \cdot \boldsymbol{\infty}$, $L = \mathbf{l} \cdot \sigma$, where $\mathbf{m}$ and $\mathbf{l}$ are real-valued vector functions. From $\mathbf{S} = \mathbf{m} + i\mathbf{l}$ and $\mathbf{S}^2 = 1$, we obtain

$$\mathbf{m}^2 - \mathbf{l}^2 = 1, \quad \mathbf{m} \cdot \mathbf{l} = 0. \tag{135}$$

Finally, we obtain the following nonlocal Kuralay-I equation

$$\mathbf{m}_t - \mathbf{m} \wedge \mathbf{m}_{xt} + \mathbf{l} \wedge \mathbf{l}_{tx} - (u_1\mathbf{m}_x - u_2\mathbf{l}_x) = 0, \tag{136}$$

$$\mathbf{l}_t - \mathbf{m} \wedge \mathbf{l}_{xt} - \mathbf{l} \wedge \mathbf{m}_{xt} - (u_1\mathbf{l}_x + u_2\mathbf{m}_x) = 0, \tag{137}$$

$$u_{1x} - \frac{1}{2}(\mathbf{m}_x^2 - \mathbf{l}_x^2) = 0, \tag{138}$$

$$u_{2x} - \mathbf{m}_x \cdot \mathbf{l}_x = 0, \tag{139}$$

where u_j are real functions and $u = u_1 + iu_2$. This nonlocal K-IE is integrable. Its Lax representation is given by

$$\Psi_x = U_4\Psi, \tag{140}$$

$$\Psi_t = V_4\Psi. \tag{141}$$

Here,

$$U_4 = -i\lambda(M + iL), \quad V_4 = \frac{2\lambda}{1 - 2\lambda}Z, \tag{142}$$

where

$$Z = 0.25(([M, M_t] - [L, L_t]) + i([M, L_t] + [L_t, M]) + 2iu(M + iL)). \qquad (143)$$

8. Dispersionless KE

To find the dispersionless limit of the Kuralay-II equation, we consider the following representation of the function $q(x, t)$:

$$q = \sqrt{f}e^{\frac{is}{\epsilon}}, \qquad (144)$$

where f, s are some functions, and ϵ is a real parameter. Substituting this expression into the K-IIE, we obtain the following set of equations

$$s_t - s_x s_t + v = 0, \qquad (145)$$
$$f_t - s_t f_x - s_x f_t = 0, \qquad (146)$$
$$v_x - 2\delta f_t = 0. \qquad (147)$$

This is the desired dispersionless Kuralay-II equation. It is integrable.

9. Some Integrable Generalizations of the K-IIE

The Kuralay equations admit several generalizations. As examples, here, we present some of them: the Zhaidary equation, the two-component Kuralay-II equation, multicomponent generalization and so on.

9.1. Integrable Zhaidary Equation
9.1.1. Case 1: Z-IIAE

One of integrable generalizations of the K-IIE is the following Zhaidary-IIA equation (Z-IIAE) [5–9]:

$$iq_t - q_{xt} + 4ic(vq)_x - 2d^2 vq = 0, \qquad (148)$$
$$ir_t + r_{xt} + 4ic(vr)_x + 2d^2 vr = 0, \qquad (149)$$
$$v_x - (rq)_t = 0. \qquad (150)$$

Hence, as $c = 0$, we have the K-IIAE

$$iq_t - q_{xt} - 2d^2 vq = 0, \qquad (151)$$
$$ir_t + r_{xt} + 2d^2 vr = 0, \qquad (152)$$
$$v_x - (rq)_t = 0. \qquad (153)$$

Thus, the K-IIAE is the particular reduction of the Z-IIAE. Note that the ZE (148)–(150) is integrable with the following LR:

$$\Phi_x = U_5 \Phi, \qquad (154)$$
$$\Phi_t = V_5 \Phi, \qquad (155)$$

where

$$U_5 = [i(c\lambda^2 + d\lambda)\sigma_3 + (2c\lambda + d)Q, \qquad (156)$$
$$V_5 = \frac{1}{1 - 2c\lambda^2 - 2d\lambda}(\lambda^2 B_2 + \lambda B_1 + B_0). \qquad (157)$$

Here,

$$B_2 = -4ic\sigma_3, \quad B_1 = -4icdv\sigma_3 - 2ic\sigma_3 Q_t - 8c^2 vQ, \quad B_0 = \frac{d}{2c}B_1 - \frac{d^2}{4c^2}B_2, \qquad (158)$$

and

$$Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad r = \epsilon \bar{q}, \quad \epsilon = \pm 1. \tag{159}$$

The compatibility condition

$$U_{5t} - V_{5x} + [U_5, V_5] = 0 \tag{160}$$

gives the ZE (148)–(150). Thus, we have proven that as $c = 0$, the Zhaidary equation reduces to the KE, so that the ZE is one of the integrable generalizations of the KE.

9.1.2. Case 2: Z-IIBE

The second form of the ZE (148)–(150) can be written as

$$iq_x - q_{xt} + 4ic(vq)_t - 2d^2vq = 0, \tag{161}$$
$$ir_x + r_{xt} + 4ic(vr)_t + 2d^2vr = 0, \tag{162}$$
$$v_t - (rq)_x = 0, \tag{163}$$

which is the Zhaidary-IIB equation (Z-IIBE). Hence, as $c = 0$, we have the following K-IIBE (15)–(17):

$$iq_x - q_{xt} - 2d^2vq = 0, \tag{164}$$
$$ir_x + r_{xt} + 2d^2vr = 0, \tag{165}$$
$$v_t - (rq)_x = 0. \tag{166}$$

As in Case 1, the Z-IIBE (161)–(163) is also integrable with the following LR:

$$\Phi_t = U_6\Phi, \tag{167}$$
$$\Phi_x = V_6\Phi, \tag{168}$$

where

$$U_6 = [i(c\lambda^2 + d\lambda)\sigma_3 + (2c\lambda + d)Q, \tag{169}$$
$$V_6 = \frac{1}{1 - 2c\lambda^2 - 2d\lambda}(\lambda^2 B_2 + \lambda B_1 + B_0). \tag{170}$$

Here,

$$B_2 = -4ic\sigma_3, \quad B_1 = -4icdv\sigma_3 - 2ic\sigma_3Q_x - 8c^2vQ, \quad B_0 = \frac{d}{2c}B_1 - \frac{d^2}{4c^2}B_2, \tag{171}$$

and

$$Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad r = \epsilon \bar{q}, \quad \epsilon = \pm 1. \tag{172}$$

The compatibility condition

$$U_{6x} - V_{6t} + [U_6, V_6] = 0 \tag{173}$$

gives the Z-IIBE (161)–(163). Thus, we have proven that as $c = 0$, the Zhaidary-I equation reduces to the K-IE in Case 1 and in Case 2.

9.2. Integrable Two-Component K-IIE

The KE admits the vector generalizations. As an example, here, we present the integrable two-component Kuralay-IIA equation (K-IIAE).

9.2.1. Integrable Two-Component K-IIAE

The integrable two-component K-IIAE has the form [5–9]

$$iq_{1t} + q_{1xt} - (v_1 + 0.5v_2)q_1 - w_1q_2 = 0, \tag{174}$$

$$iq_{2t} + q_{2xt} - (v_1 + 0.5v_2)q_2 - w_2q_1 = 0, \tag{175}$$

$$ir_{1t} - r_{1xt} + (v_1 + 0.5v_2)r_1 + w_2r_2 = 0, \tag{176}$$

$$ir_{2t} - r_{2xt} + (v_1 + 0.5v_2)r_2 + w_1r_1 = 0, \tag{177}$$

$$v_{1x} - 2b^2(r_1q_1)_t = 0, \tag{178}$$

$$v_{2x} - 2b^2(r_2q_2)_t = 0, \tag{179}$$

$$w_{1x} - b^2(r_2q_1)_t = 0, \tag{180}$$

$$w_{2x} - b^2(r_1q_2)_t = 0. \tag{181}$$

The LR of this two-component K-IIAE is given by

$$\Phi_x = U_7\Phi, \tag{182}$$

$$\Phi_t = V_7\Phi, \tag{183}$$

with

$$U_7 = -ia\lambda\Sigma + bQ, \tag{184}$$

$$V_7 = \frac{1}{1 - 2d\lambda}B. \tag{185}$$

Here,

$$B = \begin{pmatrix} 0.5i(v_1 + v_2) & ibq_{1t} & ibq_{2t} \\ -ibr_{1t} & 0.5iv_1 & iw_2 \\ -ibr_{2t} & iw_1 & 0.5iv_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q_1 & q_2 \\ r_1 & 0 & 0 \\ r_2 & 0 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{186}$$

The compatibility condition

$$U_{7t} - V_{7x} + [U_7, V_7] = 0 \tag{187}$$

gives the two-component K-IIAE (174)–(181).

9.2.2. Integrable Two-Component K-IIBE

Similarly, the integrable two-component K-IIBE is given by [5–9]

$$iq_{1x} + q_{1xt} - (v_1 + 0.5v_2)q_1 - w_1q_2 = 0, \tag{188}$$

$$iq_{2x} + q_{2xt} - (v_1 + 0.5v_2)q_2 - w_2q_1 = 0, \tag{189}$$

$$ir_{1x} - r_{1xt} + (v_1 + 0.5v_2)r_1 + w_2r_2 = 0, \tag{190}$$

$$ir_{2x} - r_{2xt} + (v_1 + 0.5v_2)r_2 + w_1r_1 = 0, \tag{191}$$

$$v_{1t} - 2b^2(r_1q_1)_x = 0, \tag{192}$$

$$v_{2t} - 2b^2(r_2q_2)_x = 0, \tag{193}$$

$$w_{1t} - b^2(r_2q_1)_x = 0, \tag{194}$$

$$w_{2t} - b^2(r_1q_2)_x = 0. \tag{195}$$

The corresponding LR of this two-component K-IIBE reads as

$$\Phi_t = U_7\Phi, \tag{196}$$

$$\Phi_x = V_7\Phi, \tag{197}$$

with

$$U_7 = -ia\lambda\Sigma + bQ, \tag{198}$$

$$V_7 = \frac{1}{1 - 2d\lambda}B. \tag{199}$$

Here,

$$B = \begin{pmatrix} 0.5i(v_1 + v_2) & ibq_{1x} & ibq_{2x} \\ -ibr_{1x} & 0.5iv_1 & iw_2 \\ -ibr_{2x} & iw_1 & 0.5iv_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q_1 & q_2 \\ r_1 & 0 & 0 \\ r_2 & 0 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{200}$$

Then, from the compatibility condition

$$U_{7x} - V_{7t} + [U_7, V_7] = 0 \tag{201}$$

we obtain the two-component K-IIBE (188)–(195).

9.3. Multicomponent KE

One of the multicomponent generalizations of the K-IIAE has the form

$$iq_{kt} + q_{kxt} - vq_k = 0, \tag{202}$$

$$ir_{kt} - r_{kxt} + vr_k = 0, \tag{203}$$

$$v_x - 2b^2 \sum_{k=1}^{N} (r_k q_k)_t = 0, \tag{204}$$

or

$$iq_{kx} + q_{kxt} - vq_k = 0, \tag{205}$$

$$ir_{kx} - r_{kxt} + vr_k = 0, \tag{206}$$

$$v_t - 2b^2 \sum_{k=1}^{N} (r_k q_k)_x = 0, \tag{207}$$

where $k = 1, 2, \ldots, N$. In particular, the two-component versions of these equations read as

$$iq_{1t} + q_{1xt} - vq_1 = 0, \tag{208}$$

$$iq_{2t} + q_{2xt} - vq_2 = 0, \tag{209}$$

$$ir_{1t} - r_{1xt} + vr_1 = 0, \tag{210}$$

$$ir_{2t} - r_{2xt} + vr_2 = 0, \tag{211}$$

$$v_x - 2b^2(r_1 q_1 + r_2 q_2)_t = 0, \tag{212}$$

or

$$iq_{1x} + q_{1xt} - vq_1 = 0, \tag{213}$$

$$iq_{2x} + q_{2xt} - vq_2 = 0, \tag{214}$$

$$ir_{1x} - r_{1xt} + vr_1 = 0, \tag{215}$$

$$ir_{2x} - r_{2xt} + vr_2 = 0, \tag{216}$$

$$v_t - 2b^2(r_1 q_1 + r_2 q_2)_x = 0. \tag{217}$$

9.4. Integrable Akbota Equation

One of the interesting integrable generalizations of the KE is the following Akbota equation (AE) [5–9]

$$iq_t + \alpha q_{xx} + \beta q_{xt} + vq = 0, \tag{218}$$
$$v_x - 2[\alpha(|q|^2)_x + \beta(|q|^2)_t] = 0, \tag{219}$$

or

$$iq_t + \alpha q_{xx} + \beta q_{xt} + vq = 0, \tag{220}$$
$$v_x - 2[\alpha(|q|^2)_x + \beta(|q|^2)_t] = 0. \tag{221}$$

In fact, as $\alpha = 0$, these AEs become

$$iq_t + \beta q_{xt} + vq = 0, \tag{222}$$
$$v_x - 2\beta(|q|^2)_t = 0. \tag{223}$$

and

$$iq_x + \beta q_{xt} + vq = 0, \tag{224}$$
$$v_t - 2\beta(|q|^2)_x = 0, \tag{225}$$

respectively. These equations, up to the simple scale transformations, coincide with the K-IIAE and the K-IIBE, respectively. The Lax representation of the A-IIAE is given by

$$\Phi_x = U_{14}\Phi, \tag{226}$$
$$\Phi_t = V_{14}\Phi, \tag{227}$$

where

$$U_{14} = \frac{i\lambda}{2}\sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix}, \quad V_{14} = \frac{1}{1-\lambda\beta}\{\frac{i\lambda^2}{2}\alpha\sigma_3 + \alpha\lambda Q + V_0\} \tag{228}$$

with

$$V_0 = \begin{pmatrix} \alpha i|q|^2 + i\beta\partial_x^{-1}|q|_t^2 & -i\beta\bar{q}_t - i\alpha\bar{q}_x \\ i\beta q_t + \alpha i q_x & -[\alpha i|q|^2 + i\beta\partial_x^{-1}|q|_t^2] \end{pmatrix}. \tag{229}$$

At the same time, the Lax representation of the A-IIBE is given by

$$\Phi_t = U_{14}\Phi, \tag{230}$$
$$\Phi_x = V_{14}\Phi, \tag{231}$$

where

$$U_{14} = \frac{i\lambda}{2}\sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix}, \quad V_{14} = \frac{1}{1-\lambda\beta}\{\frac{i\lambda^2}{2}\alpha\sigma_3 + \alpha\lambda Q + V_0\} \tag{232}$$

with

$$V_0 = \begin{pmatrix} \alpha i|q|^2 + i\beta\partial_t^{-1}|q|_x^2 & -i\beta\bar{q}_x - i\alpha\bar{q}_t \\ i\beta q_x + \alpha i q_t & -[\alpha i|q|^2 + i\beta\partial_t^{-1}|q|_x^2] \end{pmatrix}. \tag{233}$$

9.5. Integrable Zhanbota Equation

9.5.1. Zhanbota-IIA Equation

Another integrable generalization of the K-IIAE is the following Zhanbota-IIA equation [5–9]:

$$iq_t + q_{xt} - vq - 2ip = 0, \tag{234}$$

$$v_x + 2\delta_1(|q|^2)_t = 0, \tag{235}$$

$$p_x - 2i\omega p - 2\eta q = 0, \tag{236}$$

$$\eta_x + (\delta_1\bar{q}p + \delta_2\bar{p}q) = 0. \tag{237}$$

This Zhanbota-IIA equation, as $p = \eta = 0$, takes the form

$$iq_t + q_{xt} - vq = 0, \tag{238}$$

$$v_x + 2\delta_1(|q|^2)_t = 0, \tag{239}$$

which is nothing but the K-IIAE. Note that the Lax representation of the Zhanbota-IIA equation reads

$$\Phi_x = U_{12}\Phi, \tag{240}$$

$$\Phi_t = V_{12}\Phi, \tag{241}$$

where

$$U_{12} = -i\lambda\sigma_3 + A_0, \tag{242}$$

$$V_{12} = \frac{1}{1 - \kappa\lambda}\left\{B_0 + \frac{i}{\lambda + \omega}B_{-1}\right\}. \tag{243}$$

Here,

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{244}$$

$$B_0 = -\frac{i}{2}v\sigma_3 - \frac{\kappa}{2i}\begin{pmatrix} 0 & q_t \\ r_t & 0 \end{pmatrix}, \tag{245}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{246}$$

9.5.2. Zhanbota-IIB Equation

One of the integrable generalizations of the K-IIBE is the following Zhanbota-IIB equation [5–9]:

$$iq_x + q_{xt} - vq - 2ip = 0, \tag{247}$$

$$v_t + 2\delta_1(|q|^2)_x = 0, \tag{248}$$

$$p_t - 2i\omega p - 2\eta q = 0, \tag{249}$$

$$\eta_t + (\delta_1\bar{q}p + \delta_2\bar{p}q) = 0. \tag{250}$$

This Zhanbota-IIB equation, as $p = \eta = 0$, takes the form

$$iq_x + q_{xt} - vq = 0, \tag{251}$$

$$v_t + 2\delta_1(|q|^2)_x = 0, \tag{252}$$

which is nothing but the K-IIBE. The Lax representation of the Zhanbota-IIB equation reads

$$\Phi_t = U_{12}\Phi, \tag{253}$$

$$\Phi_x = V_{12}\Phi, \tag{254}$$

where

$$U_{12} = -i\lambda\sigma_3 + A_0, \tag{255}$$

$$V_{12} = \frac{1}{1 - \kappa\lambda}\{B_0 + \frac{i}{\lambda + \omega}B_{-1}\}. \tag{256}$$

Here,

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{257}$$

$$B_0 = -\frac{i}{2}v\sigma_3 - \frac{\kappa}{2i}\begin{pmatrix} 0 & q_x \\ r_x & 0 \end{pmatrix}, \tag{258}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{259}$$

9.6. Integrable Nurshuak Equation

Let us present two more examples of the integrable generalizations of the KE.

9.6.1. N-IIAE

First, let us consider the following Nurshuak-IIA equation (N-IIAE) [5–9]:

$$iq_t + \epsilon_1 q_{xt} + i\epsilon_2 q_{xxt} - vq + (wq)_x - 2ip = 0, \tag{260}$$

$$ir_t - \epsilon_1 r_{xt} + i\epsilon_2 r_{xxt} + vr + (wr)_x - 2ik = 0, \tag{261}$$

$$v_x + 2\epsilon_1(rq)_t - 2i\epsilon_2(r_{xt}q - rq_{xt}) = 0, \tag{262}$$

$$w_x - 2i\epsilon_2(rq)_t = 0, \tag{263}$$

$$p_x - 2i\omega p - 2\eta q = 0, \tag{264}$$

$$k_x + 2i\omega k - 2\eta r = 0, \tag{265}$$

$$\eta_x + rp + kq = 0. \tag{266}$$

From this N-IIAE, we obtain the K-IIAE as $\epsilon_2 = w = p = k = \eta = 0, \epsilon_1 = 1$. Note that the N-IIAE is integrable. Its Lax representation reads as

$$\Phi_x = U_8\Phi, \tag{267}$$

$$\Phi_t = V_8\Phi. \tag{268}$$

Here,

$$U_8 = -i\lambda\sigma_3 + A_0, \tag{269}$$

$$V_8 = \frac{1}{1 - (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)}\{\lambda B_1 + B_0 + \frac{i}{\lambda + \omega}B_{-1}\}, \tag{270}$$

where

$$B_1 = w\sigma_3 + 2i\epsilon_2\sigma_3 A_{0t}, \quad A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{271}$$

$$B_0 = -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & i\epsilon_1 q_t - \epsilon_2 q_{xt} + iwq \\ i\epsilon_1 r_t + \epsilon_2 r_{xt} - iwr & 0 \end{pmatrix}, \tag{272}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{273}$$

9.6.2. N-IIBE

Here, let us consider the following Nurshuak-IIB equation (N-IIBE) [5–9]:

$$iq_x + \epsilon_1 q_{xt} + i\epsilon_2 q_{ttx} - vq + (wq)_t - 2ip = 0, \tag{274}$$

$$ir_x - \epsilon_1 r_{xt} + i\epsilon_2 r_{ttx} + vr + (wr)_t - 2ik = 0, \tag{275}$$

$$v_t + 2\epsilon_1 (rq)_x - 2i\epsilon_2 (r_{xt}q - rq_{xt}) = 0, \tag{276}$$

$$w_t - 2i\epsilon_2 (rq)_x = 0, \tag{277}$$

$$p_t - 2i\omega p - 2\eta q = 0, \tag{278}$$

$$k_t + 2i\omega k - 2\eta r = 0, \tag{279}$$

$$\eta_t + rp + kq = 0. \tag{280}$$

From this N-IIBE, we obtain the K-IIBE as $\epsilon_2 = w = p = k = \eta = 0, \epsilon_1 = 1$. Note that the N-IIBE is integrable. Its Lax representation reads as

$$\Phi_t = U_8 \Phi, \tag{281}$$

$$\Phi_x = V_8 \Phi. \tag{282}$$

Here,

$$U_8 = -i\lambda\sigma_3 + A_0, \tag{283}$$

$$V_8 = \frac{1}{1 - (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)} \left\{ \lambda B_1 + B_0 + \frac{i}{\lambda + \omega} B_{-1} \right\}, \tag{284}$$

where

$$B_1 = w\sigma_3 + 2i\epsilon_2\sigma_3 A_{0x}, \quad A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{285}$$

$$B_0 = -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & i\epsilon_1 q_x - \epsilon_2 q_{xt} + iwq \\ i\epsilon_1 r_x + \epsilon_2 r_{xt} - iwr & 0 \end{pmatrix}, \tag{286}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{287}$$

10. Some Integrable Generalizations of the K-IE

In the previous section, we have presented some generalizations of the K-II equation. In this section, we consider the integrable generalizations of the K-IE. In fact, the Kuralay-I equation also admits several integrable generalizations. As examples, here, we present some of them.

10.1. Zhaidary-I Equation

The Zhaidary-I equations are integrable generalizations of the Kuralay-I equations. In this subsection, we consider these integrable generalizations.

10.1.1. Z-IAE

The Zhaidary-IA equation (Z-IAE) is given by

$$(1 + 2\beta(c\beta + d))\mathbf{S}_t - \mathbf{S} \wedge \mathbf{S}_{xt} - u\mathbf{S}_x + 4cw\mathbf{S}_x = 0, \tag{288}$$

$$u_x + \frac{1}{2}(\mathbf{S}_x^2)_t = 0, \tag{289}$$

$$w_x + \frac{1}{4(2\beta c + d)^2}(\mathbf{S}_x^2)_t = 0. \tag{290}$$

This Z-IAE is completely integrable, i.e., it can be solved by the inverse scattering transformation method (IST). It possesses all the basic characteristics of integrable equations. The corresponding Lax representation has the form

$$\Psi_x = U_1\Psi, \tag{291}$$

$$\Psi_t = V_1\Psi. \tag{292}$$

Here,

$$U_1 = [ic(\lambda^2 - \beta^2) + id(\lambda - \beta)]S + \frac{c(\lambda - \beta)}{2c\beta + d}SS_x, \tag{293}$$

$$V_1 = \frac{1}{1 - 2c\lambda^2 - 2d\lambda}\{[2c(\lambda^2 - \beta^2) + 2d(\lambda - \beta)]B + \lambda^2 F_2 + \lambda F_1 + F_0\}, \tag{294}$$

where

$$F_2 = -4ic^2wS, \quad F_1 = -4icdwS - \frac{4c^2}{2c\beta + d}wSS_x - \frac{ic}{2c\beta + d}S\{(SS_x)_t - [SS_x, B]\}, \tag{295}$$

$$F_0 = -\beta F_1 - \beta^2 F_2, \quad B = 0.25([S, S_t] + 2iuS), \quad S = \mathbf{S} \cdot \boldsymbol{\text{œ}}. \tag{296}$$

If $\beta = 0$, the ZE takes the form

$$\mathbf{S}_t - \mathbf{S} \wedge \mathbf{S}_{xt} - u\mathbf{S}_x + 4cw\mathbf{S}_x = 0, \tag{297}$$

$$u_x + \frac{1}{2}(\mathbf{S}_x^2)_t = 0, \tag{298}$$

$$w_x + \frac{1}{4d^2}(\mathbf{S}_x^2)_t = 0, \tag{299}$$

or

$$\mathbf{S}_t - \mathbf{S} \wedge \mathbf{S}_{xt} + (2cd^{-2} - 1)u\mathbf{S}_x = 0, \tag{300}$$

$$u_x + \frac{1}{2}(\mathbf{S}_x^2)_t = 0. \tag{301}$$

Note that the gauge-equivalent counterpart of the Z-IAE reads as

$$iq_t - q_{xt} + 4ic(vq)_x - 2d^2vq = 0, \tag{302}$$

$$ir_t + r_{xt} + 4ic(vr)_x + 2d^2vr = 0, \tag{303}$$

$$v_x - (rq)_t = 0, \tag{304}$$

which is, in fact, the Z-IIAE.

10.1.2. Z-IBE

The Zhaidary-IB equation (Z-IBE) has the form

$$(1 - 2\beta(c\beta + d))\mathbf{S}_x - \mathbf{S} \wedge \mathbf{S}_{xt} - u\mathbf{S}_t + 4cw\mathbf{S}_x = 0, \tag{305}$$

$$u_t + \frac{1}{2}(\mathbf{S}_x^2)_x = 0, \tag{306}$$

$$w_t + \frac{1}{4(2\beta c + d)^2}(\mathbf{S}_x^2)_x = 0. \tag{307}$$

This Z-IAE is completely integrable, i.e., it can be solved by the inverse scattering transformation method (IST). It possesses all the basic characteristics of integrable equations. The corresponding Lax representation has the form

$$\Psi_t = U_1\Psi, \tag{308}$$

$$\Psi_x = V_1\Psi. \tag{309}$$

Here,

$$U_1 = [ic(\lambda^2 - \beta^2) + id(\lambda - \beta)]S + \frac{c(\lambda - \beta)}{2c\beta + d}SS_t, \tag{310}$$

$$V_1 = \frac{1}{1 - 2c\lambda^2 - 2d\lambda}\{[2c(\lambda^2 - \beta^2) + 2d(\lambda - \beta)]B + \lambda^2 F_2 + \lambda F_1 + F_0\}, \tag{311}$$

where

$$F_2 = -4ic^2wS, \quad F_1 = -4icdwS - \frac{4c^2}{2c\beta + d}wSS_t - \frac{ic}{2c\beta + d}S\{(SS_t)_t - [SS_t, B]\}, \tag{312}$$

$$F_0 = -\beta F_1 - \beta^2 F_2, \quad B = 0.25([S, S_x] + 2iuS), \quad S = \mathbf{S} \cdot \sigma. \tag{313}$$

If $\beta = 0$, the ZE takes the form

$$\mathbf{S}_x - \mathbf{S} \wedge \mathbf{S}_{xt} - u\mathbf{S}_t + 4cw\mathbf{S}_t = 0, \tag{314}$$

$$u_t + \frac{1}{2}(\mathbf{S}_t^2)_x = 0, \tag{315}$$

$$w_t + \frac{1}{4d^2}(\mathbf{S}_t^2)_x = 0, \tag{316}$$

or

$$\mathbf{S}_x - \mathbf{S} \wedge \mathbf{S}_{xt} + (2cd^{-2} - 1)u\mathbf{S}_t = 0, \tag{317}$$

$$u_t + \frac{1}{2}(\mathbf{S}_t^2)_x = 0. \tag{318}$$

Finally, let us present the following gauge-equivalent counterpart of the Z-IBE:

$$iq_x - q_{xt} + 4ic(vq)_t - 2d^2vq = 0, \tag{319}$$

$$ir_x + r_{xt} + 4ic(vr)_t + 2d^2vr = 0, \tag{320}$$

$$v_t - (rq)_x = 0, \tag{321}$$

which is the Z-IIBE.

10.2. Shynaray Equation
10.2.1. S-IAE

Our next example is the Shynaray-IA equation (S-IAE). It has the form

$$(1 + 2c\beta^2)\mathbf{S}_t - \mathbf{S} \wedge \mathbf{S}_{xt} - u\mathbf{S}_x + 4cw\mathbf{S}_x = 0, \tag{322}$$

$$u_x + \frac{1}{2}(\mathbf{S}_x^2)_t = 0, \tag{323}$$

$$w_x + \frac{1}{16c^2\beta^2}(\mathbf{S}_x^2)_t = 0. \tag{324}$$

This S-IAE is integrable, in the sense that it has the following Lax representation

$$\Psi_x = U_5\Psi, \tag{325}$$

$$\Psi_t = V_5\Psi. \tag{326}$$

Here,

$$U_5 = ic(\lambda^2 - \beta^2)S + \frac{\lambda - \beta}{2\beta}SS_x, \tag{327}$$

$$V_5 = \frac{1}{1 - 2c\lambda^2}\{2c(\lambda^2 - \beta^2)B + \lambda^2 F_2 + \lambda F_1 + F_0\}, \tag{328}$$

where

$$F_2 = -4ic^2wS, \quad F_1 = -\frac{4c^2}{2c\beta}wSS_x - \frac{ic}{2c\beta}S\{(SS_x)_t - [SS_x, B]\}, \tag{329}$$

$$F_0 = -\beta F_1 - \beta^2 F_2, \quad B = 0.25([S, S_t] + 2iuS), \quad S = \mathbf{S} \cdot \sigma. \tag{330}$$

The gauge equivalent counterpart of the S-IAE is given by

$$iq_t - q_{xt} + 4ic(vq)_x = 0, \tag{331}$$

$$ir_t + r_{xt} + 4ic(vr)_x = 0, \tag{332}$$

$$v_x - (rq)_t = 0. \tag{333}$$

10.2.2. S-IBE

The Shynaray-IB equation (S-IBE) has the form:

$$(1 + 2c\beta^2)\mathbf{S}_x - \mathbf{S} \wedge \mathbf{S}_{xt} - u\mathbf{S}_t + 4cw\mathbf{S}_t = 0, \tag{334}$$

$$u_t + \frac{1}{2}(\mathbf{S}_t^2)_x = 0, \tag{335}$$

$$w_t + \frac{1}{16c^2\beta^2}(\mathbf{S}_t^2)_x = 0. \tag{336}$$

This S-IBE is integrable, in the sense that it admits the following Lax representation

$$\Psi_t = U_5\Psi, \tag{337}$$

$$\Psi_x = V_5\Psi. \tag{338}$$

Here,

$$U_5 = ic(\lambda^2 - \beta^2)S + \frac{\lambda - \beta}{2\beta}SS_t, \tag{339}$$

$$V_5 = \frac{1}{1 - 2c\lambda^2}\{2c(\lambda^2 - \beta^2)B + \lambda^2 F_2 + \lambda F_1 + F_0\}, \tag{340}$$

where

$$F_2 = -4ic^2wS, \quad F_1 = -\frac{4c^2}{2c\beta}wSS_x - \frac{ic}{2c\beta}S\{(SS_t)_x - [SS_t, B]\}, \tag{341}$$

$$F_0 = -\beta F_1 - \beta^2 F_2, \quad B = 0.25([S, S_x] + 2iuS), \quad S = \mathbf{S} \cdot \sigma. \tag{342}$$

The gauge-equivalent equation for the S-IBE reads as

$$iq_x - q_{xt} + 4ic(vq)_t = 0, \tag{343}$$

$$ir_x + r_{xt} + 4ic(vr)_t = 0, \tag{344}$$

$$v_t - (rq)_x = 0. \tag{345}$$

10.3. Nurshuak Equation

In this subsection, we would like to demonstrate another generalization of the K-IE. Consider the so-called Nurshuak-I equation. It has two forms: the Nurshuak-IA equation and the Nurshuak-IB equation.

10.3.1. N-IAE

Let us start from the Nurshuak-IA equation (N-IAE), which has the form

$$iS_t + 2\epsilon_1 Z_x + i\epsilon_2(S_{xt} + [S_x, Z])_x + (wS)_x + \frac{1}{\omega}[S, W] = 0, \tag{346}$$

$$u_x - \frac{i}{4}tr(S \times [S_x, S_t]) = 0, \tag{347}$$

$$w_x - \frac{i}{4}\epsilon_2[tr(S_x^2)]_t = 0, \tag{348}$$

$$iW_x + \omega[S, W] = 0, \tag{349}$$

where

$$Z = \frac{1}{4}([S, S_t] + 2iuS). \tag{350}$$

If $\epsilon_2 = 0$, from this N-IAE, we obtain the K-IAE. As an integrable equation, the N-IAE admits the LR of the form

$$\Psi_x = U_7\Psi, \tag{351}$$

$$\Psi_t = V_7\Psi. \tag{352}$$

Here,

$$U_7 = -i\lambda S, \tag{353}$$

$$V_7 = \frac{1}{1 - 2\epsilon_1\lambda - 4\epsilon_2\lambda^2}\{(2\epsilon_1\lambda + 4\epsilon_2\lambda^2)Z + \lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W\}, \tag{354}$$

where

$$V_1 = wS + i\epsilon_2(S_{xt} + [S_x, Z]), \tag{355}$$

$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}, \quad S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad S^{\pm} = S_1 \pm iS_2. \tag{356}$$

The compatibility condition $\Psi_{xt} = \Psi_{tx}$ gives the N-IAE. The gauge partner of the N-IAE is

$$iq_t + \epsilon_1 q_{xt} + i\epsilon_2 q_{xxt} - vq + (wq)_x - 2ip = 0, \tag{357}$$

$$ir_t - \epsilon_1 r_{xt} + i\epsilon_2 r_{xxt} + vr + (wr)_x - 2ik = 0, \tag{358}$$

$$v_x + 2\epsilon_1(rq)_t - 2i\epsilon_2(r_{xt}q - rq_{xt}) = 0, \tag{359}$$

$$w_x - 2i\epsilon_2(rq)_t = 0, \tag{360}$$

$$p_x - 2i\omega p - 2\eta q = 0, \tag{361}$$

$$k_x + 2i\omega k - 2\eta r = 0, \tag{362}$$

$$\eta_x + rp + kq = 0, \tag{363}$$

where $r = \delta_1\bar{q}, k = \delta_2\bar{p}, \delta_j = \pm 1$. This is the N-IIBE. Its Lax representation reads as

$$\Psi_x = U_8\Psi, \tag{364}$$

$$\Psi_t = V_8\Psi. \tag{365}$$

Here,

$$U_8 = -i\lambda\sigma_3 + A_0, \tag{366}$$

$$V_8 = \frac{1}{1 - (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)}\{\lambda B_1 + B_0 + \frac{i}{\lambda + \omega}B_{-1}\}, \tag{367}$$

where

$$B_1 = w\sigma_3 + 2i\epsilon_2\sigma_3 A_{0t}, \quad A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{368}$$

$$B_0 = -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & i\epsilon_1 q_t - \epsilon_2 q_{xt} + iwq \\ i\epsilon_1 r_t + \epsilon_2 r_{xt} - iwr & 0 \end{pmatrix}, \tag{369}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{370}$$

10.3.2. N-IBE

Now, we consider the Nurshuak-IB equation (N-IBE). It has the form

$$iS_x + 2\epsilon_1 Z_t + i\epsilon_2(S_{xt} + [S_t, Z])_t + (wS)_t + \frac{1}{\omega}[S, W] = 0, \tag{371}$$

$$u_t - \frac{i}{4}tr(S \times [S_t, S_x]) = 0, \tag{372}$$

$$w_t - \frac{i}{4}\epsilon_2[tr(S_t^2)]_x = 0, \tag{373}$$

$$iW_t + \omega[S, W] = 0, \tag{374}$$

where

$$Z = \frac{1}{4}([S, S_x] + 2iuS). \tag{375}$$

As an integrable equation, the N-IAE admits the LR of the form

$$\Psi_t = U_7\Psi, \tag{376}$$

$$\Psi_x = V_7\Psi. \tag{377}$$

Here,

$$U_7 = -i\lambda S, \tag{378}$$

$$V_7 = \frac{1}{1 - 2\epsilon_1\lambda - 4\epsilon_2\lambda^2}\left\{(2\epsilon_1\lambda + 4\epsilon_2\lambda^2)Z + \lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W\right\}, \tag{379}$$

where

$$V_1 = wS + i\epsilon_2(S_{xt} + [S_t, Z]), \tag{380}$$

$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}, \quad S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad S^{\pm} = S_1 \pm iS_2. \tag{381}$$

The gauge equivalent partner of the N-IBE is given by

$$iq_x + \epsilon_1 q_{xt} + i\epsilon_2 q_{ttx} - vq + (wq)_t - 2ip = 0, \tag{382}$$

$$ir_x - \epsilon_1 r_{xt} + i\epsilon_2 r_{ttx} + vr + (wr)_t - 2ik = 0, \tag{383}$$

$$v_t + 2\epsilon_1(rq)_x - 2i\epsilon_2(r_{xt}q - rq_{xt}) = 0, \tag{384}$$

$$w_t - 2i\epsilon_2(rq)_x = 0, \tag{385}$$

$$p_t - 2i\omega p - 2\eta q = 0, \tag{386}$$

$$k_t + 2i\omega k - 2\eta r = 0, \tag{387}$$

$$\eta_t + rp + kq = 0. \tag{388}$$

Since this N-IBE is integrable, it has the Lax representation of the form

$$\Psi_t = U_8\Psi, \tag{389}$$

$$\Psi_x = V_8\Psi. \tag{390}$$

Here,

$$U_8 = -i\lambda\sigma_3 + A_0, \tag{391}$$

$$V_8 = \frac{1}{1 - (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)}\{\lambda B_1 + B_0 + \frac{i}{\lambda + \omega}B_{-1}\}, \tag{392}$$

where

$$B_1 = w\sigma_3 + 2i\epsilon_2\sigma_3 A_{0x}, \quad A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{393}$$

$$B_0 = -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & i\epsilon_1 q_x - \epsilon_2 q_{xt} + iwq \\ i\epsilon_1 r_x + \epsilon_2 r_{xt} - iwr & 0 \end{pmatrix}, \tag{394}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{395}$$

10.4. Aizhan-I Equation

The Aizhan-I equation has two integrable forms, namely, the Aizhan-IA equation and the Aizhan-IB equation.

10.4.1. The Aizhan-IA Equation

The Aizhan-IA equation is given by

$$iS_t + i\epsilon_2(S_{xt} + [S_x, Z])_x + (wS)_x + \frac{1}{\omega}[S, W] = 0, \tag{396}$$

$$u_x - \frac{i}{4}tr(S \times [S_x, S_t]) = 0, \tag{397}$$

$$w_x - \frac{i}{4}\epsilon_2[tr(S_x^2)]_t = 0, \tag{398}$$

$$iW_x + \omega[S, W] = 0. \tag{399}$$

The Aizhan-IA equation can be considered as one of the integrable generalizations of the K-IAE. The Lax representation of the Aizhan-IA equation has the form

$$\Psi_x = U_9\Psi, \tag{400}$$

$$\Psi_t = V_9\Psi, \tag{401}$$

where

$$U_9 = -i\lambda S, \tag{402}$$

$$V_9 = \frac{1}{1 - 4\epsilon_2\lambda^2}\{4\epsilon_2\lambda^2 Z + \lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W\} \tag{403}$$

with

$$V_1 = wS + i\epsilon_2(S_{xt} + [S_x, Z]), \tag{404}$$

$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \tag{405}$$

The gauge-equivalent equation for the Aizhan-IA equation reads as

$$iq_t + i\epsilon_2 q_{xxt} - vq + (wq)_x - 2ip = 0, \tag{406}$$

$$ir_t + i\epsilon_2 r_{xxt} + vr + (wr)_x - 2ik = 0, \tag{407}$$

$$v_x - 2i\epsilon_2(r_{xt}q - rq_{xt}) = 0, \tag{408}$$

$$w_x - 2i\epsilon_2(rq)_t = 0, \tag{409}$$

$$p_x - 2i\omega p - 2\eta q = 0, \tag{410}$$

$$k_x + 2i\omega k - 2\eta r = 0, \tag{411}$$

$$\eta_x + rp + kq = 0. \tag{412}$$

This is the Aizhan-IIA equation. For this equation, the corresponding Lax representation is given by

$$\Phi_x = U_{10}\Phi, \tag{413}$$

$$\Phi_t = V_{10}\Phi, \tag{414}$$

where

$$U_{10} = -i\lambda\sigma_3 + A_0, \tag{415}$$

$$V_{10} = \frac{1}{1 - 4\epsilon_2\lambda^2}\{\lambda B_1 + B_0 + \frac{i}{\lambda + \omega}B_{-1}\}. \tag{416}$$

Here,

$$B_1 = w\sigma_3 + 2i\epsilon_2\sigma_3 A_{0t}, \tag{417}$$

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{418}$$

$$B_0 = -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & -\epsilon_2 q_{xt} + iwq \\ \epsilon_2 r_{xt} - iwr & 0 \end{pmatrix}, \tag{419}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{420}$$

10.4.2. The Aizhan-IB Equation

The Aizhan-IB equation has the form

$$iS_x + i\epsilon_2(S_{xt} + [S_t, Z])_t + (wS)_t + \frac{1}{\omega}[S, W] = 0, \tag{421}$$

$$u_t - \frac{i}{4}tr(S \times [S_t, S_x]) = 0, \tag{422}$$

$$w_t - \frac{i}{4}\epsilon_2[tr(S_t^2)]_x = 0, \tag{423}$$

$$iW_t + \omega[S, W] = 0. \tag{424}$$

The Aizhan-IB equation can be considered as one of the integrable generalizations of the K-IBE. The Lax representation of the Aizhan-IB equation has the form

$$\Psi_t = U_9\Psi, \tag{425}$$

$$\Psi_x = V_9\Psi, \tag{426}$$

where

$$U_9 = -i\lambda S, \tag{427}$$

$$V_9 = \frac{1}{1 - 4\epsilon_2\lambda^2}\{4\epsilon_2\lambda^2 Z + \lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W\} \tag{428}$$

with

$$V_1 = wS + i\epsilon_2(S_{xt} + [S_t, Z]), \tag{429}$$

$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \tag{430}$$

The gauge equivalent counterpart of the Aizhan-IB equation reads as

$$iq_x + i\epsilon_2 q_{ttx} - vq + (wq)_t - 2ip = 0, \tag{431}$$

$$ir_x + i\epsilon_2 r_{ttx} + vr + (wr)_t - 2ik = 0, \tag{432}$$

$$v_t - 2i\epsilon_2(r_{xt}q - rq_{xt}) = 0, \tag{433}$$

$$w_t - 2i\epsilon_2(rq)_x = 0, \tag{434}$$

$$p_t - 2i\omega p - 2\eta q = 0, \tag{435}$$

$$k_t + 2i\omega k - 2\eta r = 0, \tag{436}$$

$$\eta_t + rp + kq = 0. \tag{437}$$

This is the Aizhan-IIB equation. Its Lax representation is

$$\Phi_x = U_{10}\Phi, \tag{438}$$

$$\Phi_t = V_{10}\Phi, \tag{439}$$

where

$$U_{10} = -i\lambda\sigma_3 + A_0, \tag{440}$$

$$V_{10} = \frac{1}{1 - 4\epsilon_2\lambda^2}\{\lambda B_1 + B_0 + \frac{i}{\lambda + \omega}B_{-1}\}. \tag{441}$$

Here,

$$B_1 = w\sigma_3 + 2i\epsilon_2\sigma_3 A_{0t}, \tag{442}$$

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{443}$$

$$B_0 = -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & -\epsilon_2 q_{xt} + iwq \\ \epsilon_2 r_{xt} - iwr & 0 \end{pmatrix}, \tag{444}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{445}$$

10.5. Zhanbota-I Equation

In this section, we consider the Zhanbota-I equation. It has a similar form, namely the Zhanbota-IA equation and the Zhanbota-IB equation.

10.5.1. Zhanbota-IA Equation

The Zhanbota-IA equation reads

$$\mathbf{S}_t - \mathbf{S} \wedge \mathbf{S}_{xt} - u\mathbf{S}_x - \frac{1}{\omega}\mathbf{S} \wedge \mathbf{W} = 0, \tag{446}$$

$$u_x + \mathbf{S} \times (\mathbf{S}_x \wedge \mathbf{S_t}) = 0, \tag{447}$$

$$\mathbf{W}_x - \omega\mathbf{S} \wedge \mathbf{W} = 0. \tag{448}$$

The Zhanbota-IA equation is also one of the integrable generalization of the K-IAE. The matrix form of the Zhanbota-IA equation reads as

$$iS_t + \frac{1}{2}[S, S_{xt}] + iuS_x + \frac{1}{\omega}[S, W] = 0, \tag{449}$$

$$u_x - \frac{i}{4}tr(S[S_x, S_t]) = 0, \tag{450}$$

$$iW_x + \omega[S, W] = 0, \tag{451}$$

where $S = S_i\sigma_i$, $W = W_i\sigma_i$, $(i = 1, 2, 3)$ and ω is a constant parameter. The $\mathbf{W} = (W_1, W_2, W_3)$ is the vector potential. The Zhanbota-IA equation possesses the following Lax representation:

$$\Psi_x = U_{11}\Psi, \tag{452}$$

$$\Psi_t = V_{11}\Phi. \tag{453}$$

Here, the matrix operators U_{11} and V_{11} have the forms

$$U_{11} = -i\lambda S, \tag{454}$$

$$V_{11} = \frac{1}{1 - 2\lambda}\{\lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W\}, \tag{455}$$

where

$$V_1 = 2Z = \frac{1}{2}([S, S_t] + 2iuS), \tag{456}$$

$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \tag{457}$$

Let us present the gauge-equivalent counterpart of the Zhanbota-IA equation. It is not difficult to verify that the gauge-equivalent counterpart of the Zhanbota-IA equation is given by

$$q_t + \frac{\kappa}{2i}q_{xt} + ivq - 2p = 0, \tag{458}$$

$$r_t - \frac{\kappa}{2i}r_{xt} - ivr - 2k = 0, \tag{459}$$

$$v_x + \frac{\kappa}{2}(rq)_t = 0, \tag{460}$$

$$p_x - 2i\omega p - 2\eta q = 0, \tag{461}$$

$$k_x + 2i\omega k - 2\eta r = 0, \tag{462}$$

$$\eta_x + rp + kq = 0, \tag{463}$$

where q, r, p, k are some complex functions; v, η are real potential functions and κ is a constant parameter. The Lax representation for this equation reads as

$$\Phi_x = U_{12}\Phi, \tag{464}$$

$$\Phi_t = V_{12}\Phi, \tag{465}$$

where

$$U_{12} = -i\lambda\sigma_3 + A_0, \tag{466}$$

$$V_{12} = \frac{1}{1 - \kappa\lambda}\{B_0 + \frac{i}{\lambda + \omega}B_{-1}\}. \tag{467}$$

Here,

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{468}$$

$$B_0 = -\frac{i}{2} v \sigma_3 - \frac{\kappa}{2i} \begin{pmatrix} 0 & q_y \\ r_y & 0 \end{pmatrix}, \tag{469}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{470}$$

Let us consider the reduction $r = \delta_1 \bar{q}, k = \delta_2 \bar{p}$ with $\kappa = 2, \delta_j = \pm 1$, where the bar denotes the complex conjugate. Then, the system (487)–(492) takes the following more compact form

$$iq_t + q_{xt} - vq - 2ip = 0, \tag{471}$$

$$v_x + 2\delta_1(|q|^2)_t = 0, \tag{472}$$

$$p_x - 2i\omega p - 2\eta q = 0, \tag{473}$$

$$\eta_x + (\delta_1 \bar{q} p + \delta_2 \bar{p} q) = 0. \tag{474}$$

Finally, note that if $W = 0$ and $p = k = \eta = 0$, then, from (449)–(451) and (458)–(463), we obtain the K-IAE and the K-IIAE, respectively.

10.5.2. Zhanbota-IB Equation

Consider the Zhanbota-IB equation, which is given by

$$\mathbf{S}_x - \mathbf{S} \wedge \mathbf{S}_{xt} - u\mathbf{S}_t - \frac{1}{\omega}\mathbf{S} \wedge \mathbf{W} = 0, \tag{475}$$

$$u_t + \mathbf{S} \times (\mathbf{S}_t \wedge \mathbf{S}_x) = 0, \tag{476}$$

$$\mathbf{W}_t - \omega \mathbf{S} \wedge \mathbf{W} = 0. \tag{477}$$

The Zhanbota-IB equation is also one of the integrable generalization of the K-IBE. The matrix form of the Zhanbota-IB equation is given by

$$iS_x + \frac{1}{2}[S, S_{xt}] + iuS_t + \frac{1}{\omega}[S, W] = 0, \tag{478}$$

$$u_t - \frac{i}{4} tr(S[S_t, S_x]) = 0, \tag{479}$$

$$iW_t + \omega[S, W] = 0. \tag{480}$$

The Zhanbota-IB equation admits the following Lax representation:

$$\Psi_t = U_{11}\Psi, \tag{481}$$

$$\Psi_x = V_{11}\Phi. \tag{482}$$

Here, the matrix operators U_{11} and V_{11} have the forms

$$U_{11} = -i\lambda S, \tag{483}$$

$$V_{11} = \frac{1}{1 - 2\lambda}\{\lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W\}, \tag{484}$$

where

$$V_1 = 2Z = \frac{1}{2}([S, S_x] + 2iuS), \tag{485}$$

$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \tag{486}$$

Finally, let us present the gauge-equivalent counterpart of the Zhanbota-IB equation. It has the form

$$q_x + \frac{\kappa}{2i} q_{xt} + ivq - 2p = 0, \tag{487}$$

$$r_x - \frac{\kappa}{2i} r_{xt} - ivr - 2k = 0, \tag{488}$$

$$v_t + \frac{\kappa}{2}(rq)_x = 0, \tag{489}$$

$$p_t - 2i\omega p - 2\eta q = 0, \tag{490}$$

$$k_t + 2i\omega k - 2\eta r = 0, \tag{491}$$

$$\eta_t + rp + kq = 0. \tag{492}$$

Its Lax representation reads as

$$\Phi_t = U_{12}\Phi, \tag{493}$$

$$\Phi_x = V_{12}\Phi, \tag{494}$$

where

$$U_{12} = -i\lambda\sigma_3 + A_0, \tag{495}$$

$$V_{12} = \frac{1}{1 - \kappa\lambda}\left\{B_0 + \frac{i}{\lambda + \omega}B_{-1}\right\}. \tag{496}$$

Here,

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{497}$$

$$B_0 = -\frac{i}{2}v\sigma_3 - \frac{\kappa}{2i}\begin{pmatrix} 0 & q_x \\ r_x & 0 \end{pmatrix}, \tag{498}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{499}$$

For the reduction $r = \delta_1\bar{q}, k = \delta_2\bar{p}$ with $\kappa = 2, \delta_j = \pm 1$, the system (487)–(492) takes the following compact form

$$iq_x + q_{xt} - vq - 2ip = 0, \tag{500}$$

$$v_t + 2\delta_1(|q|^2)_x = 0, \tag{501}$$

$$p_t - 2i\omega p - 2\eta q = 0, \tag{502}$$

$$\eta_t + (\delta_1\bar{q}p + \delta_2\bar{p}q) = 0. \tag{503}$$

10.6. Akbota Equation

Our last example is the Akbota-I equation. As in the previous examples, it admits two cases, namely the Akbota-IA equation and the Akbota-IB equation.

10.6.1. The Akbota-IA Equation

The subject of this subsection is the following Akbota-IA equation

$$\mathbf{S}_t - \mathbf{S} \wedge (\alpha S_{xx} + \beta \mathbf{S}_{xt}) - u\mathbf{S}_x = 0, \tag{504}$$

$$u_x + \mathbf{S} \times (\mathbf{S}_x \wedge \mathbf{S}_t) = 0. \tag{505}$$

This Akbota-IA equation is one of the integrable generalizations of the K-IAE. It has the following Lax representation

$$\Psi_x - U_{13}\Psi = 0, \tag{506}$$
$$\Psi_t - V_{13}\Psi = 0, \tag{507}$$

where

$$U_{13} = \frac{i}{2}\lambda S, \quad V_{13} = \frac{1}{1 - \lambda\beta}\{\alpha(\frac{1}{2}i\lambda^2 S + \frac{1}{4}[S, S_x]) + \beta\lambda Z\}. \tag{508}$$

Note that the gauge-equivalent counterpart of the Akbota-IA equation is the following nonlinear evolution equation

$$iq_t + \alpha q_{xx} + \beta q_{xt} + vq = 0, \tag{509}$$
$$v_x - 2[\alpha(|q|^2)_x + \beta(|q|^2)_t] = 0. \tag{510}$$

This is the Akbota-IIA equation. Its Lax representation is given by

$$\Phi_x = U_{14}\Phi, \tag{511}$$
$$\Phi_t = V_{14}\Phi, \tag{512}$$

where

$$U_{14} = \frac{i\lambda}{2}\sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix}, \quad V_{14} = \frac{1}{1 - \lambda\beta}\{\frac{i\lambda^2}{2}\alpha\sigma_3 + \alpha\lambda Q + V_0\} \tag{513}$$

with

$$V_0 = \begin{pmatrix} \alpha i|q|^2 + i\beta\partial_x^{-1}|q|_t^2 & -i\beta\bar{q}_t - i\alpha\bar{q}_x \\ i\beta q_t + \alpha iq_x & -[\alpha i|q|^2 + i\beta\partial_x^{-1}|q|_t^2] \end{pmatrix}. \tag{514}$$

10.6.2. The Akbota-IB Equation

Consider the following Akbota-IB equation

$$\mathbf{S}_x - \mathbf{S} \wedge (\alpha\mathbf{S}_{tt} + \beta\mathbf{S}_{xt}) - u\mathbf{S}_t = 0, \tag{515}$$
$$u_t + \mathbf{S} \times (\mathbf{S}_t \wedge \mathbf{S}_x) = 0. \tag{516}$$

This Akbota-IB equation is an integrable generalization of the K-IBE. Its Lax representation is given by

$$\Psi_t - U_{13}\Psi = 0, \tag{517}$$
$$\Psi_x - V_{13}\Psi = 0, \tag{518}$$

where

$$U_{13} = \frac{i}{2}\lambda S, \quad V_{13} = \frac{1}{1 - \lambda\beta}\{\alpha(\frac{1}{2}i\lambda^2 S + \frac{1}{4}[S, S_t]) + \beta\lambda Z\}. \tag{519}$$

The gauge-equivalent counterpart of the Akbota-IB equation reads as

$$iq_x + \alpha q_{tt} + \beta q_{xt} + vq = 0, \tag{520}$$
$$v_t - 2[\alpha(|q|^2)_t + \beta(|q|^2)_x] = 0. \tag{521}$$

This is the Akbota-IIB equation. Its Lax representation is given by

$$\Phi_t = U_{14}\Phi, \tag{522}$$
$$\Phi_x = V_{14}\Phi, \tag{523}$$

where

$$U_{14} = \frac{i\lambda}{2}\sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix}, \quad V_{14} = \frac{1}{1-\lambda\beta}\{\frac{i\lambda^2}{2}\alpha\sigma_3 + \alpha\lambda Q + V_0\} \quad (524)$$

with

$$V_0 = \begin{pmatrix} \alpha i|q|^2 + i\beta\partial_t^{-1}|q|_x^2 & -i\beta\bar{q}_x - i\alpha\bar{q}_t \\ i\beta q_x + \alpha i q_t & -[\alpha i|q|^2 + i\beta\partial_t^{-1}|q|_x^2] \end{pmatrix}. \quad (525)$$

11. Conclusions

In this paper, the Kuralay equations, namely the Kuralay-I equation (K-IE) and the Kuralay-II equation (K-IIE), have been studied. As is known, these equations are integrable by the inverse scattering transform method. The integrable motion of space curves induced by the K-IE and K-IIE was investigated. The gauge and geometrical equivalences between these two equations were established. The Hirota bilinear form of the KE was constructed. With the help of the Hirota bilinear method, the simplest soliton solutions are also presented. Note that these simplest soliton solutions admit generalizations as the traveling wave in terms of Jacobi elliptic functions. For example, we have shown that there are two such generalizations of the 1-soliton solution. The nonlocal and dispersionless versions of the Kuralay equations have been discussed. Finally, some integrable generalizations of the KE have been presented.

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