Research Article

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Galton–Watson Theta-Processes in a Varying Environment

https://doi.org/10.1515/eqc-2024-0001
Received January 2, 2024; accepted January 24, 2024

Abstract: We consider a special class of Galton–Watson theta-processes in a varying environment fully defined by four parameters, with two of them (θ, r) being fixed over time n, and the other two (a_n, c_n) characterizing the altering reproduction laws. We establish a sequence of transparent limit theorems for the theta-processes with possibly defective reproduction laws. These results may serve as a stepping stone towards incisive general results for the Galton–Watson processes in a varying environment.

Keywords: Branching Process, Varying Environment, Limit Theorem

MSC 2020: 60J80

1 Introduction

The basic version of the Galton–Watson process (GW-process) was conceived as a stochastic model of the population growth or extinction of a single species of individuals [3, 7]. The GW-process $\{Z_n\}_{n\geq 0}$ unfolds in the discrete time setting, with Z_n standing for the population size at the generation n under the assumption that each individual is replaced by a random number of offspring. It is assumed that the offspring numbers are independent random variables having the same distribution $\{p(j)\}_{j\geq 0}$.

By allowing the offspring number distribution $\{p_n(j)\}_{j\geq 0}$ to depend on the generation number n, we arrive at the GW-process in a varying environment [4]. This more flexible model is fully described by a sequence of probability generating functions

$$f_n(s) = \sum_{j\geq 0} p_n(j)s^j, \quad 0 \leq s \leq 1,\ n \geq 1.$$

Introduce the composition of generating functions

$$F_n(s) = f_1 \circ \cdots \circ f_n(s), \quad 0 \leq s \leq 1,\ n \geq 1.$$

Given that the GW-process starts at time zero with a single individual, we get

$$\mathrm{E}(s^{Z_n}) = F_n(s), \quad \mathrm{P}(Z_n = 0) = F_n(0).$$

The state 0 of the GW-process is absorbing and the extinction probability for the modeled population is determined by

$$q = \lim F_n(0)$$

(here and throughout, all limits are taken as $n \to \infty$, unless otherwise specified). In the case of *proper* reproduction laws with $f_n(1) = 1$ for all $n \geq 1$, we get

$$\mathrm{E}(Z_n) = F_n'(1) = f_1'(1)\cdots f_n'(1), \quad \mathrm{E}(Z_n | Z_n > 0) = \frac{F_n'(1)}{1 - F_n(0)}.$$

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In [5], the usual ternary classification of the GW-processes into supercritical, critical, and subcritical processes [1], was adapted to the framework of the varying environment. Given $0 < f_n'(1) < \infty$ for all n, it was shown that under a regularity condition (A) in [5], it makes sense to distinguish among four classes of the GW-processes in a varying environment: supercritical, asymptotically degenerate, critical, and subcritical processes. In a more recent paper [10] devoted to the Markov theta-branching processes in a varying environment, the quaternary classification of [5] was further refined into a quinary classification, which can be adapted to the discrete time setting as follows:

- *supercritical case:* $q < 1$ and $\lim \mathrm{E}(Z_n) = \infty$,
- *asymptotically degenerate case:* $q < 1$ and $\liminf \mathrm{E}(Z_n) < \infty$,
- *critical case:* $q = 1$ and $\lim \mathrm{E}(Z_n|Z_n > 0) = \infty$,
- *strictly subcritical case:* $q = 1$ and a finite $\lim \mathrm{E}(Z_n|Z_n > 0)$ exists,
- *loosely subcritical case:* $q = 1$ and $\lim \mathrm{E}(Z_n|Z_n > 0)$ does not exist.

Our paper is build upon the properties of a special parametric family of generating functions [9] leading to what will be called here the Galton–Watson theta-processes or GW^θ-processes. The remarkable property of the GW^θ-processes in a varying environment is that the generating functions $F_n(s)$ have explicit expressions presented in Section 2. An important feature of the GW^θ-processes is that they allow for defective reproduction laws. If the generating function $f_i(s)$ is *defective*, in that $f_i(1) < 1$, then $F_n(1) < 1$ for all $n \geq i$. In the defective case [6, 11], a single individual, with probability $1 - f_i(1)$ may force the entire GW-process to visit to an ancillary absorbing state Δ by the observation time n with probability

$$\mathrm{P}(Z_n = \Delta) = 1 - F_n(1).$$

In Sections 3 and 4, we state ten limit theorems for the GW^θ-processes in a varying environment. These results are illuminated in Section 5 by ten examples describing different growth and extinction patterns under environmental variation. The proofs are collected in Section 6.

2 Proper and Defective Reproduction Laws

Definition 1. Consider a sequence $(\theta, r, a_n, c_n)_{n\geq 1}$ satisfying one of the following sets of conditions:
(a) $\theta \in (0, 1]$, $r = 1$, and for $n \geq 1$, $0 < a_n < \infty$, $c_n > 0$, $c_n \geq 1 - a_n$,
(b) $\theta \in (0, 1]$, $r > 1$, and for $n \geq 1$, $0 < a_n < 1$, $(1 - a_n)r^{-\theta} \leq c_n \leq (1 - a_n)(r - 1)^{-\theta}$,
(c) $\theta \in (-1, 0)$, $r = 1$, and for $n \geq 1$, $0 < a_n < 1$, $0 < c_n \leq 1 - a_n$,
(d) $\theta \in (-1, 0)$, $r > 1$, and for $n \geq 1$, $0 < a_n < 1$, $(1 - a_n)(r - 1)^{-\theta} \leq c_n \leq (1 - a_n)r^{-\theta}$,
(e) $\theta = 0$, $r = 1$, and for $n \geq 1$, $0 < a_n < 1$, $0 \leq c_n < 1$,
(f) $\theta = 0$, $r > 1$, and for $n \geq 1$, $0 < a_n < 1$, $0 \leq c_n \leq 1$.

A GW^θ-process with parameters $(\theta, r, a_n, c_n)_{n\geq 1}$ is a GW-process in a varying environment characterized by a sequence of probability generating functions $(f_n(s))_{n\geq 1}$ defined by

$$f_n(s) = r - (a_n(r - s)^{-\theta} + c_n)^{-\frac{1}{\theta}}, \quad 0 \leq s < r, \qquad f_n(r) = r, \tag{2.1}$$

for $\theta \neq 0$, and for $\theta = 0$, defined by

$$f_n(s) = r - (r - c_n)^{1-a_n}(r - s)^{a_n}, \quad 0 \leq s \leq r. \tag{2.2}$$

Definition 1 is motivated by [9, Definitions 14.1 and 14.2], which also mentions a trivial case of $\theta = -1$ not included here. Observe that in the setting of varying environment, the key parameters $\theta \in (-1, 1]$ and $r \geq 1$ stay constant over time, while the parameters (a_n, c_n) may vary. The case $\theta = r = 1$ is the well studied case of the linear-fractional reproduction law.

This section contains two key lemmas. Lemma 1 gives the explicit expressions for the generating functions $F_n(s)$ in terms of positive constants $A_n, C_n, D_n = D_n(r)$ defined by

$$A_0 = 1, \quad A_n = \prod_{i=1}^n a_i, \quad C_n = \sum_{i=1}^n A_{i-1}c_i, \quad D_n = \prod_{i=1}^n (r - c_i)^{A_{i-1}-A_i}.$$

Lemmas 2 presents the asymptotic properties of the constants A_n, C_n, D_n leading to the limit theorems stated in Sections 3 and 4.

Lemma 1. *Consider a* GW^θ*-process with parameters* (θ, r, a_n, c_n). *If* $\theta \neq 0$, *then*

$$F_n(s) = r - (A_n(r-s)^{-\theta} + C_n)^{-\frac{1}{\theta}}, \quad 0 \le s < r, \qquad F_n(r) = r, \quad n \ge 1,$$

and if $\theta = 0$, *then*

$$F_n(s) = r - (r-s)^{A_n} D_n, \quad 0 \le s \le r, \ n \ge 1.$$

Here:

(a) *for* $\theta \in (0,1]$, $r = 1$,

$$0 < A_n < \infty, \quad C_n > 0, \quad C_n \ge 1 - A_n, \quad F_n(1) = 1, \quad F_n'(1) = A_n^{-\frac{1}{\theta}}, \quad n \ge 1,$$

(b) *for* $\theta \in (0,1]$, $r > 1$,

$$0 < A_n < 1, \quad (1-A_n)r^{-\theta} \le C_n \le (1-A_n)(r-1)^{-\theta}, \quad F_n(1) \le 1, \quad n \ge 1,$$

with $F_n(1) = 1$ *if and only if* $c_k = (1-a_k)(r-1)^{-\theta}$, $1 \le k \le n$, *implying* $F_n'(1) = A_n$,

(c) *for* $\theta \in (-1,0)$, $r = 1$,

$$0 < A_n < \infty, \quad 0 < C_n \le 1 - A_n, \quad F_n(1) = 1 - C_n^{-\frac{1}{\theta}}, \quad n \ge 1,$$

(d) *for* $\theta \in (-1,0)$, $r > 1$,

$$0 < A_n < 1, \quad (1-A_n)(r-1)^{-\theta} \le C_n \le (1-A_n)r^{-\theta}, \quad F_n(1) \le 1, \quad n \ge 1,$$

with $F_n(1) = 1$ *if and only if* $c_k = (1-a_k)(r-1)^{-\theta}$, $1 \le k \le n$, *implying* $F_n'(1) = A_n$,

(e) *for* $\theta = 0$, $r = 1$,

$$0 < A_n < 1, \quad 0 < D_n \le 1, \quad F_n(1) = 1, \quad F_n'(1) = \infty, \quad n \ge 1,$$

(f) *for* $\theta = 0$, $r > 1$,

$$0 < A_n < 1, \quad (r-1)^{1-A_n} \le D_n \le r^{1-A_n}, \quad F_n(1) \le 1, \quad n \ge 1,$$

with $F_n(1) = 1$ *if and only if* $c_k = 1$, $1 \le k \le n$, *implying* $F_n'(1) = A_n$.

Lemma 2. *Denote the limits* $A = \lim A_n$, $C = \lim C_n$, $D = \lim D_n$, *whenever they exist, whether finite or infinite.*

(a) *If* $\theta \in (0,1]$, $r = 1$, *then* $C \in [1,\infty]$, *and if* $C < \infty$, *then* $A \in [0,\infty]$.
(b) *If* $\theta \in (0,1]$, $r > 1$, *then* $A \in [0,1)$ *and* $(1-A)r^{-\theta} \le C \le (1-A)(r-1)^{-\theta}$.
(c) *If* $\theta \in (-1,0)$, $r = 1$, *then* $A \in [0,1)$ *and* $0 < C \le 1 - A$.
(d) *If* $\theta \in (-1,0)$, $r > 1$, *then* $A \in [0,1)$ *and* $(1-A)(r-1)^{-\theta} \le C \le (1-A)r^{-\theta}$.
(e) *If* $\theta = 0$, $r = 1$, *then* $A \in [0,1)$ *and* $D = \prod_{n\ge1}(1-c_n)^{A_{n-1}-A_n}$ *with* $D \in [0,1]$.
(f) *If* $\theta = 0$, $r > 1$, *then* $A \in [0,1)$ *and* $D = \prod_{n\ge1}(r-c_n)^{A_{n-1}-A_n}$ *with* $(r-1)^{1-A} \le D \le r^{1-A}$.

3 Limit Theorems for the Proper GW^θ-Processes

Theorems 1–5 deal with the GW^θ-process in the case $\theta \in (0,1]$, $r = 1$, when by Lemma 1,

$$\mathrm{E}(Z_n) = A_n^{-\frac{1}{\theta}}, \quad \mathrm{P}(Z_n > 0) = (A_n + C_n)^{-\frac{1}{\theta}}.$$

Putting $B_n = \frac{C_n}{A_n}$, we obtain

$$\mathrm{E}(Z_n | Z_n > 0) = (1 + B_n)^{\frac{1}{\theta}}.$$

These five theorems fully cover the five regimes of reproduction in a varying environment and could be summarized as follows. Let $\theta \in (0,1]$, $r = 1$,

- given $C < \infty$, the GW^θ-process is
  - supercritical if $A_n \to 0$, see Theorem 1,
  - asymptotically degenerate if $A_n \to A \in (0,\infty)$, see Theorem 2,
  - strictly subcritical if $A_n \to \infty$, see Theorem 4,

- given $C = \infty$, the GW$^\theta$-process is
  - critical if $B_n \to \infty$, see Theorem 3,
  - strictly subcritical if $B_n \to B \in [0, \infty)$, see Theorem 4,
  - loosely subcritical if the $\lim B_n$ does not exist, see Theorem 5.

This section also includes Theorem 6 addressing the proper case $\theta = 0$, $r = 1$. Notice that Theorem 6 deals with the case of infinite mean values, when the above mentioned quinary classification does not apply.

Theorem 1. *Let $\theta \in (0, 1]$, $r = 1$, and $C < \infty$. If $A_n \to 0$, then $q = 1 - C^{-\frac{1}{\theta}}$ and $A_n^{\frac{1}{\theta}} Z_n$ almost surely converges to a random variable W such that*

$$\mathrm{E}(e^{-\lambda W}) = 1 - (\lambda^{-\theta} + C)^{-\frac{1}{\theta}}, \quad \lambda \ge 0.$$

Theorem 2. *Let $\theta \in (0, 1]$, $r = 1$, and $C < \infty$. If $A_n \to A \in (0, \infty)$, then*

$$q = 1 - (A + C)^{-\frac{1}{\theta}}, \qquad \mathrm{E}(Z_n) \to A^{-\frac{1}{\theta}},$$

and Z_n almost surely converges to a random variable Z_∞ such that

$$\mathrm{E}(Z_\infty) = A^{-\frac{1}{\theta}}, \quad \mathrm{E}(s^{Z_\infty}) = 1 - (A(1 - s)^{-\theta} + C)^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$$

Theorem 3. *Let $\theta \in (0, 1]$, $r = 1$, and $C = \infty$. If $B_n \to \infty$, then $q = 1$,*

$$\mathrm{P}(Z_n > 0) \sim C_n^{-\frac{1}{\theta}}, \quad \mathrm{E}(Z_n | Z_n > 0) \sim B_n^{\frac{1}{\theta}},$$

and with $\lambda_n = \lambda B_n^{-\frac{1}{\theta}}$,

$$\mathrm{E}(e^{-\lambda_n Z_n} | Z_n > 0) \to 1 - (1 + \lambda^{-\theta})^{-\frac{1}{\theta}}, \quad \lambda \ge 0.$$

Theorem 4. *Let $\theta \in (0, 1]$ and $r = 1$. If $A_n \to \infty$ and $B_n \to B \in [0, \infty)$, then $q = 1$,*

$$\mathrm{P}(Z_n > 0) \sim (1 + B)^{-\frac{1}{\theta}} A_n^{-\frac{1}{\theta}}, \quad \mathrm{E}(Z_n | Z_n > 0) \to (1 + B)^{\frac{1}{\theta}},$$

and

$$\mathrm{E}(s^{Z_n} | Z_n > 0) \to 1 - ((1 + B)(1 - s)^{-\theta} + B + B^2)^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$$

Theorem 5. *Let $\theta \in (0, 1]$, $r = 1$, and assume that* $\lim B_n$ *does not exist. Then $q = 1$ and letting*

$$B_{k_n} \to B \in [0, \infty]$$

along a subsequence $k_n \to \infty$, we get:

(i) *if $B = \infty$, then*

$$\mathrm{P}(Z_{k_n} > 0) \sim C_{k_n}^{-\frac{1}{\theta}}, \quad \mathrm{E}(Z_{k_n} | Z_{k_n} > 0) \sim B_{k_n}^{\frac{1}{\theta}},$$

and with $\lambda_n = \lambda B_n^{-\frac{1}{\theta}}$,

$$\mathrm{E}(e^{-\lambda_{k_n} Z_{k_n}} | Z_{k_n} > 0) \to 1 - (1 + \lambda^{-\theta})^{-\frac{1}{\theta}}, \quad \lambda \ge 0,$$

(ii) *if $B \in [0, \infty)$, then $A_{k_n} \to \infty$,*

$$\mathrm{P}(Z_{k_n} > 0) \sim (1 + B)^{-\frac{1}{\theta}} A_{k_n}^{-\frac{1}{\theta}}, \quad \mathrm{E}(Z_{k_n} | Z_{k_n} > 0) \to (1 + B)^{\frac{1}{\theta}},$$

and

$$\mathrm{E}(s^{Z_{k_n}} | Z_{k_n} > 0) \to 1 - ((1 + B)(1 - s)^{-\theta} + B + B^2)^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$$

Theorem 6. *Suppose $\theta = 0$ and $r = 1$. Then $\mathrm{P}(Z_n > 0) = D_n$, so that $q = 1 - D$, with D given by Lemma 2* (e). *Furthermore:*

(i) *if $A = 0$ and $D = 0$, then $q = 1$ and*

$$\mathrm{P}(A_n \ln Z_n \le x | Z_n > 0) \to 1 - e^{-x}, \quad x \ge 0,$$

(ii) *if $A = 0$ and $D > 0$, then $q < 1$ and*

$$\mathrm{P}(A_n \ln Z_n \le x) \to 1 - e^{-x} D, \quad x \ge 0,$$

(iii) *if* $A \in (0,1)$ *and* $D = 0$*, then* $q = 1$ *and*

$$\mathrm{E}(s^{Z_n}|Z_n > 0) \to 1 - (1-s)^A, \quad 0 \le s \le 1,$$

(iv) *if* $A \in (0,1)$ *and* $D > 0$*, then* $q < 1$ *and* Z_n *almost surely converges to a random variable* Z_∞ *such that*

$$\mathrm{E}(Z_\infty) = \infty, \quad \mathrm{E}(s^{Z_\infty}) = 1 - (1-s)^A D, \quad 0 \le s \le 1.$$

Remarks

We make the following observations.

(i) It is a straightforward exercise to check that the above mentioned regularity condition (A) in [5] is valid for the GW^θ-process in the case $\theta \in (0,1]$, $r = 1$.

(ii) The limiting distribution obtained in Theorem 3 coincides with that of [12] obtained for the critical GW-processes in a constant environment with a possibly infinite variance for the offspring number.

(iii) Statement (ii) of Theorem 6 is of the Darling–Seneta-type limit theorem obtained in [2] for GW-processes with infinite mean.

(iv) Part (iv) of Theorem 6 presents the pattern of limit behavior similar to the asymptotically degenerate regime in the case of infinite mean values. The conditions of Theorem 6 (iv) hold if and only if

$$\sum_{n\ge1}(1-a_n) < \infty \tag{3.1}$$

and

$$\sum_{n\ge1}(1-a_n)\ln\frac{1}{1-c_n} < \infty. \tag{3.2}$$

4 Limit Theorems for the Defective GW^θ-Process

In the defective case, there are two kinds of absorption times:

(i) τ_0 the absorption time of the GW^θ-process at 0,

(ii) τ_Δ the absorption time of the GW^θ-process at the state Δ.

Let $\tau = \min(\tau_0, \tau_\Delta)$ be the absorption time of the GW^θ-process either at 0 or at the state Δ. Let us recall that $q = \mathrm{P}(\tau_0 < \infty)$ and denote

$$q_\Delta = \mathrm{P}(\tau_\Delta < \infty), \quad Q = \mathrm{P}(\tau < \infty) = q + q_\Delta.$$

Clearly,

$$\mathrm{P}(\tau \le n) = \mathrm{P}(\tau_0 \le n) + \mathrm{P}(\tau_\Delta \le n) = F_n(0) + 1 - F_n(1),$$

implying

$$\mathrm{P}(\tau > n) = F_n(1) - F_n(0).$$

Furthermore,

$$\mathrm{E}(Z_n; \tau_\Delta > n) = F_n'(1), \qquad \mathrm{E}(s^{Z_n}; \tau_\Delta > n) = F_n(s), \quad 0 \le s \le 1,$$

so that

$$\mathrm{E}(Z_n|\tau > n) = \frac{F_n'(1)}{F_n(1) - F_n(0)}, \qquad \mathrm{E}(s^{Z_n}|\tau > n) = \frac{F_n(s) - F_n(0)}{F_n(1) - F_n(0)}, \quad 0 \le s \le 1.$$

Theorems 7–10 present the transparent asymptotical results on these absorption probabilities and the limit behavior of the GW^θ-process in the four defective cases. Corollaries of Theorems 7–9 deal with the proper subcases, where $\tau = \tau_0$. All three corollaries describe a strictly subcritical case, when $A = 0$, and an asymptotically degenerate case, when $A \in (0,1)$.

Theorem 7. *Consider the case $\theta \in (0,1]$, $r > 1$. Then*

$$q = r - (Ar^{-\theta} + C)^{-\frac{1}{\theta}}, \quad q_\Delta = 1 - r + (A(r-1)^{-\theta} + C)^{-\frac{1}{\theta}},$$

where $A \in [0,1)$ and $(1-A)r^{-\theta} \le C \le (1-A)(r-1)^{-\theta}$.

(i) *If $A = 0$, then*

$$q = 1 - q_\Delta = r - C^{-\frac{1}{\theta}} \in [0,1],$$

so that $Q = 1$. Furthermore,

$$A_n^{-1}\mathrm{P}(\tau > n) \to ((r-1)^{-\theta} - r^{-\theta})\theta^{-1}C^{-\frac{1}{\theta}-1},$$

$$\mathrm{E}(Z_n | \tau > n) \to \frac{(r-1)^{-\theta-1}}{(r-1)^{-\theta} - r^{-\theta}}, \quad \mathrm{E}(s^{Z_n} | \tau > n) \to \frac{(r-s)^{-\theta} - r^{-\theta}}{(r-1)^{-\theta} - r^{-\theta}}, \quad 0 \le s \le 1.$$

(ii) *If $A \in (0,1)$, then $Q \in [0,1)$,*

$$\mathrm{E}(Z_n; \tau_\Delta > n) \to A(A + C(r-1)^\theta)^{-\frac{1}{\theta}-1},$$

and Z_n almost surely converges to a random variable Z_∞taking values in the set $\{\Delta, 0, 1, 2, \dots\}$, with

$$\mathrm{P}(Z_\infty = \Delta) = 1 - r + (A(r-1)^{-\theta} + C)^{-\frac{1}{\theta}},$$

$$\mathrm{E}(s^{Z_\infty}; Z_\infty \ne \Delta) = r - (A(r-s)^{-\theta} + C)^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$$

Corollary. *Consider the case $\theta \in (0,1]$, $r > 1$ assuming*

$$c_n = (1 - a_n)(r-1)^{-\theta}, \quad n \ge 1, \tag{4.1}$$

so that $C = (1-A)(r-1)^{-\theta}$ implying $q_\Delta = 0$.

(i) *If $A = 0$, then $q = 1$ with*

$$A_n^{-1}\mathrm{P}(Z_n > 0) \to ((r-1)^{-\theta} - r^{-\theta})\theta^{-1}(r-1)^{\theta+1}.$$

Furthermore,

$$\mathrm{E}(Z_n | Z_n > 0) \to \frac{(r-1)^{-\theta-1}}{\theta((r-1)^{-\theta} - r^{-\theta})}, \qquad \mathrm{E}(s^{Z_n} | Z_n > 0) \to \frac{(r-s)^{-\theta} - r^{-\theta}}{(r-1)^{-\theta} - r^{-\theta}}, \quad 0 \le s \le 1.$$

(ii) *If $A \in (0,1)$, then*

$$q = 1 - r + (Ar^{-\theta} + C)^{-\frac{1}{\theta}}, \quad \mathrm{E}(Z_n) \to A,$$

so that $q \in (0,1)$, and Z_n almost surely converges to a random variable Z_∞ such that

$$\mathrm{E}(Z_\infty) = A, \qquad \mathrm{E}(s^{Z_\infty}) = r - (A(r-s)^{-\theta} + (1-A)(r-1)^{-\theta})^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$$

Theorem 8. *Consider the case $\theta \in (-1,0)$, $r > 1$ and put $\alpha = -\frac{1}{\theta}$, so that $\alpha > 1$. Then*

$$q = r - (Ar^{\frac{1}{\alpha}} + C)^\alpha, \quad q_\Delta = 1 - r + (A(r-1)^{\frac{1}{\alpha}} + C)^\alpha,$$

where $A \in [0,1)$ and $(1-A)(r-1)^{\frac{1}{\alpha}} \le C \le (1-A)r^{\frac{1}{\alpha}}$.

(i) *If $A = 0$, then*

$$q = 1 - q_\Delta = r - C^\alpha \in [0,1],$$

so that $Q = 1$. Furthermore,

$$A_n^{-1}\mathrm{P}(\tau > n) \to \alpha C^{\alpha-1}(r^{\frac{1}{\alpha}} - (r-1)^{\frac{1}{\alpha}}),$$

$$\mathrm{E}(Z_n | \tau > n) \to \frac{(r-1)^{\frac{1}{\alpha}-1}}{r^{\frac{1}{\alpha}} - (r-1)^{\frac{1}{\alpha}}}, \quad \mathrm{E}(s^{Z_n} | \tau > n) \to \frac{r^{\frac{1}{\alpha}} - (r-s)^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}} - (r-1)^{\frac{1}{\alpha}}}, \quad 0 \le s \le 1.$$

(ii) *If $A \in (0,1)$, then $Q \in [0,1)$,*

$$\mathrm{E}(Z_n; \tau_\Delta > n) \to A(A + C(r-1)^{-\frac{1}{\alpha}})^{\alpha-1},$$

and Z_n almost surely converges to a random variable Z_∞ taking values in the set $\{\Delta, 0, 1, 2, \dots\}$, with

$$\mathrm{P}(Z_\infty = \Delta) = 1 - r + (A(r-1)^{\frac{1}{\alpha}} + C)^\alpha,$$

$$\mathrm{E}(s^{Z_\infty}; Z_\infty \ne \Delta) = r - (A(r-s)^{\frac{1}{\alpha}} + C)^\alpha, \quad 0 \le s \le 1.$$

Corollary. *Consider the case $\theta \in (-1,0)$, $r > 1$ assuming (4.1), so that $C = (1-A)(r-1)^{\frac{1}{\alpha}}$ implying $q_\Delta = 0$.*
(i) *If $A = 0$, then $q = 1$ with*

$$A_n^{-1}\mathrm{P}(Z_n > 0) \to \alpha(r-1)^{1-\frac{1}{\alpha}}(r^{\frac{1}{\alpha}} - (r-1)^{\frac{1}{\alpha}}).$$

Furthermore,

$$\mathrm{E}(Z_n|Z_n > 0) \to \frac{(r-1)^{-\theta-1}}{\theta((r-1)^{-\theta} - r^{-\theta})}, \qquad \mathrm{E}(s^{Z_n}|Z_n > 0) \to \frac{(r-s)^{-\theta} - r^{-\theta}}{(r-1)^{-\theta} - r^{-\theta}}, \quad 0 \le s \le 1.$$

(ii) *If $A \in (0,1)$, then*

$$q = 1 - r + (Ar^{\frac{1}{\alpha}} + (1-A)(r-1)^{\frac{1}{\alpha}})^\alpha, \quad \mathrm{E}(Z_n) \to A,$$

so that $q \in (0,1)$, and Z_n almost surely converges to a random variable Z_∞ such that

$$\mathrm{E}(Z_\infty) = A, \qquad \mathrm{E}(s^{Z_\infty}) = r - (A(r-s)^{\frac{1}{\alpha}} + (1-A)(r-1)^{\frac{1}{\alpha}})^\alpha, \quad 0 \le s \le 1.$$

Theorem 9. *Consider the case $\theta = 0$, $r > 1$ implying*

$$q = r - r^A D, \quad q_\Delta = 1 - r + (r-1)^A D, \quad Q = 1 - (r^A - (r-1)^A)D,$$

where D is given by Lemma 2 (f)*.*
(i) *If $A = 0$, then $Q = 1$, and*

$$\mathrm{P}(\tau > n) \sim (\ln r - \ln(r-1))A_n D_n.$$

Moreover,

$$\mathrm{E}(Z_n|\tau > n) \to \frac{(r-1)^{-1}}{\ln r - \ln(r-1)}, \qquad \mathrm{P}(s^{Z_n}|\tau > n) \to \frac{\ln r - \ln(r-s)}{\ln r - \ln(r-1)}, \quad 0 \le s \le 1.$$

(ii) *If $A \in (0,1)$, then $Q < 1$,*

$$(r-1)^{1-A} \le D \le r^{1-A}, \quad \mathrm{E}(Z_n; \tau_\Delta > n) \to A(r-1)^{A-1}D,$$

and Z_n almost surely converges to a random variable Z_∞ taking values in the set $\{\Delta, 0, 1, 2, \dots\}$, with

$$\mathrm{P}(Z_\infty = \Delta) = 1 - r + (r-1)^A D, \qquad \mathrm{E}(s^{Z_\infty}; Z_\infty \ne \Delta) = r - (r-s)^A D, \quad 0 \le s \le 1.$$

Corollary. *Given $\theta = 0$, $r > 1$, assume $c_n \equiv 1$. Then $D = (r-1)^{1-A}$ implying $q_\Delta = 0$.*
(i) *If $A = 0$, then $q = 1$, and*

$$\mathrm{P}(Z_n > 0) \sim (\ln r - \ln(r-1))A_n D_n.$$

Moreover,

$$\mathrm{E}(Z_n|Z_n > 0) \to \frac{(r-1)^{-1}}{\ln r - \ln(r-1)}, \qquad \mathrm{P}(s^{Z_n}|Z_n > 0) \to \frac{\ln r - \ln(r-s)}{\ln r - \ln(r-1)}, \quad 0 \le s \le 1.$$

(ii) *If $A \in (0,1)$, then*

$$q = r - r^A(r-1)^{1-A}, \quad \mathrm{E}(Z_n) \to A,$$

so that $q \in (0,1)$, and Z_n almost surely converges to a proper random variable Z_∞, such that

$$\mathrm{E}(Z_\infty) = A, \qquad \mathrm{E}(s^{Z_\infty}) = r - (r-s)^A(r-1)^{1-A}, \quad 0 \le s \le 1.$$

Theorem 10. *In the case $\theta \in (-1,0)$, $r = 1$, put $\alpha = -\frac{1}{\theta}$, so that $\alpha > 1$. Then*

$$q = 1 - (A+C)^\alpha, \quad q_\Delta = C^\alpha, \quad Q = 1 - (A+C)^\alpha + C^\alpha,$$

where $A \in [0,1)$ and $0 < C \le 1 - A$.
(i) *If $A = 0$, then $q = 1 - q_\Delta = 1 - C^\alpha$, $Q = 1$, and*

$$A_n^{-1}\mathrm{P}(\tau > n) \to \alpha C^{\alpha-1}.$$

Moreover,

$$\mathrm{E}(s^{Z_n}|\tau > n) \to 1 - (1-s)^{\frac{1}{\alpha}}, \quad 0 \le s \le 1.$$

(ii) *If $A \in (0,1)$, then $Q < 1$,*

$$\mathrm{E}(Z_n; \tau_\Delta > n) = \infty,$$

and Z_n almost surely converges to a random variable Z_∞ taking values in the set $\{\Delta, 0, 1, 2, \dots\}$, with

$$\mathrm{P}(Z_\infty = \Delta) = C^\alpha, \qquad \mathrm{E}(s^{Z_\infty}; Z_\infty \ne \Delta) = 1 - (A(1-s)^{\frac{1}{\alpha}} + C)^\alpha, \quad 0 \le s \le 1.$$

Remarks

We make the following observations.

(i) Theorem 7 (ii) should be compared to the more general [6, Theorem 1], which allows the limit Z_∞ to take the value ∞ with a positive probability. The convergence results for the conditional expectation should be compared to the statements of [6, Theorems 3 and 4].

(ii) The conditional convergence in distribution stated in Theorem 7 (i) should be compared to [11, Theorem 2a ($k = 0$)] in the more general setting under the assumption of constant environment.

5 Examples

The following ten examples illustrate each of the ten theorems of this paper. Observe that given

$$c_n = (1 - a_n)\sigma, \quad n \geq 1, \tag{5.1}$$

for some suitable positive constant σ, we get $C_n = (1 - A_n)\sigma$, $n \geq 1$. Similarly, if

$$c_n = (a_n - 1)\sigma, \quad n \geq 1, \tag{5.2}$$

for some suitable positive constant σ, then $C_n = (A_n - 1)\sigma$, $n \geq 1$.

Example 1. Suppose $\theta \in (0, 1]$, $r = 1$, and

$$a_n = \frac{n}{n+1}, \quad A_n = \frac{1}{n+1}, \quad n \geq 1. \tag{5.3}$$

If (5.1) holds for some $\sigma \geq 1$, then by Theorem 1,

$$q = 1 - \sigma^{-\frac{1}{\theta}}, \quad n^{-\frac{1}{\theta}} \mathrm{E}(Z_n) \to 1,$$

and $n^{-\frac{1}{\theta}} Z_n \to W$ almost surely, with

$$\mathrm{E}(e^{-\lambda W}) = 1 - (\lambda^{-\theta} + \sigma)^{-\frac{1}{\theta}}, \quad \lambda \geq 0.$$

Example 2. Suppose $\theta \in (0, 1]$, $r = 1$, and

$$a_n = \frac{n(n+3)}{(n+1)(n+2)}, \quad A_n = \frac{n+3}{3(n+1)}, \quad n \geq 1. \tag{5.4}$$

If (5.1) holds for some $\sigma \geq 1$, then by Theorem 2,

$$q = 1 - \Big(\frac{3}{1+2\sigma}\Big)^{\frac{1}{\theta}}, \quad \mathrm{E}(Z_n) \to 3^{\frac{1}{\theta}},$$

and $Z_n \to Z_\infty$ almost surely, with

$$\mathrm{E}(Z_\infty) = 3^{\frac{1}{\theta}}, \qquad \mathrm{E}(s^{Z_\infty}) = 1 - 3^{\frac{1}{\theta}}(2\sigma + (1-s)^{-\theta})^{-\frac{1}{\theta}}, \quad 0 \leq s \leq 1.$$

Example 3. Suppose $\theta \in (0, 1]$ and $r = 1$. Let

$$a_1 = \frac{1}{2}, \qquad a_{2n} = 4, \quad a_{2n+1} = \frac{1}{4},$$
$$c_{2n-1} = 1, \qquad c_{2n} = 2,$$
$$A_{2n-1} = \frac{1}{2}, \qquad A_{2n} = 2, \quad n \geq 1.$$

Then $C = \infty$ and $B_n \to \infty$ implying the conditions of Theorem 3. Observe that for this example, $\lim A_n$ does not exist.

Example 4. Suppose $\theta \in (0,1]$ and $r = 1$. Recall that Theorem 4 is the only one among Theorems 1–5 which may hold both with $C < \infty$ and $C = \infty$. For this reason, we present two examples (1) and (2) for each of these two situations:

(1) Let

$$a_n = \frac{n+1}{n}, \quad c_n = \frac{1}{n^2(n+1)}, \quad n \geq 1,$$

implying

$$A_n = n+1, \quad C_n = \frac{n}{n+1}, \quad B_n = \frac{n}{(n+1)^2}, \quad n \geq 1.$$

In this case, according to Theorem 4,

$$\mathrm{P}(Z_n > 0) \sim n^{-\frac{1}{\theta}}, \quad \mathrm{E}(Z_n | Z_n > 0) \to 1,$$

and

$$\mathrm{E}(s^{Z_n} | Z_n > 0) \to s, \quad 0 \leq s \leq 1.$$

(2) Let

$$a_n = \frac{n+1}{n}, \quad A_n = n+1, \quad n \geq 1,$$

and (5.2) hold for some $\sigma > 0$. Then

$$C_n = \sigma n, \quad B_n = \frac{\sigma n}{n+1}, \quad n \geq 1.$$

In this case, according to Theorem 4,

$$\mathrm{P}(Z_n > 0) \sim (1+\sigma)^{-\frac{1}{\theta}} n^{-\frac{1}{\theta}}, \quad \mathrm{E}(Z_n | Z_n > 0) \to (1+\sigma)^{\frac{1}{\theta}},$$

and

$$\mathrm{E}(s^{Z_n} | Z_n > 0) \to 1 - ((1+\sigma)(1-s)^{-\theta} + \sigma + \sigma^2)^{-\frac{1}{\theta}}, \quad 0 \leq s \leq 1.$$

Example 5. Suppose $\theta \in (0,1]$ and $r = 1$. Let

$$a_n = \begin{cases} n & \text{for } n = 2^k - 1,\ k \geq 1, \\ \frac{1}{n-1} & \text{for } n = 2^k,\ k \geq 1, \\ 1 & \text{otherwise,} \end{cases} \qquad A_n = \begin{cases} n & \text{for } n = 2^k - 1,\ k \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Taking

$$c_n = \begin{cases} 1 & \text{for } n = 2^k,\ k > 1, \\ \frac{1}{n^2} & \text{otherwise,} \end{cases}$$

we get

$$C_n = \sum_{k:\, 2 \leq 2^k \leq n} (2^k - 1 - 2^{-2k}) + \sum_{k=1}^{n} k^{-2}, \quad n \geq 1,$$

implying $C_{k_n} \sim 2^{n+1}$, provided $2^n - 1 \leq k_n < 2^{n+1} - 1$. Thus, by Theorem 5, for $k_n = 2^n$, $\lambda_n = \lambda(2n)^{-\frac{1}{\theta}}$,

$$\mathrm{P}(Z_{k_n} > 0) \sim (2k_n)^{-\frac{1}{\theta}}, \qquad \mathrm{E}(e^{-\lambda_{k_n} Z_{k_n}} | Z_{k_n} > 0) \to 1 - (1 + \lambda^{-\theta})^{-\frac{1}{\theta}}, \quad \lambda \geq 0,$$

and on the other hand, for $k_n = 2^n - 1$,

$$\mathrm{P}(Z_{k_n} > 0) \sim (3k_n)^{-\frac{1}{\theta}}, \qquad \mathrm{E}(s^{Z_{k_n}} | Z_{k_n} > 0) \to 1 - (3(1-s)^{-\theta} + 6)^{-\frac{1}{\theta}}, \quad 0 \leq s \leq 1.$$

Example 6. Suppose $\theta = 0$, $r = 1$, and assume $c_n = 1 - e^{-n^{\sigma}}$, $-\infty < \sigma < \infty$, $n \geq 1$, yielding

$$D_n = \exp\bigg(-\sum_{i=1}^{n} i^{\sigma}(A_{i-1} - A_i)\bigg), \quad n \geq 1.$$

Notice that (5.3) implies $A = 0$ and

$$D_n = \exp\Big(-\sum_{i=1}^{n} \frac{i^{\sigma-1}}{i+1}\Big), \quad n \geq 1,$$

on the other hand, (5.4) implies $A = \frac{1}{3}$ and

$$D_n = \exp\Big(-\sum_{i=1}^{n} \frac{2i^{\sigma-1}}{3(i+1)}\Big), \quad n \geq 1.$$

(i) If (5.3) holds and $\sigma \geq 1$, then

$$A_n \sim n^{-1}, \quad D_n = \exp\Big(-\sum_{i=1}^{n} \frac{i^{\sigma-1}}{i+1}\Big) \to 0,$$

so that the conditions of Theorem 6 (i) are satisfied.

(ii) If (5.3) holds and $\sigma < 1$, then

$$A_n \sim n^{-1}, \quad D = \exp\Big(-\sum_{i=1}^{\infty} \frac{i^{\sigma-1}}{i+1}\Big),$$

so that the conditions of Theorem 6 (ii) are satisfied.

(iii) If (5.4) holds and $\sigma \geq 1$, then

$$A = \frac{1}{3}, \quad D_n = \exp\Big(-\sum_{i=1}^{n} \frac{2i^{\sigma-1}}{3(i+1)}\Big) \to 0,$$

so that the conditions of Theorem 6 (iii) are satisfied.

(iv) If (5.4) holds and $\sigma < 1$, then

$$A = \frac{1}{3}, \quad D = \exp\Big(-\sum_{i=1}^{\infty} \frac{2i^{\sigma-1}}{3(i+1)}\Big),$$

so that the conditions of Theorem 6 (iv) are satisfied.

Example 7. Suppose $\theta \in (0,1]$, $r > 1$ assuming (5.1) with $r^{-\theta} \leq \sigma \leq (r-1)^{-\theta}$.

(i) If (5.3), then the conditions of Theorem 7 (i) hold with $A_n \sim n^{-1}$ and $C = \sigma$.

(ii) If (5.4), then the conditions of Theorem 7 (ii) hold with $A = \frac{1}{3}$ and $C = \frac{2\sigma}{3}$.

Example 8. Suppose $\theta \in (-1,0)$, $r > 1$ assuming (5.1) with $r - 1 \leq \sigma^{\alpha} \leq r$, where $\alpha = -\frac{1}{\theta}$.

(i) If (5.3), then the conditions of Theorem 8 (i) hold with $A_n \sim n^{-1}$ and $C = \sigma$.

(ii) If (5.4), then the conditions of Theorem 8 (ii) hold with $A = \frac{1}{3}$ and $C = \frac{2\sigma}{3}$.

Example 9. Suppose $\theta = 0$ and $r > 1$ and assume

$$c_n = \sigma, \quad 0 \leq \sigma \leq 1, \quad n \geq 1,$$

which implies

$$D_n = (r-\sigma)^{1-A_n}, \quad n \geq 1.$$

(i) If (5.3), then by Theorem 9 (i), we get in particular,

$$\mathrm{P}(\tau > n) \sim \gamma n^{-1}, \quad \gamma = (r-\sigma)\ln\frac{r}{r-1}.$$

(ii) If (5.3), then by Theorem 9 (ii), we get in particular,

$$q = r - r^{\frac{1}{3}}(r-\sigma)^{\frac{2}{3}}, \quad q_\Delta = 1 - r + (r-1)^{\frac{1}{3}}(r-\sigma)^{\frac{2}{3}}, \quad Q = 1 - (r^{\frac{1}{3}} - (r-1)^{\frac{1}{3}})(r-\sigma)^{\frac{2}{3}}.$$

Example 10. Suppose $\theta \in (-1,0)$, $r = 1$. Put $\alpha = -\frac{1}{\theta}$ and assume (5.1) with $0 < \sigma \leq 1$.

(i) If (5.3), then by Theorem 10 (i), we get in particular, $q_\Delta = \sigma^{\alpha}$ and

$$\mathrm{P}(\tau > n) \sim \alpha\sigma^{\alpha-1}n^{-1}.$$

(ii) If (5.4), then by Theorem 10 (ii), we get in particular, $Q = 1 - (\frac{1}{3} + \frac{2\sigma}{3})^{\alpha} + \frac{2\sigma}{3}^{\alpha}$.

6 Proofs

In this section we sketch the proofs of lemmas and theorems of this paper. The corollaries to Theorems 7–9 are easily obtained from the corresponding theorems.

Proof of Lemma 1

Relations (2.1) and (2.2) imply respectively

$$(r - f_k \circ f_{k+1}(s))^{-\theta} = a_k(r - f_{k+1}(s))^{-\theta} + c_k = a_k a_{k+1}(r - s)^{-\theta} + c_k + a_k c_{k+1},$$

and

$$r - f_k \circ f_{k+1}(s) = (r - c_k)^{1-a_k}(r - f_{k+1}(s))^{a_k} = (r - c_k)^{1-a_k}(r - c_{k+1})^{(1-a_{k+1})a_k}(r - s)^{a_k a_{k+1}},$$

entailing the main claims of Lemma 1. Parts (a)–(f) follow from the respective restrictions (a)–(f) on (a_n, c_n) stated in Definition 1.

Proof of Lemma 2

(a) In the case $\theta \in (0, 1]$, $r = 1$, the claim follows from the existence of $\lim C_n$ and $\lim(A_n + C_n)$, which in turn, follows from monotonicity of the two sequences. To see that $A_n + C_n \le A_{n+1} + C_{n+1}$, it suffices to observe that

$$A_n - A_{n+1} = A_n(1 - a_{n+1}) \le A_n c_{n+1} = C_{n+1} - C_n.$$

The second part of Lemma 2 is a direct implication of the definition of C_n.

(b)–(f) The rest of the stated results follows immediately from the restrictions (b)–(f) imposed on (a_n, c_n) in Definition 1.

Church–Lindvall Condition for the GW$^\theta$-Process

In [8] it was shown for the GW-processes in a varying environment that the almost surely convergence $Z_n \xrightarrow{\text{a.s.}} Z_\infty$ holds with $\mathrm{P}(0 < Z_\infty < \infty) > 0$ if and only if the following condition holds:

$$\sum_{n\ge1}(1 - p_n(1)) < \infty. \tag{6.1}$$

Relation (6.1) is equivalent to

$$\prod_{n\ge n_0} p_n(1) > 0 \tag{6.2}$$

for some $n_0 \ge 1$. For the GW$^\theta$-process, the equality $p_n(1) = f_n'(0)$ implies

$$p_n(1) = a_n(a_n + c_n r^\theta)^{-\frac{1}{\theta}-1} \tag{6.3}$$

for $\theta \ne 0$, and for $\theta = 0$,

$$p_n(1) = a_n(1 - c_n r^{-1})^{1-a_n}. \tag{6.4}$$

Lemma 3. *In the case $\theta \in (0, 1]$ and $r = 1$, relation* (6.1) *holds if and only if*

$$A_n \to A \in (0, \infty) \tag{6.5}$$

and

$$\sum_{n\ge1} c_n < \infty. \tag{6.6}$$

Proof. In view of (6.3), we have

$$\prod_{i=1}^{n} p_i(1) = A_n G_n^{-\frac{1}{\theta}-1}, \quad G_n := \prod_{i=1}^{n}(a_i + c_i).$$

Since $a_n + c_n \geq 1$, we have

$$\lim G_n = G \in [1, \infty].$$

If $G = \infty$, then (6.2) is not valid, implying that (6.1) is equivalent to (6.5) plus $G < \infty$. It remains to verify that under (6.5), the inequality $G < \infty$ is equivalent to (6.6). Suppose (6.5) holds, and observe that in this case, $G < \infty$ is equivalent to

$$\prod_{n\geq 1}\left(1 + \frac{c_n}{a_n}\right) < \infty,$$

which is true if and only if

$$\sum_{n\geq 1} \frac{c_n}{a_n} < \infty.$$

Since under (6.5), $a_n \to 1$, the latter condition is equivalent to (6.6). □

Lemma 4. *In the case $\theta = 0$ and $r = 1$, relation* (6.1) *holds if and only if $A \in (0, 1)$ and $D \in (0, 1)$.*

Proof. In view of (6.4), we have

$$\prod_{n\geq 1} p_n(1) = A \prod_{n\geq 1}(1 - c_n)^{1-a_n}.$$

It remains to observe that given $A \in (0, 1)$ the relation $D \in (0, 1)$ is equivalent to

$$\prod_{n\geq 1}(1 - c_n)^{1-a_n} > 0.$$ □

Lemma 5. *Assume that $\theta \neq 0$ and $r > 1$, and consider $\{\tilde{Z}_n\}$, a GW-process in a varying environment with the proper probability generating functions*

$$\tilde{f}_n(s) = \frac{f_n(s)}{f_n(1)} = \frac{r - (a_n(r - s)^{-\theta} + c_n)^{-\frac{1}{\theta}}}{r - (a_n(r - 1)^{-\theta} + c_n)^{-\frac{1}{\theta}}}.$$

Relation (3.1) *implies*

$$\sum_{n=1}^{\infty}(1 - \tilde{p}_n(1)) < \infty.$$

Proof. Assume $\theta \in (0, 1]$ and $r > 1$ together with (3.1). Then $A_n \to A \in (0, 1)$, $a_n \to 1$, and $c_n \to 0$. We have

$$\tilde{p}_n(1) = \tilde{f}_n'(0) = a_n h_n^{-\frac{1}{\theta}-1} k_n^{-1},$$

where

$$h_n = a_n + c_n r^{\theta}, \quad k_n = r - (a_n(r - 1)^{-\theta} + c_n)^{-\frac{1}{\theta}}$$

are such that $h_n \geq 1$ and $k_n \in (0, 1]$. The statement follows from the representation

$$\prod_{n\geq 1} p_n(1) = AH^{-\frac{1}{\theta}-1}K^{-1},$$

where $H = \prod_{n\geq 1} h_n$ and $K = \prod_{n\geq 1} k_n$. It is easy to show that (3.1) and $(1 - a_n)r^{-\theta} \leq c_n \leq (1 - a_n)(r - 1)^{-\theta}$ yield

$$\sum_{n\geq 1}(h_n - 1) \leq r^{\theta} \sum_{n\geq 1} c_n \leq r^{\theta}(r - 1)^{-\theta} \sum_{n\geq 1}(1 - a_n) < \infty,$$

implying $H \in [1, \infty)$. On the other hand, $K \in (0, 1]$, since

$$\sum_{n\geq 1}(1 - k_n) < \infty,$$

which follows from

$$1 - k_n \leq (r - 1)(a_n + c_n(r - 1)^{\theta})^{-\frac{1}{\theta}} - 1) \leq r(1 - (a_n + c_n(r - 1)^{\theta})^{\frac{1}{\theta}}) \leq r\theta^{-1}(1 - a_n).$$

In the other case, when (3.1) holds together with $\theta \in (-1, 0)$ and $r > 1$, the lemma is proven similarly. □

Proof of Theorems 1–5

The proofs of these theorems are done using the usual for these kind of results arguments applied to the explicit expressions available for $F_n(s)$. In particular, the following standard formula is a starting point for computing the conditional limit distributions:

$$\mathrm{E}(s^{Z_n}|Z_n>0)=\frac{\mathrm{E}(s^{Z_n})-\mathrm{P}(Z_n=0)}{\mathrm{P}(Z_n>0)}=1-\frac{1-F_n(s)}{1-F_n(0)}. \tag{6.7}$$

Thus in the case $\theta\in(0,1]$ and $r>1$, Lemma 1 and (6.7) imply

$$\mathrm{E}(s^{Z_n}|Z_n>0)=1-\frac{((1-s)^{-\theta}+B_n)^{-\frac{1}{\theta}}}{(1+B_n)^{-\frac{1}{\theta}}}\to 1-\frac{((1-s)^{-\theta}+B)^{-\frac{1}{\theta}}}{(1+B)^{-\frac{1}{\theta}}},$$

proving the main statement of Theorem 4. The almost sure convergence stated in Theorem 2 follows from Lemma 3 and the earlier cited criterium of [8].

Proof of Theorem 6

Suppose $\theta=0$, $r=1$, in which case $A\in[0,1)$ and $D\in[0,1]$.

(i) Suppose $A=D=0$. In this case $q=1-D=1$, and by (6.7) and Lemma 1,

$$\mathrm{E}(s^{Z_n}|Z_n>0)=1-(1-s)^{A_n}.$$

Putting here $s_n=\exp(-\lambda e^{-\frac{x}{A_n}})$, we get as $n\to\infty$,

$$\mathrm{E}(s_n^{Z_n}|Z_n>0)=1-(1-\exp(-\lambda e^{-\frac{x}{A_n}}))^{A_n}=1-\exp(A_n\ln(\lambda e^{-\frac{x}{A_n}}(1+o(1)))\to 1-e^{-x}.$$

This implies a convergence in distribution

$$(Z_ne^{-\frac{x}{A_n}}|Z_n>0)\xrightarrow{\mathrm{d}} W(x),$$

where the limit $W(x)$ has a degenerate distribution with

$$\mathrm{P}(W(x)\le w)=(1-e^{-x})1_{\{0\le w<\infty\}}.$$

In other words,

$$\mathrm{P}(Z_n\le we^{\frac{x}{A_n}}|Z_n>0)\to(1-e^{-x})1_{\{0\le w<\infty\}}.$$

After taking the logarithm of Z_n, we arrive at the statement of Theorem 6 (i).

(ii) Statement (ii) follows from Lemma 1 and relation (6.7) in a similar way as statement (i).

(iii) If $A\in(0,1)$ and $D=0$, then $q=1$ and by relation (6.7) and Lemma 1,

$$\mathrm{E}(s^{Z_n}|Z_n>0)=1-(1-s)^{A_n}\to 1-(1-s)^{A}.$$

(iv) Let $A>0$ and $D>0$. Since $q=1-D$, similarly to part (iii), we obtain

$$\mathrm{E}(s^{Z_n})\to 1-(1-s)^{A}D.$$

By Lemma 4, the convergence in distribution $Z_n\xrightarrow{\mathrm{d}}Z_\infty$ can be upgraded to the almost surely convergence $Z_n\xrightarrow{\mathrm{a.s.}}Z_\infty$.

Proof of Theorems 7 and 8

In this section we prove only Theorem 7. Theorem 8 is proven similarly.

By Lemma 1,

$$F_n(0)=r-(A_nr^{-\theta}+C_n)^{-\frac{1}{\theta}},\quad F_n(1)=r-(A_n(r-1)^{-\theta}+C_n)^{-\frac{1}{\theta}}.$$

It follows that

$$P(\tau > n) = (A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n(r-1)^{-\theta} + C_n)^{-\frac{1}{\theta}},$$

$$E(Z_n|\tau > n) = \frac{F_n'(1)}{F_n(1) - F_n(0)} = \frac{\theta^{-1}A_n(A_n + C_n(r-1)^{\theta})^{-\frac{1}{\theta}-1}}{(A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n(r-1)^{-\theta} + C_n)^{-\frac{1}{\theta}}},$$

$$E(s^{Z_n}|\tau > n) = \frac{F_n(s) - F_n(0)}{F_n(1) - F_n(0)} = \frac{(A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n(r-s)^{-\theta} + C_n)^{-\frac{1}{\theta}}}{(A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n(r-1)^{-\theta} + C_n)^{-\frac{1}{\theta}}}.$$

(i) Assume that $A = 0$. Then the sequence of positive numbers

$$V_n = A_n^{-1}(C - C_n) = c_{n+1} + c_{n+2}a_{n+1} + c_{n+3}a_{n+2}a_{n+1} + \cdots$$

satisfies

$$r^{-\theta} \le \liminf V_n \le \limsup V_n \le (r-1)^{-\theta}.$$

For a given $x \in (0, \infty)$, put

$$W_n(x) = A_n^{-1}(C^{-\frac{1}{\theta}} - (A_n x + C_n)^{-\frac{1}{\theta}}).$$

Since

$$W_n(x) = A_n^{-1}(C^{-\frac{1}{\theta}} - (A_n(x - V_n) + C)^{-\frac{1}{\theta}}) = \theta^{-1}C^{-\frac{1}{\theta}-1}(x - V_n + o(1)),$$

the representation

$$A_n^{-1}P(\tau > n) = W_n((r-1)^{-\theta}) - W_n(r^{-\theta})$$

yields the first asymptotic result stated in part (i) of Theorem 7. The other two asymptotic results follow from the representations

$$E(Z_n|\tau > n) = \frac{\theta^{-1}(A_n + C_n(r-1)^{\theta})^{-\frac{1}{\theta}-1}}{W_n((r-1)^{-\theta}) - W_n(r^{-\theta})},$$

$$E(s^{Z_n}|\tau > n) = \frac{W_n((r-s)^{-\theta}) - W_n(r^{-\theta})}{W_n((r-1)^{-\theta}) - W_n(r^{-\theta})}.$$

(ii) The second claim follows from the equality

$$E(s^{Z_n}; \tau_\Delta > n) = r - (A_n(r-s)^{-\theta} + C_n)^{-\frac{1}{\theta}}.$$

Proof of Theorem 9

If $\theta = 0$ and $r > 1$, then by Lemma 1

$$P(Z_n = 0) = r - r^{A_n}D_n, \quad P(Z_n \neq \Delta) = r - (r-1)^{A_n}D_n.$$

It follows that

$$q = r - r^A D, \quad q_\Delta = 1 - r + (r-1)^A D, \quad Q = 1 - (r^A - (r-1)^A)D.$$

(i) If $A = 0$, then clearly

$$q = r - D, \quad q_\Delta = 1 - r + D, \quad Q = 1,$$

and

$$P(\tau > n) = (r^{A_n} - (r-1)^{A_n})D_n \sim (\ln r - \ln(r-1))A_n D_n.$$

Furthermore,

$$E(Z_n|\tau > n) = \frac{F_n'(1)}{F_n(1) - F_n(0)} = \frac{A_n(r-1)^{A_n-1}}{r^{A_n} - (r-1)^{A_n}} \to \frac{(r-1)^{-1}}{\ln r - \ln(r-1)},$$

$$P(s^{Z_n}|\tau > n) = \frac{F_n(s) - F_n(0)}{F_n(1) - F_n(0)} = \frac{r^{A_n} - (r-s)^{A_n}}{r^{A_n} - (r-1)^{A_n}} \to \frac{\ln r - \ln(r-s)}{\ln r - \ln(r-1)}.$$

(ii) In the case $A > 0$, the main claim is obtained as

$$E(s^{Z_n}; \tau_\Delta > n) = r - (r-s)^{A_n}D_n \to r - (r-s)^A D.$$

Proof of Theorem 10

If $\theta \in (-1, 0)$ and $r = 1$, then by Lemmas 1 and 2,

$$F_n(0) = 1 - (A_n + C_n)^\alpha, \quad F_n(1) = 1 - C_n^\alpha$$

and

$$q = 1 - (A + C)^\alpha, \quad q_\Delta = C^\alpha,$$

where $\alpha = -\frac{1}{\theta}$ and $0 < C \leq 1 - A$.

(i) Suppose $A = 0$. Then the sequence of positive numbers $V_n = A_n^{-1}(C - C_n)$ satisfies

$$0 \leq \liminf V_n \leq \limsup V_n \leq 1.$$

For a given $x \in (0, \infty)$, put

$$W_n(x) = A_n^{-1}((A_n x + C_n)^\alpha - C^\alpha).$$

Since

$$W_n(x) = \alpha C^{\alpha-1}(x - V_n + o(1)),$$

the representation

$$A_n^{-1}\mathrm{P}(\tau > n) = W_n(1) - W_n(0)$$

yields the first asymptotic result stated in part (i) of Theorem 10. The other asymptotic result follows from the representation

$$\mathrm{E}(s^{Z_n} | \tau > n) = \frac{W_n(1) - W_n((1-s)^{\frac{1}{\alpha}})}{W_n(1) - W_n(0)}.$$

(ii) Claim (ii) is derived as

$$\mathrm{P}(s^{Z_n}; \tau > n) = 1 - (A_n(1-s)^{\frac{1}{\alpha}} + C_n)^\alpha \to 1 - (A(1-s)^{\frac{1}{\alpha}} + C)^\alpha.$$

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