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Wiener–Beurling spaces and their properties ☆

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In this paper, we introduce the Wiener–Beurling space, which generalize the Wiener space of absolutely convergent Fourier series, the Berling space as well as the Cesàro and Copson spaces. We investigate their main properties: embeddings, duality, and interpolation. We also discuss the product and convolution operators as well as various average operators in the Wiener–Beurling spaces.

1. Introduction

The Wiener algebra $A = A^1$ of absolutely convergent Fourier series on the circle $\mathbb{T}^1$ (see, e.g., [15]) and its ℓ^β-versions, the β-Wiener spaces A^β, are defined as follows

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$$A^{\beta}=A^{\beta}(\mathbb{T}^1)=\left\{f\in L^1(\mathbb{T}^1):\ \|f\|_{A^{\beta}(\mathbb{T}^1)}=\left(\sum_{j=-\infty}^{\infty}|\widehat{f}(j)|^{\beta}\right)^{1/\beta}<\infty\right\}.$$

Beurling [10] introduced the spaces A^*, nowadays called the Beurling space, to study contraction properties of functions. By definition,

$$A^*(\mathbb{T}^1)=\left\{f\in L^1(\mathbb{T}^1):\ \|f\|_{A^*(\mathbb{T}^1)}=\sum_{j=0}^{\infty}\sup_{j\leqslant|k|}|\widehat{f}(k)|<\infty\right\}.$$

Important properties of the Wiener and Beurling spaces and their comparison can be found in [5,15]. In the case of function on $\mathbb{R}$ the Wiener algebra was studied in [20].

Recently [22], the β-Beurling spaces have been introduced:

$$A^{*,\beta}=A^{*,\beta}(\mathbb{T}^1)=\left\{f\in L^1(\mathbb{T}^1):\ \|f\|_{A^{*,\beta}(\mathbb{T}^1)}=\left(\sum_{s=0}^{\infty}2^s\sup_{2^s\leqslant|k|}|\widehat{f}(k)|^{\beta}\right)^{1/\beta}<\infty\right\}.$$

It is easy to see that $A^{*,1}=A^*$. Both Wiener and Beurling function spaces play an important role in harmonic analysis, in particular, in the summability theory and in the Fourier synthesis (see, e.g., the texts [8], [28, Theorems 1.25 and 1.16], and [32, Theorem 8.1.3, Ch. 6]).

The main goal of this paper is to introduce a more general scale of spaces, which we call the Wiener–Beurling spaces, and investigate their basic properties.

As usual, we will write $F_1\asymp F_2$ if $c^{-1}F_1\leqslant F_2\leqslant cF_1$ for some positive constant c independent of essential parameters. Moreover, $F_1\lesssim F_2$ stands for $F_1\leqslant cF_2$. We will assume that $\frac{1}{p}+\frac{1}{p'}=1$ for any $p\in(0,\infty]$. By $[a]$ we denote the integer part of a.

Definition 1.1. Let $\alpha\in\mathbb{R}$ and $0<p,q\leqslant\infty$. We define the space $WB^{\alpha}_{p,q}$ to be the collection of formal trigonometric series (not necessarily the Fourier series)

$$f(x)=\sum_{m\in\mathbb{Z}}c_me^{imx},\qquad x\in\mathbb{T},$$

such that

$$\|f\|_{WB^{\alpha}_{p,q}}=\Big(\sum_{k=0}^{\infty}\Big[2^{\alpha k}\Big(\sum_{[2^{k-1}]\leqslant|m|<2^k}|c_m|^p\Big)^{1/p}\Big]^q\Big)^{1/q}<\infty,\tag{1.1}$$

with the usual modification for $p=\infty$ or/and $q=\infty$. Moreover, $\|f\|_{WB^{\alpha}_{p,q}}$ is a (quasi-)norm (a norm for $1\leqslant p,q\leqslant\infty$).

Several remarks are in order.

(i) A similar definition (with specific values of the parameters α, p, and q) has first appeared in the paper by D. Borwein [11] devoted to summability problems. Various applications related to some special cases of Wiener–Beurling norm can be found in, e.g., [4,8,14,16,17,19,26,29].

(ii) Clearly, $\|f\|_{WB^{\alpha}_{p,q}}$ can be equivalently defined by

$$\Big(\sum_{k=0}^{\infty}\big[\gamma_k^{\alpha}\big(\sum_{[\gamma_k]\leqslant|m|<\gamma_{k+1}}|c_m|^p\big)^{1/p}\big]^q\Big)^{1/q},$$

where $0<\gamma_0<1$ such that $1<\gamma<\gamma_{k+1}/\gamma_k<\gamma^*<\infty$.

(iii) The parameter α in definition (1.1) can be seen as the smoothness parameter since $f\in WB^{\alpha+\beta}_{p,q}$ with $f(x)=\sum_{m\in\mathbb{Z}}c_me^{imx}$ if and only if $f^{(\beta)}\in WB^{\alpha}_{p,q}$, where $f^{(\beta)}$ is the Weyl fractional derivative of f [33, Ch. XII, §8].

Examples. It is clear that both Wiener and Beurling spaces are particular cases of the Wiener–Beurling spaces, namely, $WB^{0}_{\beta,\beta}=A^{\beta}$ and $WB^{1/\beta}_{\infty,\beta}=A^{*,\beta}$. The latter follows from the fact that

$$\|f\|_{A^{*,\beta}}\asymp\left(\sum_{s=0}^{\infty}2^s\sup_{2^s\leqslant|k|<2^{s+1}}|c_k|^{\beta}\right)^{1/\beta}.$$

It is also easy to see that, for $1<q\leqslant\infty$, we have

$$\|f\|_{WB^{\frac{1}{q}-1}_{1,q}}\asymp\|\{c_n\}\|_{ces(q)},$$

where the discrete Cesàro spaces $ces(q)$ is given by

$$ces(q)=\Big\{\{c_m\}_{m=0}^{\infty},c_m\in\mathbb{R}:\|\{c_n\}\|_{ces(q)}=\Big(\sum_{s=0}^{\infty}\Big[\frac{1}{2s+1}\sum_{m=-s}^{s}|c_m|\Big]^q\Big)^{1/q}<\infty\Big\}.$$

Indeed, using for $\alpha<0$ Hardy's inequality

$$\sum_{k=0}^{\infty}2^{\alpha k}\Big(\sum_{m=0}^{k}A_m\Big)^{\gamma}\lesssim\sum_{k=0}^{\infty}2^{\alpha k}A_k^{\gamma},\qquad A_m\geqslant0,\quad\gamma>0,\tag{1.2}$$

gives

$$\|f\|_{WB^{\alpha}_{p,q}}\asymp\Big(\sum_{k=0}^{\infty}2^{\alpha kq}\big(\sum_{0\leqslant|m|<2^k}|c_m|^p\big)^{q/p}\Big)^{1/q}.\tag{1.3}$$

For the case $p=1$ and $\alpha=\frac{1}{q}-1<0$ we then have

$$\|f\|_{WB^{\alpha}_{p,q}} \asymp \Big(\sum_{k=0}^{\infty}\big(\frac{1}{k+1}\sum_{0\leqslant|m|<k}|c_m|\big)^q\Big)^{1/q} \asymp \|\{c_n\}\|_{ces(q)}.$$

Similarly, one can show that for any $0<q<\infty$ we have

$$\|f\|_{WB^{\frac{1}{q}-1}_{1,q}} = \|\{c_n\}\|_{cop(q)},$$

where the discrete Copson spaces $cop(q)$ is given by

$$cop(q) = \Big\{\{c_m\}_{m=0}^{\infty}, c_m\in\mathbb{R} : \Big(\sum_{s=0}^{\infty}\Big[\sum_{m=s}^{\infty}\frac{|c_m|+|c_{-m}|}{m}\Big]^q\Big)^{1/q}<\infty\Big\}.$$

In particular, this immediately implies the following important fact due to Bennett [8, (6.8)]: $ces(q)=cop(q)$ for $1<q<\infty$.

For the survey of Cesàro and Copson spaces see [4].

Structure of the paper. Section 2 consists of the key embeddings between Wiener–Beurling spaces with various parameters and Lebesgue and Besov spaces. In Section 3 we prove that the Köthe dual space of $WB^{\alpha}_{p,q}$ is $WB^{-\alpha}_{p',q'}$ for $\alpha\in\mathbb{R}$ and $1\leq p,q\leqslant\infty$. This, in particular, extends the results on duality of Cesaro space obtained by Bennett. As a corollary, we derive that $(A^{*,\beta})' = WB^{-1/\beta}_{1,\beta'}$, cf. [5, Prop. 2] for the case $\beta=1$.

Section 4 studies the convolution and product operator for functions from Wiener–Beurling spaces. We claim that $WB^{\alpha}_{p,1}$ is an algebra for $1\leqslant p\leqslant\infty$ and $\alpha\geqslant\frac{1}{p'}$, significantly extending the results by Belinskii, Liflyand and Trigub [5].

In Section 5 two interpolation theorems of Wiener–Beurling spaces are obtained. We prove that $(WB^{\alpha_0}_{p,q_0}, WB^{\alpha_1}_{p,q_1})_{\theta,q} = WB^{\alpha}_{p,q}$ for $0<p,q,q_0,q_1\leqslant\infty$, $-\infty<\alpha_1<\alpha_0<\infty$ and $\alpha=(1-\theta)\alpha_0+\theta\alpha_1$, $0<\theta<1$. In the diagonal case, i.e., $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, we derive that $(WB^{\alpha_0}_{p_0,q_0}, WB^{\alpha_1}_{p_1,q_1})_{\theta,q} = WB^{\alpha}_{q,q}$ for $0<p_0,p_1\leq\infty$, $0<q,q_0,q_1<\infty$, and $\alpha=(1-\theta)\alpha_0+\theta\alpha_1$, $0<\theta<1$. In particular, we obtain the interpolation properties of Cesaro and Copson spaces, investigated recently by Astashkin and Maligranda [3,4], as well as properties of the $A_{p,q}$ spaces studied by Lerner and Liflyand [17]. Recall that the $\|\{c_m\}\|_{A_{p,q}}$ norm is given by $\Big(\sum_{k=0}^{\infty}\big(\frac{1}{k+1}\sum_{k\leqslant|m|<2k}|c_m|^p\big)^{q/p}\Big)^{1/q}$.

Finally, in Section 6 we obtain Bernstein and Nikol'skii type polynomial inequalities in Wiener–Beurling spaces. Moreover, we study the Fourier series with averages of Fourier coefficients substantially improving the remarkable result by Bennett [8, Theorem 20.31] for Cesáro and Copson spaces.

2. Embedding properties of Wiener–Beurling spaces

2.1. Embeddings between Wiener–Beurling and Lebesgue spaces

We start with the simple embeddings, which follow from the properties of l^p spaces and Hölder's inequality. *Let* $\alpha\in\mathbb{R}$, $\varepsilon>0$, $0<p\leqslant\infty$, *and* $0<q,q_1\leqslant\infty$. *Then*

(1) $WB^{\alpha}_{p,q} \hookrightarrow WB^{\alpha}_{p,q+\varepsilon}$,
(2) $WB^{\alpha}_{p,q} \hookrightarrow WB^{\alpha+\varepsilon}_{p,q_1}$, $0 < p \leqslant \infty$, $0 < q$, $q_1 \leqslant \infty$,
(3) $WB^{\alpha}_{p,q} \hookrightarrow WB^{\alpha}_{p+\varepsilon,q}$,
(4) *if* $-\infty < \alpha_1 < \alpha_2 < \infty$ *and* $\alpha_1 + \frac{1}{p_1} = \alpha_2 + \frac{1}{p_2}$, *then*

$$WB^{\alpha_2}_{p_2,q} \hookrightarrow WB^{\alpha_1}_{p_1,q}. \tag{2.1}$$

The latter implies that $WB^{\alpha}_{p,\beta} \hookrightarrow A^{\beta}$ with $\frac{1}{\beta} = \alpha + \frac{1}{p}$.

Let us now recall Pitt's inequality for Fourier coefficients [24,27]:

$$\|f\|_{WB^{-\beta}_{q,q}} \asymp \left(\sum_{n\in\mathbb{Z}} \Big((|n|+1)^{-\beta} |\widehat{f}_n| \Big)^q \right)^{1/q} \leqslant C \left(\int_{\mathbb{T}} \Big(|x|^{\alpha} f(x) \Big)^p dx \right)^{1/p}$$

provided

$$1 < p \leqslant q < \infty, \qquad 0 \leqslant \alpha < \frac{1}{p'}, \qquad \beta = \alpha + \frac{1}{p} + \frac{1}{q} - 1 \geqslant 0.$$

The reverse inequality is given as follows: suppose

$$1 < p \leqslant q < \infty, \qquad 0 \leqslant \alpha < \frac{1}{q}, \qquad \beta = \alpha - \frac{1}{p} - \frac{1}{q} + 1 \geqslant 0,$$

then

$$\left(\int_{\mathbb{T}} \Big(|x|^{-\alpha} |f(x)| \Big)^q dx \right)^{1/q} \leq C \left(\sum_{n\in\mathbb{Z}} \Big((|n|+1)^{\beta} |\widehat{f}_n| \Big)^p \right)^{1/p} \asymp \|f\|_{WB^{\beta}_{p,p}}.$$

In particular, we derive that $L_p \hookrightarrow WB^{\frac{1}{p'}-\frac{1}{p}}_{p,p}$ for $1 < p \leqslant 2$ and $WB^{\frac{1}{p'}-\frac{1}{p}}_{p,p} \hookrightarrow L_p$ for $2 \leq p < \infty$.

However, the latter two embeddings can be strengthened using the results of Kellogg [16]. Namely, we have

(5) *if* $1 < p \leqslant 2$, *then*

$$L_p \hookrightarrow WB^{0}_{p',2},$$

(6) *if* $p \geqslant 2$, *then*

$$WB^{0}_{p',2} \hookrightarrow L_p.$$

The next theorem discusses the well-known Wiener's problem on functions with positive Fourier coefficients. Recall that a space of functions X is called *solid* if it satisfies the following property: For every $f = \sum c_n e^{in\theta}$ in X, if another function $g = \sum d_n e^{in\theta}$ satisfies $|d_n| \le |c_n|$ for every n, then g is also in X.

Theorem 2.1. *[2] Let X be a space of functions. If $L^p \hookrightarrow X$ and X is solid, then*

$$L^p_{loc+} \hookrightarrow X,$$

where

$$L^p_{loc+} = \left\{ f : \text{ all } \hat{f}(n) \ge 0 \text{ and } \int_{-\delta}^{\delta} |f|^p \, dx < \infty \text{ for some } \delta = \delta(f) > 0 \right\}.$$

The space $WB^{\alpha}_{p,q}$ is clearly solid, which, together with embedding (5), implies that

(5') *if* $1 < p \leqslant 2$, *then*

$$L^p_{loc+} \hookrightarrow WB^0_{p',2}.$$

2.2. Embeddings between Besov and Wiener–Beurling spaces

We define the Besov space $B^{\alpha}_{p,q}$, $\alpha \in \mathbb{R}$, $0 < p, q \leqslant \infty$, to be the collection of formal trigonometric series (not necessarily the Fourier series) $f(x) = \sum_{m \in \mathbb{Z}} c_m e^{imx}$ such that

$$\|f\|_{B^{\alpha}_{p,q}} = \left(\sum_{k=0}^{\infty} 2^{\alpha q k} \Big\| \sum_{[2^{k-1}] \leqslant |m| < 2^k - 1} c_m e^{imx} \Big\|_p^q \right)^{1/q} < \infty.$$

It is well known (see, e.g., [21]) that for $\alpha > 0$ and $1 < p < \infty$ this definition is equivalent to the classical definitions of the Besov space given in terms of moduli of smoothness or Fourier-analytical decompositions. In light of Nikol'skii's inequality for trigonometric polynomials it is easy to see the following (Jawerth–Franke-type) embedding: if $-\infty < \alpha_1 < \alpha_2 < \infty$, $0 < p_2 < p_1 \leqslant \infty$, and $\alpha_1 - \frac{1}{p_1} = \alpha_2 - \frac{1}{p_2}$, then $B^{\alpha_2}_{p_2,q} \hookrightarrow B^{\alpha_1}_{p_1,q}$, cf. embedding (4) for Wiener-Beurling spaces.

To compare Besov and Wiener–Beurling spaces, we start with the simple application of Parseval's identity:

$$B^{\alpha}_{2,q} = WB^{\alpha}_{2,q}, \qquad \alpha \in \mathbb{R}, \quad 0 < q \leqslant \infty. \tag{2.2}$$

Let us recall the classical Bernstein theorem and its generalization by Szasz [30]: if $f \in B^{1/p-1/2}_{2,p}$, $1 \leqslant p \leqslant 2$, then $\{\widehat{f_n}\} \in l_p$. In fact, using (2.2) this result trivially follows from the embedding properties of Wiener-Beurling spaces (see (2.1)):

$$B_{2,p}^{1/p-1/2} = WB_{2,p}^{1/p-1/2} \hookrightarrow WB_{p,p}^{0} = A^p.$$

Further, the Hausdorff–Young and Nikol'skii's inequalities immediately give

$$B_{p,q}^{\alpha} \hookrightarrow \begin{cases} WB_{\infty,q}^{\alpha+1-\frac{1}{p}}, & 0 < p \leqslant 1; \\ WB_{p',q}^{\alpha}, & 1 < p \leqslant 2; \\ WB_{2,q}^{\alpha}, & 2 < p \leqslant \infty; \end{cases}$$

and

$$WB_{p,q}^{\alpha} \hookrightarrow \begin{cases} B_{\infty,q}^{\alpha}, & 0 < p \leqslant 1; \\ B_{p',q}^{\alpha}, & 1 < p \leqslant 2; \\ B_{2,q}^{\alpha+\frac{1}{p}-\frac{1}{2}}, & 2 < p \leqslant \infty. \end{cases}$$

Pitt's inequalities imply the following embeddings

$$B_{p,q}^{\alpha+\beta} \hookrightarrow WB_{r,q}^{\alpha}, \quad 1 < p \leqslant r \leqslant p', \quad \beta = \frac{1}{p} + \frac{1}{r} - 1 \geqslant 0, \ 0 < q \leqslant \infty, \quad \forall \alpha \in \mathbb{R}$$

and

$$WB_{p,q}^{\alpha+\beta} \hookrightarrow B_{r,q}^{\beta}, \qquad 1 < r' \leqslant p \leqslant r < \infty, \quad \alpha = 1 - \frac{1}{p} - \frac{1}{r} \geqslant 0, \ 0 < q \leqslant \infty, \quad \forall \beta \in \mathbb{R}.$$

Sharpness of these embeddings can be seen with the help of functions with lacunary and monotone Fourier coefficients. For trigonometric series with lacunary coefficients, i.e., $f(x) = \sum_{m\in\mathbb{Z}} c_m e^{imx}$ such that $c_k = 0$ for $|k| \neq n_s$ with $n_{s+1}/n_s \geqslant \lambda > 1$, we have

$$\|f\|_{WB_{p_1,q}^{\alpha}} \asymp \|f\|_{B_{p_2,q}^{\alpha}}, \qquad \alpha \in \mathbb{R}, \quad 0 < p_1, p_2, q \leqslant \infty.$$

Now we consider series with monotone coefficients.

Theorem 2.2. *Let $1 < p < \infty$, $\alpha \in \mathbb{R}$, and $0 < q, q_1 \leqslant \infty$. Let $f(x) = \sum_{m\in\mathbb{Z}} c_m e^{imx}$ be such that $c_k \leqslant c_m$ for $|m| \leq |k|$, then*

$$\|f\|_{WB_{p,q}^{\alpha}} \asymp \|f\|_{B_{p',q}^{\alpha}}.$$

Proof. Since the coefficients c_k are decreasing, then by the Hardy–Littlewood theorem on monotone Fourier coefficients [33, V.2, XII, §6], we have that

$$\|f\|_{B_{p',q}^{\alpha}} = \left(\sum_{k=0}^{\infty} \left(2^{\alpha k} \Big\| \sum_{[2^{k-1}] \leqslant |m| < 2^k} c_m e^{imx} \Big\|_{L_{p'}} \right)^q \right)^{1/q}$$

$$\asymp \left(\sum_{k=0}^{\infty} 2^{\alpha q k} \left(\sum_{m=1}^{2^{k-1}} \left(m^{1/p} (|c_{2^{k-1}+m}| + |c_{-2^{k-1}-m}|) \right)^{p'} \frac{1}{m} \right)^{q/p'} \right)^{1/q}$$

$$\asymp \left(\sum_{k=0}^{\infty} 2^{(\alpha+1/p)qk} \left(|c_{2^k}| + |c_{-2^k}| \right)^q \right)^{1/q},$$

where in the last step we have used the monotonicity of c_k.

On the other hand, the monotonicity of coefficients implies that

$$\|f\|_{WB^{\alpha}_{p,q}} = \left(\sum_{k=0}^{\infty} 2^{\alpha q k} \left(\sum_{[2^{k-1}] \leqslant |m| < 2^k - 1} |c_m|^p \right)^{q/p} \right)^{1/q}$$

$$\asymp \left(\sum_{k=0}^{\infty} 2^{(\alpha+1/p)qk} \left(|c_{2^k}| + |c_{-2^k}| \right)^q \right)^{1/q},$$

completing the proof. □

Remark 2.1. Analyzing the proof of this theorem, we observe that the same result holds under less restrictive conditions on the coefficients $\{c_n\}$. In particular, monotonicity can be assumed only on each dyadic block (that is, $\{c_n\}_{2^{k-1} \leqslant |n| < 2^k}$ is decreasing or increasing) or, what is more, monotonicity can be replaced by the general monotonicity, see, e.g., [31].

3. Duality of Wiener-Beurling spaces

The problem to describe the dual of the Cesaro space $ces(q)$, which is a particular case of the Wiener-Beurling space, has a long history (see, e.g., [8, Sect. 2]). Bennett proved that, for $1 < q < \infty$,

$$\left(ces(q) \right)' = \left\{ \{c_n\} : \sum_n \sup_{k \geqslant n} |c_k|^{q'} < \infty \right\}, \tag{3.1}$$

where the Köthe dual of a space E is given by $E' = \Big\{ \{d_n\} : \sum_n |c_n d_n| < \infty \quad \text{for all} \quad \{c_n\} \in E \Big\}$. We extend this result to the Wiener-Beurling space.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$ and $1 \leq p, q \leqslant \infty$. We have*

$$\left(WB^{\alpha}_{p,q} \right)' = WB^{-\alpha}_{p',q'},$$

where $\|f\|_{X'} := \sup_{\|g\|_X = 1} \int_{\mathbb{R}} f(x) \overline{g(x)} dx$.

In particular, if $p = 1$, $\alpha = \frac{1}{q} - 1$, and $1 < q < \infty$ we recover (3.1). For the Beurling space $A^{*,1}$ (the case $p = \infty$ and $\alpha = q = 1$), this result was proved in [5, Proposition 2].

Proof. For $f = \sum_n \xi_n e^{inx}$ and $g = \sum_{n\in\mathbb{Z}} c_n e^{inx}$, we have

$$\|f\|_{\left(WB^{\alpha}_{p,q}\right)'} := \sup_{\|g\|_{WB^{\alpha}_{p,q}}=1} \int\limits_{\mathbb{R}} f(x)\overline{g(x)}dx = \sup_{\|g\|_{WB^{\alpha}_{p,q}}=1} \sum_{n\in\mathbb{Z}} \xi_n \overline{c_n}.$$

Then Hölder's inequality implies

$$\|f\|_{\left(WB^{\alpha}_{p,q}\right)'} \leq \sup_{\|g\|_{WB^{\alpha}_{p,q}}=1} \|f\|_{WB^{-\alpha}_{p',q'}} \|g\|_{WB^{\alpha}_{p,q}} = \|f\|_{WB^{-\alpha}_{p',q'}}.$$

To prove the reverse, for a given function $f = \sum_n \xi_n e^{inx} \in WB^{-\alpha}_{p',q'}$, we define $g = \sum_{n\in\mathbb{Z}} c_n e^{inx}$ with

$$c_n = d_k \xi_n |\xi_n|^{p'-1} \left(\sum_{[2^{k-1}]\leqslant|m|<2^k} |\xi_m|^{p'} \right)^{-1/p} \quad \text{for} \quad [2^{k-1}] \leq |n| < 2^k, \quad k \in \mathbb{Z}_+,$$

$$d_k = 2^{-\alpha k q'} \left(\sum_{[2^{k-1}]\leqslant|m|<2^k} |\xi_m|^{p'} \right)^{\frac{q'-1}{p'}} \left(\|f\|_{WB^{-\alpha}_{p',q'}} \right)^{-\frac{q'}{q}}.$$

Then we have

$$\|g\|_{WB^{\alpha}_{p,q}} = \left(\sum_{k=0}^{\infty} \left(2^{\alpha k} d_k \frac{\left(\sum_{[2^{k-1}]\leqslant|n|<2^k} |\xi_n|^{(p'-1)p}\right)^{1/p}}{\left(\sum_{[2^{k-1}]\leqslant|n|<2^k} |\xi_n|^{p'}\right)^{1/p}} \right)^q \right)^{1/q} = \left(\sum_{k=0}^{\infty} \left(2^{\alpha k} d_k\right)^q \right)^{1/q}$$

$$= \left(\sum_{k=0}^{\infty} \left(2^{\alpha k} 2^{-\alpha k q'} \left(\sum_{[2^{k-1}]\leqslant|m|<2^k} |\xi_m|^{p'} \right)^{\frac{q'-1}{p'}} \right)^q \right)^{1/q} \left(\|f\|_{WB^{-\alpha}_{p',q'}} \right)^{-\frac{q'}{q}} = 1.$$

Thus,

$$\|f\|_{\left(WB^{\alpha}_{p,q}\right)'} \geq \sum_{n\in\mathbb{Z}} \xi_n \overline{c_n} = \|f\|_{WB^{-\alpha}_{p',q'}},$$

where the last equality follows by the choice of c_n with the help of the Parseval identity. □

4. Convolution and product of functions from Wiener–Beurling spaces

For functions $f = \sum_k c_k e^{ikx}$ and $g = \sum_k d_k e^{ikx}$, we define their convolution $f * g$ and product $f \cdot g$ as follows:

$$f * g = \sum_k c_k d_k e^{ikx}, \qquad f \cdot g = \sum_k \sum_\nu c_\nu d_{k-\nu} e^{ikx}.$$

Holder's inequality immediately yields the following result for the convolution of functions in the Beurling–Wiener space.

Theorem 4.1. *Let $\alpha, \alpha_i \in \mathbb{R}$ and $0 < p, p_i, q, q_i \leqslant \infty$ for $i = 1, 2$. Let*

$$\alpha = \alpha_1 + \alpha_2, \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

If $f \in WB^{\alpha_1}_{p_1,q_1}$ and $g \in WB^{\alpha_2}_{p_2,q_2}$, then $f * g \in WB^{\alpha}_{p,q}$ and

$$\|f * g\|_{WB^{\alpha}_{p,q}} \leqslant \|f\|_{WB^{\alpha_1}_{p_1,q_1}} \|g\|_{WB^{\alpha_2}_{p_2,q_2}}.$$

This result is closely connected to the multiplier theory of H^p spaces; see [16,26].

The product of functions from the Beurling–Wiener space is treated by the following result.

Theorem 4.2. *Let $1 \leqslant p \leqslant \infty$, $\alpha > 0$, and $\bar{\alpha} = \max(\frac{1}{p'}, \alpha)$. Then*

$$\|f \cdot g\|_{WB^{\alpha}_{p,1}} \lesssim \|f\|_{WB^{\bar{\alpha}}_{p,1}} \|g\|_{WB^{\bar{\alpha}}_{p,1}}.$$

Proof. Let $f = \sum_k c_k e^{ikx}$ and $g = \sum_k d_k e^{ikx}$. First, we note that

$$\|f \cdot g\|_{WB^{\alpha}_{p,1}} = \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{[2^{k-1}] \leqslant |n| < 2^k} \left| \sum_{\nu \in \mathbb{Z}} c_\nu d_{n-\nu} \right|^p \right)^{1/p}$$

$$\leq \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n=[2^{k-1}]}^{2^k-1} \left| \sum_{\nu \in \mathbb{Z}} c_\nu d_{n-\nu} \right|^p \right)^{1/p} + \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n=-[2^{k-1}]}^{-2^k+1} \left| \sum_{\nu \in \mathbb{Z}} c_\nu d_{n-\nu} \right|^p \right)^{1/p}$$

$$=: I_1 + I_2.$$

We will estimate only the first term, the second one can be treated similarly. We have

$$I_1 = \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n=[2^{k-1}]}^{2^k-1} \left| \sum_{\nu=-\infty}^{\frac{n}{2}} c_\nu d_{n-\nu} + \sum_{\nu=\frac{n}{2}}^{\infty} c_\nu d_{n-\nu} \right|^p \right)^{1/p}$$

$$\leqslant \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n=[2^{k-1}]}^{2^k-1} \left| \sum_{\nu=-\infty}^{\frac{n}{2}} c_\nu d_{n-\nu} \right|^p \right)^{1/p} + \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n=[2^{k-1}]}^{2^k-1} \left| \sum_{\nu=-\infty}^{\frac{n}{2}} c_{n-\nu} d_\nu \right|^p \right)^{1/p}$$
$$=: I_{11} + I_{12}.$$

Changing variable and using Minkowski's inequality, we derive

$$\left(\sum_{n=[2^{k-1}]}^{2^k-1} \left| \sum_{\nu \leqslant \frac{n}{2}} c_\nu d_{n-\nu} \right|^p \right)^{1/p}$$
$$\leqslant \sum_{\nu=-\infty}^{2^{k-2}-1} |c_\nu| \left(\sum_{n=[2^{k-1}]}^{2^k-1} |d_{n-\nu}|^p \right)^{1/p} + \sum_{\nu=2^{k-2}}^{2^{k-1}-1} |c_\nu| \left(\sum_{n=2\nu}^{2^k-1} |d_{n-\nu}|^p \right)^{1/p}$$
$$\leqslant \sum_{\nu=-\infty}^{2^{k-2}-1} |c_\nu| \left(\sum_{\xi \geqslant 2^{k-2}} |d_\xi|^p \right)^{1/p} + \sum_{\nu=2^{k-2}}^{2^{k-1}-1} |c_\nu| \left(\sum_{\xi=2^{k-2}}^{2^k} |d_\xi|^p \right)^{1/p}$$
$$\leqslant \sum_{\nu \in \mathbb{Z}} |c_\nu| \left(\sum_{\xi \geqslant 2^{k-2}} |d_\xi|^p \right)^{1/p}.$$

Thus we have

$$I_{11} \leqslant \sum_{\nu \in \mathbb{Z}} |c_\nu| \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n \geqslant [2^{k-2}]} |d_n|^p \right)^{1/p}.$$

Since, for $\alpha > 0$, taking into account Hardy's inequality

$$\sum_{k=0}^{\infty} 2^{k\alpha} \Big(\sum_{m=k}^{\infty} A_m \Big)^{\gamma} \lesssim \sum_{k=0}^{\infty} 2^{k\alpha} A_k^{\gamma}, \qquad A_m \geqslant 0, \quad \gamma > 0, \tag{4.1}$$

we have

$$\sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{[2^{k-2}] \leq n} |d_n|^p \right)^{1/p} \lesssim \|g\|_{WB^{\alpha}_{p,1}}$$

and

$$\sum_{\nu \in \mathbb{Z}} |c_\nu| \leqslant \sum_{n=0}^{\infty} \left(\sum_{n=[2^{k-1}]}^{2^k-1} |c_\nu|^p \right)^{1/p} 2^{\frac{k}{p'}} \leqslant \|f\|_{WB^{\frac{1}{p'}}_{p,1}},$$

we finally have

$$I_{11} \lesssim \|f\|_{WB^{1/p'}_{p,1}} \|g\|_{WB^{\alpha}_{p,1}} \leq \|f\|_{WB^{\bar{\alpha}}_{p,1}} \|g\|_{WB^{\bar{\alpha}}_{p,1}}.$$

To conclude the proof, we note that I_{12} and I_2 can be estimated similarly. □

Corollary 4.1. *Let $1 \leqslant p \leqslant \infty$, $\alpha \geqslant \frac{1}{p'}$. Then $f \cdot g \in WB^{\alpha}_{p,1}$ provided that $f, g \in WB^{\alpha}_{p,1}$, i.e., $WB^{\alpha}_{p,1}$ is an algebra.*

When $p = \infty$ and $\alpha = q = 1$ this result was obtained in [5, Proposition 1]. We also mention the recent paper [25], where the author studied the boundedness of the convolution operator from the Cesáro space $ces(q)$ into $ces(q)$.

5. Interpolation properties of Wiener–Beurling spaces

Consider a compatible couple (X_0, X_1) of Banach spaces X_0 and X_1 (see [7, Ch. 3]). Let

$$K(f,t) := K(f,t; X_0, X_1) := \inf_{f=f_0+f_1} (\|f_0\|_{X_0} + t\|f_1\|_{X_1}), \quad f \in X_0 + X_1,$$

be the K-functional. The interpolation space $(0 < \theta < 1)$ is given by

$$(X_0, X_1)_{\theta,q} = \Big\{ f \in X_0 + X_1 : \|f\|_{(X_0,X_1)_{\theta,q}} = \Big(\int_0^\infty (t^{-\theta} K(f,t))^q \frac{dt}{t} \Big)^{\frac{1}{q}} < \infty \Big\}$$

for $0 < q < \infty$, with the usual modification for $q = \infty$.

The main interpolation properties of Cesaro and Copson sequence spaces have been recently studied in [3,4]:

$$(ces(q_0), ces(q_1))_{\theta,q} = ces(q), \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \ 1 < q_0 < q_1 \leqslant \infty, \ 0 < \theta < 1, \tag{5.1}$$

$$(cop(q_0), cop(q_1))_{\theta,q} = cop(q), \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \ 1 \leqslant q_0 < q_1 \leqslant \infty, \ 0 < \theta < 1. \tag{5.2}$$

The similar result for the weighted Cesáro space was proved in [18, Cor.5] for $q \in [1, \infty]$.

In [17], the authors considered the $A_{p,q}$ spaces with the norm

$$\|\{c_m\}\|_{A_{p,q}} = \Big(\sum_{k=0}^{\infty} \Big(\frac{1}{k+1} \sum_{k \leqslant |m| < 2k} |c_m|^p \Big)^{q/p} \Big)^{1/q} < \infty$$

and proved that

$$(A_{p,q_0}, A_{p,q_1})_{\theta,q} = A_{p,q}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 0 < \theta < 1, \tag{5.3}$$

provided $1 \leqslant q_0 < q_1 \leqslant \infty$, $1 \leqslant p \leqslant \infty$ and

$$(A_{p_0,p}, A_{p_1,p})_{\theta,p} = A_{p,p} = \ell_p, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1, \tag{5.4}$$

provided $1 \leqslant p_0 < p_1 \leqslant \infty$. The $A_{p,q}$ spaces are used in various problems of Fourier analysis; see, e.g., the recent monograph [19].

It is easy to see that $\|c_n\|_{A_{p,q}} < \infty$ if and only if $f \in WB_{p,q}^{1/q-1/p}$, where $f = \sum c_k e^{ikx}$. In fact we have that $f \in WB_{p,q}^{\alpha}$, $f = \sum c_k e^{ikx}$, if and only if

$$\Big(\sum_{k=0}^{\infty} 2^{\alpha q k} \Big(\sum_{[2^{k-1}] \leqslant |m| < 2^k} |c_m|^p \Big)^{q/p} \Big)^{1/q} \asymp \Big(\sum_{k=0}^{\infty} k^{\alpha q - 1} \Big(\sum_{k \leqslant |m| < 2k} |c_m|^p \Big)^{q/p} \Big)^{1/q} < \infty.$$

We extend (5.1)–(5.4) in the following

Theorem 5.1. *Let $-\infty < \alpha_1 < \alpha_0 < \infty$.*

(i) *If $0 < p, q, q_0, q_1 \leqslant \infty$, then*

$$(WB_{p,q_0}^{\alpha_0}, WB_{p,q_1}^{\alpha_1})_{\theta,q} = WB_{p,q}^{\alpha}, \tag{5.5}$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ and $0 < \theta < 1$.

(ii) *If $0 < p_0, p_1 \leq \infty$ and $0 < q, q_0, q_1 < \infty$, then*

$$(WB_{p_0,q_0}^{\alpha_0}, WB_{p_1,q_1}^{\alpha_1})_{\theta,q} = WB_{q,q}^{\alpha}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{5.6}$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ and $0 < \theta < 1$.

In particular, if $\alpha_0 = 1/q_0 - 1/p$ and $\alpha_1 = 1/q_1 - 1/p$ we recover (5.3) and if $q = q_0 = q_1$, $\alpha_0 = 1/q - 1/p_0$, and $\alpha_1 = 1/q - 1/p_1$ we recover (5.4) with q in place of p. Note also that (5.6) with $q = q_0 = q_1 = \infty$ becomes a well-known result ([9, Ch. 5]).

Proof. Let $f \in (WB_{p,q_0}^{\alpha_0}, WB_{p,q_1}^{\alpha_1})_{\theta q}$ be such that $f = f_0 + f_1$, where $f_0 = \sum_k d_k e^{ikx} \in WB_{p,q_0}^{\alpha_0}$, $f_1 = \sum_k \xi_k e^{ikx} \in WB_{p,q_1}^{\alpha_1}$, and $c_k = d_k + \xi_k$. Then for any $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} 2^{\alpha k} \left(\sum_{[2^{k-1}] \leqslant |m| < 2^k} |c_m|^p \right)^{1/p} &\lesssim 2^{(\alpha-\alpha_0)k} \Big(2^{\alpha_0 k} \Big(\sum_{[2^{k-1}] \leqslant |m| < 2^k} |d_m|^p \Big)^{1/p} \\ &+ 2^{(\alpha_0-\alpha_1)k} 2^{\alpha_1 k} \Big(\sum_{[2^{k-1}] \leqslant |m| < 2^k} |\xi_m|^p \Big)^{1/p} \Big) \\ &\lesssim 2^{(\alpha-\alpha_0)k} \Big(\|f_0\|_{WB_{p,\infty}^{\alpha_0}} + 2^{(\alpha_0-\alpha_1)k} \|f_1\|_{WB_{p,\infty}^{\alpha_1}} \Big). \end{aligned}$$

Thus we derive that

$$2^{\alpha k}\left(\sum_{[2^{k-1}]\leqslant|m|<2^k}|c_m|^p\right)^{1/p}\lesssim 2^{(\alpha-\alpha_0)k}K(f,2^{(\alpha_0-\alpha_1)k};WB^{\alpha_0}_{p,\infty},WB^{\alpha_1}_{p,\infty}),$$

which yields

$$\|f\|_{WB^{\alpha}_{p,q}}\lesssim\left(\sum_{k=0}^{\infty}\left(2^{(\alpha-\alpha_0)k}K(f,2^{(\alpha_0-\alpha_1)k};WB^{\alpha_0}_{p,\infty},WB^{\alpha_1}_{p,\infty})\right)^q\right)^{\frac{1}{q}}$$
$$\lesssim\left(\int_0^{\infty}(t^{-\theta}K(f,t))^q\frac{dt}{t}\right)^{\frac{1}{q}}=\|f\|_{(WB^{\alpha_0}_{p,\infty},WB^{\alpha_1}_{p,\infty})_{\theta q}}.$$

Using now the embedding properties of Beurling–Wiener spaces, we obtain

$$(WB^{\alpha_0}_{p,q_0},WB^{\alpha_1}_{p,q_1})_{\theta,q}\hookrightarrow(WB^{\alpha_0}_{p,\infty},WB^{\alpha_1}_{p,\infty})_{\theta,q}\hookrightarrow WB^{\alpha}_{p,q}.$$

To show the reverse embedding, let $f\in WB^{\alpha}_{p,q}$ and $\tau=\min(q_0,q_1,q)$, $r\in\mathbb{Z}$. Define functions f_0 and f_1 as follows

$$f_0=\begin{cases}\sum_{|m|<2^r}c_m(f)e^{imx} & \text{for}\quad r\in\mathbb{Z}_+=\{z\in\mathbb{Z},z\geqslant 0\},\\ 0 & \text{for}\quad r\notin\mathbb{Z}_+,\end{cases}$$

and $f_1=f-f_0$. Then, letting $\Delta_k:=\Delta_k(p):=\left(\sum_{[2^{k-1}]\leqslant|m|<2^k}|c_m|^p\right)^{1/p}$, we obtain

$$K(f,2^{(\alpha_0-\alpha_1)r};WB^{\alpha_0}_{p,q_0},WB^{\alpha_1}_{p,q_1})\lesssim K(f,2^{(\alpha_0-\alpha_1)r};WB^{\alpha_0}_{p,\tau},WB^{\alpha_1}_{p,\tau})$$
$$\lesssim\|f_0\|_{WB^{\alpha_0}_{p,\tau}}+2^{(\alpha_0-\alpha_1)r}\|f_1\|_{WB^{\alpha_1}_{p,\tau}}$$
$$=\Big(\sum_{k=0}^{r}2^{\alpha_0k\tau}\Delta_k^{\tau}\Big)^{1/\tau}+2^{(\alpha_0-\alpha_1)r}\Big(\sum_{k=r+1}^{\infty}2^{\alpha_1k\tau}\Delta_k^{\tau}\Big)^{1/\tau}. \tag{5.7}$$

It follows that

$$\|f\|_{(WB^{\alpha_0}_{p,q_0},WB^{\alpha_1}_{p,q_1})_{\theta,q}}\lesssim\|f\|_{(WB^{\alpha_0}_{p,\tau},WB^{\alpha_1}_{p,\tau})_{\theta,q}}=\left(\int_0^{\infty}(t^{-\theta}K(f,t))^q\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$\lesssim\left(\int_0^{\infty}\left(t^{-\theta(\alpha_0-\alpha_1)}K(f,t^{(\alpha_0-\alpha_1)})\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$\lesssim \left(\int_0^1 \left(t^{-\theta(\alpha_0-\alpha_1)} t^{(\alpha_0-\alpha_1)} \|f\|_{WB^{\alpha_1}_{p,\tau}} \right)^q \frac{dt}{t} + \int_1^\infty \left(t^{-\theta(\alpha_0-\alpha_1)} K(f, t^{(\alpha_0-\alpha_1)}) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_{WB^{\alpha_1}_{p,\tau}} + \left(\sum_{k=0}^\infty \left(2^{(\alpha_1-\alpha_0)k} K(f, 2^{(\alpha_0-\alpha_1)k}) \right)^q \right)^{\frac{1}{q}}.$$

Using $\|f\|_{WB^{\alpha_1}_{p,\tau}} \lesssim \|f\|_{WB^{\alpha}_{p,q}}$ for $\alpha_1 < \alpha$ and estimate (5.7), we arrive at

$$\|f\|_{(WB^{\alpha_0}_{p,q_0}, WB^{\alpha_1}_{p,q_1}))_{\theta q}} \lesssim \|f\|_{WB^{\alpha}_{p,q}}$$
$$+ \Big(\sum_{r=0}^\infty 2^{(\alpha-\alpha_0)rq} \Big(\Big(\sum_{k=0}^r 2^{\alpha_0 k\tau} \Delta_k^\tau \Big)^{1/\tau} + 2^{(\alpha_0-\alpha_1)r} \Big(\sum_{k=r+1}^\infty 2^{\alpha_1 k\tau} \Delta_k^\tau \Big)^{1/\tau} \Big)^q \Big)^{\frac{1}{q}}.$$

Finally, taking into account $\alpha_1 < \alpha < \alpha_0$ and using Hardy's inequalities (1.2) and (4.1), we derive

$$\|f\|_{(WB^{\alpha_0}_{p,q_0}, WB^{\alpha_1}_{p,q_1})_{\theta,q}} \lesssim \|f\|_{WB^{\alpha}_{p,q}}.$$

To obtain (5.6), we first apply the power theorem for interpolation spaces (see [9, Th. 3.11.6]):

$$\left((WB^{\alpha_0}_{p_0,q_0}, WB^{\alpha_1}_{p_1,q_1})_{\theta,q} \right)^q = \left((WB^{\alpha_0}_{p_0,q_0})^{q_0}, (WB^{\alpha_1}_{p_1,q_1})^{q_1} \right)_{\eta,1}, \tag{5.8}$$

where

$$\eta = \frac{q\theta}{q_1}, \qquad q = (1-\eta)q_0 + \eta q_1, \quad 0 < \eta < 1.$$

Observe that for $f(x) = \sum_{m\in\mathbb{Z}} c_m e^{imx}$ we have

$$K(t, f; (WB^{\alpha_0}_{p_0,q_0})^{q_0}, (WB^{\alpha_1}_{p_1,q_1})^{q_1}) = \inf_{f=f_0+f_1} \left(\|f_0\|^{q_0}_{WB^{\alpha_0}_{p_0,q_0}} + t\|f_1\|^{q_1}_{WB^{\alpha_1}_{p_1,q_1}} \right)$$
$$= \sum_{k=0}^\infty \inf_{b_k=b_{0,k}+b_{1,k}} \left(2^{\alpha_0 q_0 k} \|b_{0,k}\|^{q_0}_{l_{p_0}} + t 2^{\alpha_1 q_1 k} \|b_{1,k}\|^{q_1}_{l_{p_1}} \right),$$

where $b_k = \{c_m\}_{m=2^k}^{2^{k+1}-1}$, $b_{0,k} = \{c^0_m\}_{m=2^k}^{2^{k+1}-1}$, $b_{1,k} = \{c^1_m\}_{m=2^k}^{2^{k+1}-1}$, $c_m = c^0_m + c^1_m$, $m \in \mathbb{Z}$.

This together with (5.8) implies

$$\|f\|^q_{(WB^{\alpha_0}_{p_0,q_0}, WB^{\alpha_1}_{p_1,q_1})_{\theta,q}} \asymp \|f\|_{((WB^{\alpha_0}_{p_0,q_0})^{q_0}, (WB^{\alpha_1}_{p_1,q_1})^{q_1})_{\eta,1}}$$
$$= \int_0^\infty t^{-\eta} K(t, f; (WB^{\alpha_0}_{p_0,q_0})^{q_0}, (WB^{\alpha_1}_{p_1,q_1})^{q_1}) \frac{dt}{t}$$

$$= \sum_{k=0}^{\infty} 2^{\alpha_0 q_0 k} \int_0^{\infty} t^{-\eta} K(t 2^{(\alpha_1 q_1 - \alpha_0 q_0)k}, b_k; (l_{p_0})^{q_0}, (l_{p_1})^{q_1}) \frac{dt}{t}.$$

Changing variables (note that $\alpha_0 q_0 + (\alpha_1 q_1 - \alpha_0 q_0)\eta = \alpha q$) and applying again the power theorem, we obtain

$$\|f\|^q_{(WB^{\alpha_0}_{p_0,q_0}, WB^{\alpha_1}_{p_1,q_1})_{\theta,q}} = \sum_{k=0}^{\infty} 2^{\alpha q k} \|b_k\|_{((l_{p_0})^{q_0},(l_{p_1})^{q_1})_{\eta,1}} \asymp \sum_{k=0}^{\infty} 2^{\alpha q k} \|b_k\|^q_{(l_{p_0},l_{p_1})_{\theta,q}}$$

$$\asymp \sum_{k=0}^{\infty} 2^{\alpha q k} \|b_k\|^q_{l_q}, \qquad \frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \qquad \square$$

Theorem 5.1 immediately implies the following

Corollary 5.1. *Let $0 < q \leqslant \infty$, $\beta > 0$. We have*

(i) $\left(WB^{\alpha_1}_{\beta,q_1}, A^{\beta}\right)_{\theta,q} = WB^{\alpha}_{\beta,q}, \qquad \alpha = (1-\theta)\alpha_1, \quad 0 < \theta < 1, \quad \alpha_1 \neq 0.$

(ii) $\left(A^{*\beta}, WB^{\alpha_1}_{\infty,q}\right)_{\theta,q} = WB^{\alpha}_{\infty,q}, \qquad \alpha = \frac{1-\theta}{\beta} + \theta\alpha_1, \quad 0 < \theta < 1, \quad \alpha_1 \neq 1/\beta.$

(iii) $\left(A^{*\beta_0}, A^{*\beta_1}\right)_{\theta,q} = WB^{\frac{1}{\beta}}_{\infty,q}, \qquad \frac{1}{\beta} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}, \quad 0 < \theta < 1, \quad 0 < \beta_0 < \beta_1 < \infty.$
In particular,

$$\left(A^{*\beta_0}, A^{*\beta_1}\right)_{\theta,\beta} = A^{*\beta}, \qquad \frac{1}{\beta} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}, \quad 0 < \theta < 1.$$

(iv) *Let $0 < q_0, q_1 < \infty$, $q_0 \neq q_1$, and $0 < r \leqslant \infty$. Then $f(x) = \sum c_n e^{inx} \in WB^{\frac{1}{q}-1}_{1,r}$ if and only if*

$$\{c_n\} \in (cop(q_0), cop(q_1))_{\theta,r}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{5.9}$$

In particular, taking $r = q$, we arrive at (5.2).

(v) *Similarly, for $1 < q_0, q_1 < \infty$, $q_0 \neq q_1$, and $0 < r \leqslant \infty$, (5.9) can be equivalently replaced by*

$$\{c_n\} \in (ces(q_0), ces(q_1))_{\theta,r}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

In particular, taking $r = q$, we arrive at (5.1).

6. Final remarks

1. Note that the classical Bernstein inequality $\|T_N^{(\alpha)}\|_p \leqslant N^{\alpha}\|T_N\|_p$ for trigonometric polynomials $T_N(x) = \sum_{|k|\leqslant N} c_k e^{ikx}$ has the usual form in the Wiener-Beurling spaces, namely,

$$\|T_N^{(\alpha)}\|_{WB^{\alpha}_{p,q}} \leqslant N^{\alpha}\|T_N\|_{WB^{\alpha}_{p,q}}.$$

2. Let us also discuss the Nikolskii inequality for trigonometric polynomials. Its analogue for the Wiener-Beurling spaces reads as follows: *If* $0 < r \le p \le \infty$ *and* $\alpha + 1/r - \beta - 1/p \ge 0$, *then*

$$\|T_N\|_{WB^{\alpha}_{r,q}} \lesssim N^{(\alpha+1/r)-(\beta+1/p)}\|T_N\|_{WB^{\beta}_{p,q}}. \tag{6.1}$$

Indeed, let $2^{m-1} \le N < 2^m$ and $\gamma = \alpha + 1/r - 1/p$. Taking into account (2.1), we derive that

$$\begin{aligned}
\|T_N\|_{WB^{\alpha}_{r,q}} &\lesssim \|T_N\|_{WB^{\gamma}_{p,q}} \\
&\lesssim \Big(\sum_{k=0}^{m-1} 2^{\gamma qk}\Big(\sum_{[2^{k-1}]\leqslant|n|<2^k} |c_n|^p\Big)^{q/p} + 2^{\gamma qm}\Big(\sum_{[2^{m-1}]\leqslant|n|\le N} |c_n|^p\Big)^{q/p}\Big)^{1/q} \\
&\lesssim N^{(\alpha+1/r)-(\beta+1/p)}\|T_N\|_{WB^{\beta}_{p,q}}.
\end{aligned}$$

Note that if the first integrability parameters are the same but the second ones are different, then the Nikolskii inequality has a different form. Namely, *if* $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q_1 < q \leqslant \infty$, *then*

$$\|T_N\|_{WB^{\alpha}_{p,q_1}} \leqslant ([\ln N] + 1)^{\frac{1}{q}-\frac{1}{q_1}} \|T_N\|_{WB^{\alpha}_{p,q}}. \tag{6.2}$$

This follows from Hölder's inequality.

Thus, in view of (6.1) and (6.2), the Nikolskii inequalities in the Wiener–Beurling spaces are similar to those in the Lorentz spaces, cf. [23].

3. Now we study the Fourier series with averages of Fourier coefficients. Given a series $f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}$, define

$$Hf(x) = \sum_{k\in\mathbb{Z}} c_k^h e^{ikx}, \qquad c_k^h = \frac{1}{2|k|+1}\sum_{m=-|k|}^{|k|} |c_m|$$

and

$$Bf(x) = \sum_{k\in\mathbb{Z}} c_k^b e^{ikx}, \qquad c_k^b = \sum_{n=|k|}^{\infty} \frac{|c_n| + |c_{-n}|}{2n} < \infty.$$

This topic was originated by Hardy [13], who proved that if $f \in L_p$, $1 < p < \infty$, then $Hf \in L_p$. See also [1,6]; for the recent results see [12].

We will see that Wiener–Beurling spaces have the remarkable property that series with Hardy-Cesáro and Copson averages belong to Wiener–Beurling spaces if and only if the original series belong to similar spaces with different values of parameters.

Theorem 6.1. *If* $0 < p, q \leqslant \infty$, *then*

$$Hf \in WB^{\alpha}_{p,q} \Longleftrightarrow f \in WB^{\alpha+\frac{1}{p}-1}_{1,q} \quad \textit{if and only if} \quad \alpha < 1 - \frac{1}{p} \tag{6.3}$$

and

$$Bf \in WB^{\alpha}_{p,q} \Longleftrightarrow f \in WB^{\alpha+\frac{1}{p}-1}_{1,q} \quad \textit{if and only if} \quad \alpha + \frac{1}{p} > 0. \tag{6.4}$$

Remark 6.1. In a particular case $p = 1$ and $\alpha = \frac{1}{q} - 1 < 0$ we arrive at the known results for Cesáro and Copson sequence spaces by Bennett [8, Theorem 20.31]:

$$\{c_m\} \in ces(q) \Longleftrightarrow \{c^h_m\} \in ces(q) \Longleftrightarrow \{c^b_m\} \in ces(q), \qquad 1 < q < \infty$$

and

$$\{c_m\} \in cop(q) \Longleftrightarrow \{c^h_m\} \in cop(q) \Longleftrightarrow \{c^b_m\} \in cop(q), \qquad 1 < q < \infty.$$

Proof. To show (6.3), we observe that

$$\|Hf\|_{WB^{\alpha}_{p,q}} \asymp \Big(\sum_{k=0}^{\infty} 2^{k\alpha q} \Big(\sum_{m=2^k}^{2^{k+1}} \Big(\frac{1}{m} \sum_{|s|=0}^{m} |c_s| \Big)^p \Big)^{q/p} \Big)^{1/q}$$
$$\asymp \Big(\sum_{k=0}^{\infty} 2^{kq(\alpha+\frac{1}{p}-1)} \Big(\sum_{|s|=0}^{2^k} |c_s| \Big)^q \Big)^{1/q}.$$

The latter is equivalent to $\|f\|_{WB^{\alpha+\frac{1}{p}-1}_{1,q}}$ for $\alpha + \frac{1}{p} - 1 < 0$, cf. (1.3). To verify the sharpness of this condition, assume that $\beta = \alpha + \frac{1}{p} - 1 \geqslant 0$ and define

$$c_k = \frac{1}{(|k|+1)^{1+\beta+\varepsilon}}, \quad \varepsilon > 0.$$

It is routine to show that $\|f\|_{WB^{\beta}_{p,q}} < \infty$ and $\|Hf\|_{WB^{\alpha}_{p,q}} = \infty$ for $f = \sum c_k e^{ikx}$.

The proof of (6.4) is similar. The sharpness can be seen considering $f = \sum c_k e^{ikx}$ with

$$c_k = \frac{1}{(\ln(|k|+1))^{\frac{1+\varepsilon}{q}} (|k|+1)^{\alpha+\frac{1}{p}}}, \qquad \alpha + \frac{1}{p} \leqslant 0, \qquad 0 < \varepsilon \leqslant \frac{1}{q},$$

for which one has $\|f\|_{WB^{\alpha+\frac{1}{p}-1}_{1,q}} < \infty$ but $\|Bf\|_{WB^{\alpha}_{p,q}} = \infty$. $\square$

Declaration of competing interest

No competing interest.

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