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Some examples of nonassociative coalgebras and supercoalgebras



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ABSTRACT

Locally finiteness of some varieties of nonassociative coalgebras is studied and the Gelfand-Dorfman construction for Novikov coalgebras and the Kantor construction for Jordan super-coalgebras are given. We give examples of a non-locally finite differential coalgebra, Novikov coalgebra, Lie coalgebra, Jordan super-coalgebra, and right-alternative coalgebra. The dual algebra of each of these examples satisfies very strong additional identities. We also constructed examples of an infinite dimensional simple differential coalgebra, Novikov coalgebra, Lie coalgebra, and Jordan super-coalgebra over a field of characteristic zero.

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1. Introduction

The notion of coalgebra is dual to the notion of algebra. The theory of (co)associative coalgebras has been developed for a long time within the framework of the theory of Hopf algebras [18]. The study of coalgebras, bialgebras, and Hopf algebras received a new impetus when the term “quantum group”, along with revolutionary new examples, was launched by V. Drinfeld in 1986 [3]. Lie bialgebras, which are simultaneously Lie algebras and Lie coalgebras, were introduced as one of the most important notions of quantum group theory [4]. The study of Lie coalgebras, investigated earlier by W. Michaelis [11], was intensified. It is well known that the dual of an associative coalgebra is an associative algebra and the dual of a Lie coalgebra is a Lie algebra. In 1994 J. Anquella, T. Cortes, and F. Montaner [1] called a coalgebra C an \mathfrak{M} -coalgebra if the dual algebra C^* belongs to the variety of algebras \mathfrak{M} . This allows to define alternative, Jordan, Malcev, left-symmetric, Novikov coalgebras, and so on.

The Fundamental Theorem on Coalgebras asserts that every finitely generated associative coalgebra over a field is finite-dimensional. An analogue of this result is true for alternative and Jordan coalgebras [1], for structurable coalgebras [20], for Jordan copairs [24], for right alternative Malcev admissible coalgebras and binary $(-1,1)$ -coalgebras [15].

Lie coalgebras are not locally finite and the first example of an infinite dimensional finitely generated Lie coalgebra was given in [11]. In 1995 A. Slinko [17] found some necessary and sufficient conditions for Lie coalgebras to be locally finite. A connection between Jordan and Lie (super)coalgebras, which is an analog of the well known Kantor-Koecher-Tits construction for usual (super)algebras, was found in [21,22]. M.E. Goncharov and V.N. Zhelyabin [6,7] showed that every Malcev coalgebra embeds into a Lie coalgebra with triality. Unlike Jordan coalgebras, Jordan super-coalgebras are not locally finite [22].

In 2000 D. Kozybaev [9] constructed an example of a non-locally finite right-symmetric coalgebra and an example of a non-locally finite right-alternative coalgebra. I. Shestakov reported [16] that the example of a right-alternative coalgebra given in [9] is incorrect. This report attracted the attention of the authors to these old examples. First of all we noticed that the left-symmetric analogue of the non-locally finite right-symmetric coalgebra from [9] is a Novikov coalgebra. Moreover, we noticed that the commutator coalgebra of this coalgebra is exactly the non-locally finite Lie coalgebra given by W. Michaelis [11]. In order to understand the nature of these examples we started to study codifferential coalgebras.

To any associative and commutative differential algebra A one relates the following three algebras:

- (1) a Novikov algebra obtained from A by the Gelfand-Dorfman construction;
- (2) a Lie algebra obtained as the commutator algebra of the Novikov algebra mentioned in (1); and
- (3) a Jordan superalgebra obtained from A by the Kantor construction.

The notion of coderivation allows us to define the notion of a (co)differential coalgebra. We constructed a very easy example C of a non-locally finite associative and commutative differential coalgebra. Using this example we constructed three examples of non-locally finite coalgebras. We define an analogue of the Gelfand-Dorfman construction for coalgebras and using this we give an example of a non-locally finite Novikov coalgebra obtained from C , and the dual algebra of this coalgebra satisfies the identity $(xy)z = 0$. This example coincides with Kozybaev's example mentioned above. Moreover, the commutator coalgebra of this Novikov coalgebra is exactly the non-locally finite Lie coalgebra given by Michaelis [11]. This Lie coalgebra is metabelian. Using C and an analogue of the Kantor construction for super coalgebras, we give an example of a non-locally finite Jordan super-coalgebra. The dual of this super-coalgebra satisfies the super identities $xy = yx, xz = zx$, and $(zz_1)(z_2z_3) = 0$ for even variables x, y and for odd variables z, z_1, z_2, z_3 .

We repeated the same route starting from the simple differential algebra $(F[x], \partial = \frac{d}{dx})$ over a field F of characteristic zero. The graded dual of this algebra is a simple infinite dimensional differential coalgebra. Recall that a coalgebra is called simple if it does not have any proper subcoalgebras. Applying the Gelfand-Dorfman construction to this coalgebra, we get an example of a simple infinite dimensional Novikov coalgebra. This coalgebra is the graded dual of the Novikov-Witt algebra \mathcal{L}_1 [10]. The commutator Lie coalgebra of this coalgebra is also simple and coincides with the graded dual of the Witt algebra W_1 . And finally, using the Kantor construction, we constructed an example of a simple infinite dimensional Jordan super-coalgebra.

We also noticed that the example of a non-locally finite right-alternative coalgebra from [9] can be fixed only by interchanging the indexes $3n - 2$ and $3n - 1$ on lines 3 and 4 of the formula (12). The fixed example is given in Section 6. A much more complicated example of a non-locally finite right-alternative coalgebra was recently given in [16].

The paper is organized as follows. In Section 2 we give some necessary terminology of coalgebras, notations, and general statements. In Section 3 we give examples of non-locally finite differential, Novikov, and Lie coalgebras. Simple infinite dimensional differential, Novikov, and Lie coalgebras are given in Section 4. Section 5 is devoted to Jordan super-coalgebras and Section 6 is devoted to right-alternative coalgebras.

2. Coalgebras and coderivations

Let F be an arbitrary field. For any vector space V over F denote by

$$V^* = \text{Hom}_F(V, F)$$

its dual vector space, i.e., the vector space of all linear forms on V .

Denote by

$$V^{\otimes n} = \underbrace{V \otimes \dots \otimes V}_n$$

the n -th tensor power of the vector space V over F .

The map

$$\rho : (V^*)^{\otimes n} \rightarrow (V^{\otimes n})^*$$

defined by

$$\rho(f_1 \otimes \dots \otimes f_n)\left(\sum_{i_1 \dots i_n} e_{i_1} \otimes \dots \otimes e_{i_n}\right) = \sum_{i_1 \dots i_n} f_1(e_{i_1}) \dots f_n(e_{i_n})$$

is injective. For this reason we can assume that

$$(V^*)^{\otimes n} \subseteq (V^{\otimes n})^*.$$

If $\phi : V \rightarrow U$ is a linear map of vector spaces then the *transpose* $\phi^* : U^* \rightarrow V^*$ of ϕ is defined by

$$\phi^*(u^*)(v) = u^*(\phi(v)), \quad v \in V, u^* \in U^*.$$

A vector space C over F with a linear map

$$\Delta : C \rightarrow C \otimes_F C$$

is called a *coalgebra*. The map Δ is called its *comultiplication*. We often call the pair (C, Δ) a coalgebra in order to emphasize the comultiplication in question. For any $a \in C$, using the Sweedler notation (see [18]), we write

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}.$$

If C is a coalgebra, then

$$(fg)(a) = \rho(f \otimes g)(\Delta(a)) = \sum_a f(a_{(1)})g(a_{(2)}), \quad f, g \in C^*, a \in C,$$

defines a product on C^* and this product turns C^* into an algebra. Denote this product by

$$m_\Delta : C^* \otimes C^* \rightarrow C^*.$$

The algebra C^* or (C^*, m_Δ) is called the *dual algebra* of the coalgebra (C, Δ) .

A coalgebra (C, Δ) is called *coassociative* if

$$(\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta = 0,$$

i.e., for any $a \in C$ we have

$$\sum_a (\Delta(a_{(1)}) \otimes a_{(2)} - a_{(1)} \otimes \Delta(a_{(2)})) = 0.$$

It is well known that a coalgebra (C, Δ) is coassociative if and only if its dual algebra C^* is associative. Moreover, a coalgebra (C, Δ) is a Lie coalgebra if and only if its dual C^* is a Lie algebra [11]. Following these results, the definition of coalgebras from any variety of algebras was given in [1]:

Let \mathfrak{M} be an arbitrary variety of algebras. A coalgebra (C, Δ) is called an \mathfrak{M} -coalgebra if its dual algebra C^* belongs to \mathfrak{M} .

Let V be a vector space and let $\tau : V \otimes V \mapsto V \otimes V$ be the *ordinary flip*, i.e., a linear map with $\tau(x \otimes y) = y \otimes x$ for all $x, y \in V$.

A coalgebra (C, Δ) is called *cocommutative* if

$$\Delta = \tau\Delta,$$

i.e.,

$$\sum_a a_{(1)} \otimes a_{(2)} = \sum_a a_{(2)} \otimes a_{(1)}$$

for any $a \in C$.

Let (C, Δ) be an arbitrary coalgebra. A subspace B of C is called *subcoalgebra* of the coalgebra (C, Δ) if $\Delta(B) \subseteq B \otimes B$.

A subcoalgebra B of a coalgebra (C, Δ) is called *proper* if $B \neq \{0\}, C$. A coalgebra (C, Δ) without proper subcoalgebras is called *simple* [14].

It is well known [1] that C admits a C^* -bimodule structure. The left and right actions of C^* on C are defined by

$$\alpha \cdot a = \sum_{(a)} a_{(1)} \alpha(a_{(2)}), \quad a \cdot \alpha = \sum_{(a)} \alpha(a_{(1)}) a_{(2)}, \quad \alpha \in C^*, a \in C.$$

Moreover, a vector subspace B of a coalgebra C is a subcoalgebra if and only if B is a C^* -subbimodule of C . Therefore the intersection of any set of subcoalgebras of C is again a subcoalgebra.

Let S be a subset of a coalgebra C . The smallest subcoalgebra $Coalg(S)$ of C that contains S is called the subcoalgebra *generated by* S . In other words, $Coalg(S)$ is the C^* -subbimodule of C generated by S . If S is a finite set then $Coalg(S)$ is called *finitely generated*.

A coalgebra (C, Δ) is called *locally finite* if every finitely generated subcoalgebra of C is finite dimensional.

A linear map $d : C \mapsto C$ is called a *coderivation* of the coalgebra (C, Δ) if

$$\Delta d = (d \otimes id + id \otimes d)\Delta,$$

i.e.,

$$\Delta(d(a)) = \sum_a d(a_{(1)}) \otimes a_{(2)} + a_{(1)} \otimes d(a_{(2)}), \quad a \in C.$$

A triple (C, Δ, d) is called a *codifferential coalgebra* if (C, Δ) is a coalgebra and d is its coderivation. A subspace B of a codifferential coalgebra C is called a *subcoalgebra* if B is a subcoalgebra of (C, Δ) and $d(B) \subseteq B$, i.e., B is a codifferentially closed subcoalgebra.

Lemma 1. *Let d be a coderivation of the coalgebra (C, Δ) . Then its transpose d^* is a derivation of the dual algebra C^* , i.e.,*

$$d^*(fg) = d^*(f)g + fd^*(g), \quad f, g \in C^*.$$

Proof. Let $f, g \in C^*, a \in C$. Then

$$\begin{aligned} (d^*(fg))(a) &= (fg)(d(a)) = \rho(f \otimes g)\Delta(d(a)) = \sum_a f(d(a_{(1)}))g(a_{(2)}) + f(a_{(1)})g(d(a_{(2)})) \\ &= \sum_a (d^*(f))(a_{(1)})g(a_{(2)}) + f(a_{(1)})(d^*(g))(a_{(2)}) = (d^*(f)g + fd^*(g))(a), \end{aligned}$$

which proves the statement of the lemma. \square

From now on, in many places we use the “co” suffix only once, instead of using it repeatedly. For example, we will write “an associative and commutative differential coalgebra” instead of “a coassociative and cocommutative codifferential coalgebra”.

Corollary 1. *If (C, Δ, d) is an associative and commutative differential coalgebra then (C^*, m_Δ, d^*) is an associative and commutative differential algebra.*

Let A be an algebra over F with the multiplication

$$m : A \otimes A \rightarrow A,$$

i.e., $m(a \otimes b) = ab$ for all $a, b \in A$. Let

$$m^* : A^* \mapsto (A \otimes A)^*$$

be the transpose of m .

Unfortunately, the image $m^*(A^*)$ of m^* does not always belong to $A^* \otimes A^* \equiv \rho(A^* \otimes A^*) \subseteq (A \otimes A)^*$. If A is finite dimensional then $A^* \otimes A^* \equiv \rho(A^* \otimes A^*) = (A \otimes A)^*$ and the dual space A^* is a coalgebra with the comultiplication $\Delta = \rho^{-1}m^*$.

If A is infinite dimensional then the structure of the dual coalgebra (A°, Δ°) is a little complicated (see [1,12]). A subspace V of A^* is called a *good subspace* if $m^*(V) \subseteq$

$\rho(V \otimes V)$. If V is a good subspace then $\Delta_V = \rho^{-1}m^*$ is a comultiplication on V . Let A° be the sum of all good subspaces of A^* . Then A° is the largest good subspace of A^* and the pair (A°, Δ°) is a coalgebra with the comultiplication $\Delta^\circ = \Delta_{A^\circ}$. The coalgebra (A°, Δ°) is called *the finite dual* of A . It is known that if A is an associative algebra then

$$A^\circ = \{\alpha \in A^* \mid \ker \alpha \text{ contains a finite codimensional ideal of } A\}.$$

Let

$$A = \bigoplus_{i \in \mathbb{Z}} A_i, \quad A_i A_j \subseteq A_{i+j},$$

be a \mathbb{Z} -graded algebra such that there exists an integer m with $A_i = 0$ for all $i < m$ and A_i is finite dimensional for all $i \geq m$. In this case

$$(A \otimes A)_k = \bigoplus_{k=i+j} A_i \otimes A_j$$

is finite dimensional for all k and

$$(A \otimes A)_k^* = \bigoplus_{k=i+j} \rho(A_i^* \otimes A_j^*).$$

The coproduct $\Delta = \rho^{-1}m^*$ turns the graded space

$$\bigoplus_{i \in \mathbb{Z}} A_i^*$$

into a coalgebra. This coalgebra is called the *graded dual* of the graded algebra A .

The following lemma is useful for studying subcoalgebras.

Lemma 2. *Let V be a vector space and let W be a subspace of V . Let $a = \sum_{i=1}^n a_i \otimes b_i$ and let the vectors a_1, \dots, a_n be linearly independent. Assume that $a \in W \otimes W$. Then $b_1, b_2, \dots, b_n \in W$.*

Proof. Let $\alpha_1, \dots, \alpha_n$ be a system of linear forms dual to a_1, \dots, a_n , i.e., $\alpha_i(a_j) = \delta_{ij}$ for all i, j , where δ is the Kronecker delta function.

Set $\phi = \alpha_1 \otimes \text{id} : V \otimes V \mapsto V$. Then

$$\phi\left(\sum e_i \otimes f_i\right) = \sum_i \alpha_1(e_i) f_i.$$

Obviously, $\phi(W \otimes W) \subseteq W$ and $\phi(a) = \sum_i \alpha_1(a_i) b_i = b_1$. Since $a \in W \otimes W$ it follows that $\phi(a) \in W$. Consequently, $b_1 \in W$. Similarly, we get $b_2, \dots, b_n \in W$. \square

3. Examples of non locally finite differential, Novikov, and Lie coalgebras

An algebra A is called a *Novikov algebra* if it satisfies the following identities:

$$x(yz) - (xy)z = y(xz) - (yx)z, \tag{1}$$

$$(xy)z = (xz)y. \tag{2}$$

Recall that an algebra satisfying the identity (1) is called *left-symmetric*. Left-symmetric algebras are Lie-admissible, i.e., if A is a left-symmetric algebra then A with respect to the commutator $[x, y] := xy - yx$ is a Lie algebra. This algebra is called the *commutator algebra* of A and is denoted by $A^{(-)}$.

The identity (1) can be written as

$$(x, y, z) = (y, x, z), \tag{3}$$

where $(x, y, z) := (xy)z - x(yz)$ is the associator of elements x, y, z .

The Gelfand-Dorfman construction [5]. Let A be an associative and commutative algebra with a derivation d . Define a new multiplication (\circ) on A by

$$x \circ y = xd(y),$$

where $x, y \in A$. Then (A, \circ) is a Novikov algebra.

Moreover, the vector space A with respect to the bracket

$$[x, y] = xd(y) - yd(x)$$

is a Lie algebra. Obviously, $(A, [\cdot, \cdot])$ is the commutator algebra of the Novikov algebra (A, \circ) .

Thus, any associative and commutative differential algebra A generates the Novikov algebra (A, \circ) and the Lie algebra $(A, [\cdot, \cdot])$. Over a field of characteristic zero every Novikov algebra can be embedded into a Novikov algebra (A, \circ) for a suitable associative commutative differential algebra A [2]. The class of Lie algebras embeddable into Lie algebras of the type $(A, [\cdot, \cdot])$ is not described yet.

Proposition 1. *A pair (C, Δ) is a Novikov coalgebra if and only if the following (co)identities hold:*

$$(\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta = (\tau \otimes \text{id})(\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta, \tag{4}$$

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \tau)(\Delta \otimes \text{id})\Delta. \tag{5}$$

Proof. Let (C, Δ) be a coalgebra and let $\alpha, \beta, \gamma \in C^*$ and $c \in C$. Then

$$(\alpha, \beta, \gamma)(c) = (\alpha \otimes \beta \otimes \gamma)((\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta(c))$$

and

$$(\beta, \alpha, \gamma)(c) = (\alpha \otimes \beta \otimes \gamma)((\tau \otimes \text{id})(\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta(c)).$$

Consequently,

$$\begin{aligned} & [(\alpha, \beta, \gamma) - (\beta, \alpha, \gamma)](c) \\ &= (\alpha \otimes \beta \otimes \gamma)[(\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta - (\tau \otimes \text{id})(\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta](c). \end{aligned}$$

This implies that the identity (3) in C^* is equivalent to the identity (4) in C .

Similarly, the identity (2) in C^* is equivalent to the identity (5) in C . \square

The Gelfand-Dorfman construction for coalgebras. Let (C, Δ, d) be an associative and commutative differential coalgebra. Define on the space C a new comultiplication Δ_N by

$$\Delta_N = (\text{id} \otimes d)\Delta.$$

This means

$$\Delta_N(a) = \sum_a a_{(1)} \otimes d(a_{(2)})$$

for any $a \in C$. Set also

$$\Delta_L = \Delta_N^{(-)} = (1 - \tau)\Delta_N,$$

i.e.,

$$\Delta_L(a) = \sum_a (a_{(1)} \otimes d(a_{(2)}) - d(a_{(2)}) \otimes a_{(1)})$$

for any $a \in C$.

Proposition 2. (1) *The coalgebra (C, Δ_N) is a Novikov coalgebra and the product in its dual algebra is defined by*

$$\alpha \circ \beta = \alpha d^*(\beta), \quad \alpha, \beta \in C^*.$$

(2) *The coalgebra (C, Δ_L) is a Lie coalgebra and the bracket in its dual algebra is defined by*

$$[\alpha, \beta] = \alpha d^*(\beta) - d^*(\alpha)\beta, \quad \alpha, \beta \in C^*.$$

Proof. By Corollary 1, the dual (C^*, m_Δ, d^*) of the differential coalgebra (C, Δ, d) is a differential algebra. We have

$$(fg)(a) = \rho(f \otimes g)(\Delta(a)),$$

where $f, g \in C^*, a \in C$. By the Gelfand-Dorfman construction, the algebra (C^*, \circ) is a Novikov algebra, where $f \circ g = fd^*(g)$ for all $f, g \in C^*$.

On the other hand,

$$\begin{aligned} (f \circ g)(a) &= (fd^*(g))(a) = \sum_a f(a_{(1)})d^*(g)(a_{(2)}) = \sum_a f(a_{(1)})g(d(a_{(2)})) \\ &= \rho(f \otimes g)(\Delta_N(a)), \end{aligned}$$

i.e.,

$$f \circ g = fd^*(g) = \rho(f \otimes g)\Delta_N.$$

Hence (C^*, \circ) is the dual algebra of the coalgebra (C, Δ_N) . Since (C^*, \circ) is a Novikov algebra it follows that (C, Δ_N) is a Novikov coalgebra.

The second statement of the lemma can be checked similarly. \square

Example 1. Let C be a vector space with a linear basis

$$e, f_1, f_2, \dots, f_n, \dots$$

Define a comultiplication $\Delta : C \rightarrow C \otimes C$ on C by

$$\Delta(e) = e \otimes e, \quad \Delta(f_i) = f_i \otimes e + e \otimes f_i, \quad i \geq 1.$$

Define also a linear map $d : C \rightarrow C$ by

$$d(e) = 0, \quad d(f_i) = f_{i+1}, \quad i \geq 1.$$

Lemma 3. *The triple (C, Δ, d) is an associative and commutative differential coalgebra.*

Proof. Obviously (C, Δ) is cocommutative. Direct calculations give that

$$\begin{aligned} (\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta(e) &= e \otimes e \otimes e - e \otimes e \otimes e = 0, \\ (\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta(f_i) &= f_i \otimes e \otimes e + e \otimes f_i \otimes e \\ &+ e \otimes e \otimes f_i - f_i \otimes e \otimes e - e \otimes f_i \otimes e - e \otimes e \otimes f_i = 0, \end{aligned}$$

i.e., (C, Δ) is coassociative.

We also have

$$\begin{aligned} \Delta(d(e)) &= 0 = d(e) \otimes e + e \otimes d(e), \\ \Delta(d(f_i)) &= \Delta(f_{i+1}) = f_{i+1} \otimes e + e \otimes f_{i+1} = \\ &(d(f_i) \otimes e + d(e) \otimes f_{i+1}) + (f_{i+1} \otimes d(e) + e \otimes d(f_i)), \quad i \geq 1. \end{aligned}$$

This means that d is a coderivation of the coalgebra (C, Δ) . \square

By Lemma 3, (C, Δ, d) is an associative and commutative differential coalgebra. Consequently, (C^*, m_Δ, d^*) is an associative and commutative differential algebra. Following the tradition in the theory of ordinary differential algebras, we denote the derivative of $x \in C^*$ by x' , i.e., $x' = d^*(x)$.

Proposition 3. *The differential coalgebra (C, Δ, d) is not locally finite and the differential algebra (C^*, m_Δ, d^*) satisfies the differential identity*

$$x'y' = 0.$$

Proof. Obviously the codifferential subcoalgebra B of C generated by f_1 contains f_i for all i . Since

$$\Delta(f_1) = f_1 \otimes e + e \otimes f_1 \in B \otimes B$$

it follows that $e \in B$ by Lemma 2. Therefore $B = C$ is infinite dimensional.

Let $\alpha, \beta \in C^*$ and $c \in C$. Then

$$(d^*(\alpha)d^*(\beta))(c) = \sum_c \alpha(d(c_{(1)}))\beta(d(c_{(2)})).$$

Consequently

$$(d^*(\alpha)d^*(\beta))(e) = \alpha(d(e))\beta(d(e)) = 0$$

and

$$(d^*(\alpha)d^*(\beta))(f_i) = \alpha(d(f_i))\beta(d(e)) + \alpha(d(e))\beta(d(f_i)) = 0$$

since $d(e) = 0$.

This means that C^* satisfies the differential identity $x'y' = 0$. \square

Example 2. Consider the comultiplication $\Delta_N = (\text{id} \otimes d)\Delta$ on C . By Proposition 2, (C, Δ_N) is a Novikov coalgebra. We have

$$\Delta_N(e) = (\text{id} \otimes d)\Delta(e) = e \otimes d(e) = 0$$

and

$$\Delta_N(f_i) = (\text{id} \otimes d)\Delta(f_i) = e \otimes d(f_i) = e \otimes f_{i+1}$$

for all $i \geq 1$.

Theorem 1. *The Novikov coalgebra (C, Δ_N) is not locally finite and the Novikov algebra C^* satisfies the identity*

$$(x \circ y) \circ z = 0.$$

Proof. Let B be the subcoalgebra of (C, Δ_N) generated by f_1 . If $f_i \in B$ then

$$\Delta_N(f_i) = (\text{id} \otimes d)\Delta(f_i) = e \otimes f_{i+1} \in B \otimes B$$

implies that $e, f_{i+1} \in B$ by Lemma 2. Consequently, $B = C$ is infinite dimensional.

Let $\alpha, \beta, \gamma \in C^*$. Then

$$((\alpha \circ \beta) \circ \gamma)(e) = (\alpha \otimes \beta \otimes \gamma)(\Delta_N \otimes \text{id})\Delta_N(e) = 0,$$

$$((\alpha \circ \beta) \circ \gamma)(f_i) = (\alpha \otimes \beta \otimes \gamma)(\Delta_N \otimes \text{id})\Delta_N(f_i) = (\alpha \otimes \beta \otimes \gamma)(\Delta_N(e) \otimes f_{i+1}) = 0,$$

since $\Delta_N(e) = 0$. Hence the algebra C^* satisfies the identity $(x \circ y) \circ z = 0$. \square

Example 2 is the left-symmetric analogue of the non-locally finite right-symmetric coalgebra from [9]. Theorem 1 reinforces the validity of Shestakov’s conjecture [15,16] on the relation between the locally finite property of coalgebras in a variety of algebras and the existence of locally nilpotent radical in this variety of algebras.

Example 3. Now consider the Lie coalgebra (C, Δ_L) . Recall that $\Delta_L = \Delta_N^{(-)}$ and, consequently,

$$\Delta_L(e) = (d \otimes \text{id} + \text{id} \otimes d)\Delta(e) = 0,$$

$$\Delta_L(f_i) = (d \otimes \text{id} + \text{id} \otimes d)\Delta(f_i) = e \otimes f_{i+1} - f_{i+1} \otimes e$$

for all $i \geq 1$.

Theorem 2. *The Lie coalgebra (C, Δ_L) is not locally finite and the Lie algebra C^* satisfies the identity*

$$[[x, y], [z, t]] = 0.$$

Proof. Let B be the subcoalgebra of (C, Δ_L) generated by f_1 . If $f_i \in B$ then

$$\Delta_L(f_i) = e \otimes f_{i+1} - f_{i+1} \otimes e \in B \otimes B$$

implies that $e, f_{i+1} \in B$ by Lemma 2. Consequently, $B = C$ is infinite dimensional.

Let $\alpha, \beta, \gamma, \delta \in C^*$. Then

$$([\alpha, \beta])(e) = (\alpha \otimes \beta)\Delta_L(e) = 0$$

and

$$\begin{aligned} ([[\alpha, \beta], [\gamma, \delta]])(f_i) &= (\alpha \otimes \beta \otimes \gamma \otimes \delta)(\Delta_L \otimes \Delta_L)\Delta_L(f_i) \\ &= (\alpha \otimes \beta \otimes \gamma \otimes \delta)(\Delta_L(e) \otimes \Delta_L(f_{i+1}) - \Delta_L(f_{i+1}) \otimes \Delta_L(e)) = 0 \end{aligned}$$

since $\Delta_L(e) = 0$. Therefore $[[\alpha, \beta], [\gamma, \delta]] = 0$. \square

Example 3 is Michaelis’s example of a non-locally finite Lie coalgebra from [11].

4. Infinite dimensional simple coalgebras

Let F be a field of characteristic zero and let $F[x]$ be the algebra of polynomials over F in one variable x . Then $(F[x], m, \partial)$, where m is the polynomial multiplication and $\partial = \frac{d}{dx}$, is a simple differential algebra. Consider the natural grading

$$F[x] = F1 \oplus Fx \oplus \dots \oplus Fx^n \oplus \dots$$

The following example is the graded dual of the differential algebra $(F[x], m, \partial)$.

Example 4. Let

$$C = Fx_0 \oplus Fx_1 \oplus \dots \oplus Fx_n \oplus \dots,$$

where $x_i \in F[x]^*$ is defined by $x_i(x^j) = \delta_{ij}$ for all i, j . It is easy to check that the dual comultiplication

$$\Delta = \rho^{-1}m^* : C \rightarrow C \otimes C$$

is defined by

$$\Delta(x_n) = \sum_{i=0}^n x_i \otimes x_{n-i}$$

and the coderivation

$$d = \partial^* : C \rightarrow C$$

is defined by

$$d(x_n) = (n + 1)x_{n+1}$$

for all $n \geq 0$.

Consequently, (C, Δ, d) is an associative and commutative differential coalgebra.

Proposition 4. *The differential coalgebra (C, Δ, d) is simple.*

Proof. Let B be a nonzero subcoalgebra of (C, Δ, d) and let

$$f = x_n + \sum_{i < n} \alpha_i x_i \in B.$$

Then

$$\Delta(f) = x_n \otimes x_0 + \sum_{i < n} x_i \otimes f_i \in B, \quad f_i \in C.$$

By Lemma 2 this implies that $x_0 \in B$. Since B is differentially closed it follows that $x_i \in B$ for all $i \geq 0$. Consequently, $B = C$. \square

Applying the Gelfand-Dorfman construction to the differential algebra $(F[x], m, \partial)$ we get a Novikov algebra $(F[x], \circ)$, where $f \circ g = f\partial(g)$. This Novikov algebra was denoted by \mathcal{L}_1 in [10] and is the first algebra in the list of left-symmetric Witt algebras \mathcal{L}_n . Notice that the commutator algebra of \mathcal{L}_n is the Witt algebra W_n [10]. Since \mathcal{L}_1 is a Novikov algebra, we call \mathcal{L}_1 the *Novikov-Witt* algebra. Recall that

$$\mathcal{L}_1 = F\partial \oplus Fx\partial \oplus \dots \oplus Fx^n\partial \oplus \dots$$

is the algebra of all derivations of $F[x]$ with respect to the product

$$x^i\partial \circ x^j\partial = jx^{i+j-1}\partial, \quad i, j \geq 0.$$

The following example is the graded dual of the Novikov-Witt algebra \mathcal{L}_1 .

Example 5. Let (C, Δ, d) be the differential coalgebra from Example 4. Applying the Gelfand-Dorfman construction, we get a Novikov coalgebra (C, Δ_N) , where

$$\Delta_N = (\text{id} \otimes d)\Delta,$$

i.e.,

$$\Delta_N(x_n) = \sum_{i=0}^n (n - i + 1)x_i \otimes x_{n-i+1}$$

Theorem 3. *The Novikov coalgebra (C, Δ_N) is simple.*

Proof. Let B be a nonzero subcoalgebra of C and let

$$f = x_n + \sum_{i < n} \alpha_i x_i \in B.$$

Then

$$\Delta_N(f) = (n + 1)x_0 \otimes x_{n+1} + \sum_{i < n+1} f_i \otimes x_i \in B, \quad f_i \in C.$$

By Lemma 2 this implies that $x_0 \in B$. Suppose that $x_n \in B$ for some $n \geq 0$. Then

$$\Delta_N(x_n) = \sum_{i=0}^n (n - i + 1)x_i \otimes x_{n-i+1} \in B$$

implies that $x_0, \dots, x_{n+1} \in B$ by Lemma 2. Consequently, $B = C$. \square

The following example is the graded dual of the Witt algebra W_1 .

Example 6. Let (C, Δ, d) be the Novikov coalgebra from Example 5. Then the commutator coalgebra (C, Δ_L) is a Lie coalgebra, where

$$\Delta_L = \Delta_N^{(-)} = (1 - \tau)\Delta_N,$$

i.e.,

$$\begin{aligned} \Delta_L(x_n) &= \sum_{i=0}^n (n - i + 1)x_i \otimes x_{n-i+1} - (i + 1)x_{i+1} \otimes x_{n-i} \\ &= \sum_{i=0}^{n+1} (n + 1 - 2i)x_i \otimes x_{n+1-i}. \end{aligned}$$

Theorem 4. *The Lie coalgebra (C, Δ_L) is simple.*

Proof. Let B be a nonzero subcoalgebra of C and let

$$f = x_n + \sum_{i < n} \alpha_i x_i \in B.$$

Then, as in the proof of Theorem 3, we get $x_0 \in B$. If $x_n \in B$ then applying Lemma 2 to the inclusion $\Delta_L(x_n) \in B \otimes B$, we get $x_{n+1} \in B$. Consequently, $B = C$. \square

Notice that an example of an infinite dimensional Lie coalgebra without finite dimensional subcoalgebras was constructed in [12] and [13].

5. Non locally finite Jordan supercoalgebras

Let G be the Grassman algebra with identity. Then $G = G_0 + G_1$ is a Z_2 -graded algebra. Let $J = J_0 + J_1$ be a Z_2 -graded algebra. Then $G(J) = G_0 \otimes J_0 + G_1 \otimes J_1$ is a subalgebra of the algebra $G \otimes J$. The subalgebra $G(J)$ is called *Grassman envelope* of the algebra J .

An algebra J is called a *Jordan superalgebra*, if its Grassman envelope $G(J)$ is a Jordan algebra, i.e., $G(J)$ satisfies the following identities:

$$xy = yx, \\ (x^2y)x = x^2(yx).$$

The Kantor construction [8]. Let A be an associative commutative algebra over F with a derivation D . Denote by \bar{A} an isomorphic copy of the vector space A with an isomorphism $a \mapsto \bar{a}$. On the direct sum of the vector spaces

$$J(A, D) = A \oplus \bar{A}$$

define a product (\cdot) by

$$a \cdot b = ab, a \cdot \bar{b} = \overline{ab}, \bar{a} \cdot b = \overline{ab}, \bar{a} \cdot \bar{b} = aD(b) - D(a)b,$$

where $a, b \in A$ and ab is the product of elements in A . Then $J(A, D)$ is a Jordan superalgebra. The superalgebra $J(A, D)$ is called a superalgebra of the *vector type*.

The Kantor construction for coalgebras. Let (C, Δ, d) be an associative and commutative differential coalgebra. Let \bar{C} be an isomorphic copy of the vector space C with an isomorphism $c \mapsto \bar{c}$. On the direct sum of vector spaces

$$J(C, d) = C \oplus \bar{C}$$

define a coproduct Δ_J by

$$\Delta_J(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} + \overline{c_{(1)}} \otimes \overline{d(c_{(2)})} - \overline{d(c_{(1)})} \otimes \overline{c_{(2)}},$$

$$\Delta_J(\bar{c}) = \sum_{(c)} \overline{c_{(1)}} \otimes c_{(2)} + c_{(1)} \otimes \overline{c_{(2)}},$$

where $c \in C$ and $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$.

Proposition 5. *The coalgebra $(J(C, d), \Delta_J)$ is a Jordan supercoalgebra and its dual $J(C^*, d^*)$ is a Jordan superalgebra of the vector type.*

Proof. By Corollary 1, (C^*, m_Δ, d^*) is an associative and commutative differential algebra. We have $J(C, d)^* = C^* + (\overline{C})^*$. The isomorphism of C and \overline{C} induces the isomorphism of C^* and $(\overline{C})^*$. Under this isomorphism, for any $\alpha \in C^*$ there corresponds $\overline{\alpha} \in (\overline{C})^*$ such that $\alpha(c) = \overline{\alpha}(\overline{c})$ for any $c \in C$. Therefore, we can write that $J(C, d)^* = C^* + \overline{C}^*$.

Let $\alpha, \beta \in C^*$, and $c \in C$. Denote by (\cdot) the multiplication of the algebra $(J(C, d), \Delta_J)^*$. Then we have

$$\begin{aligned} (\alpha \cdot \beta)(c) &= \rho(\alpha \otimes \beta)\Delta_J(c), & (\alpha \cdot \overline{\beta})(\overline{c}) &= \rho(\alpha \otimes \overline{\beta})\Delta_J(\overline{c}), \\ (\overline{\alpha} \cdot \beta)(\overline{c}) &= \rho(\overline{\alpha} \otimes \beta)\Delta_J(\overline{c}), & (\overline{\alpha} \cdot \overline{\beta})(c) &= \rho(\overline{\alpha} \otimes \overline{\beta})\Delta_J(c). \end{aligned}$$

From this we get

$$\alpha \cdot \beta = \alpha\beta, \quad \alpha \cdot \overline{\beta} = \overline{\alpha\beta}, \quad \overline{\alpha}\beta = \overline{\alpha\beta}, \quad \overline{\alpha} \cdot \overline{\beta} = \alpha d^*(\beta) - d^*(\alpha)\beta,$$

where $\alpha\beta$ is the product of elements in the dual algebra (C^*, m_Δ, d^*) .

Consequently, $(J(C, d), \Delta_J)^* = J(C^*, d^*)$. \square

Example 7. Let (C, Δ, d) be the differential coalgebra from Example 1 and let $(J(C, d), \Delta_J)$ be the Jordan supercoalgebra obtained from (C, Δ, d) by the Kantor construction for coalgebras. Notice that

$$\begin{aligned} \Delta_J(e) &= e \otimes e, & \Delta_J(f_i) &= e \otimes f_i + f_i \otimes e + \overline{e} \otimes \overline{f_{i+1}} - \overline{f_{i+1}} \otimes \overline{e}, \\ \Delta_J(\overline{e}) &= e \otimes \overline{e} + \overline{e} \otimes e, & \Delta_J(\overline{f_i}) &= e \otimes \overline{f_i} + \overline{f_i} \otimes e + \overline{e} \otimes f_i + f_i \otimes \overline{e}, \end{aligned}$$

for all $i \geq 1$.

Theorem 5. *The Jordan super-coalgebra $(J(C, d), \Delta_J)$ is not locally finite and its dual superalgebra satisfies the super identities*

$$xy = yx, \quad xz = zx, \quad (z_1 z_2)(z_3 z_4) = 0$$

for all even x, y and odd z, z_1, z_2, z_3, z_4 .

Proof. Let B be the subcoalgebra of $J(C, d)$ generated by $\overline{f_1}$. If $\overline{f_i} \in B$, then $\Delta_J(\overline{f_i}) \in B \otimes B$ implies that $e, \overline{e}, f_i \in B$ by Lemma 2. If $f_i \in B$, then $\Delta_J(f_i) \in B \otimes B$ implies that $\overline{f_{i+1}} \in B$. Consequently, $B = J(C, d)$ is infinite dimensional.

By Proposition 5, the dual of the supercoalgebra $(J(C, d), \Delta_J)$ is the Jordan superalgebra $J(C^*, d^*)$ obtained from the differential algebra (C^*, m_Δ, d^*) by the Kantor

construction. The first two identities of $J(C^*, d^*)$, mentioned in the lemma, directly follow from the Kantor construction since C^* is an associative and commutative algebra. Notice that the product of two odd elements z_1, z_2 from $J(C^*, d^*)$ belongs to the ideal $d^*(C^*)C^*$. By Proposition 3, we have $d^*(C^*)^2 = 0$. Consequently, $(z_1z_2)(z_3z_4) = 0$ for all odd elements z_1, z_2, z_3, z_4 . \square

The following example is the graded dual of the simple Jordan superalgebra $J(F[x], \partial)$ obtained from the simple differential algebra $(F[x], \partial)$ by the Kantor construction.

Example 8. Let (C, Δ, d) be the codifferential coalgebra from Example 4 and let $(J(C, d), \Delta_J)$ be the Jordan super-coalgebra obtained from (C, Δ, d) by the Kantor construction for coalgebras. Direct calculations give that

$$\Delta_J(x_n) = \sum_{i=0}^n x_i \otimes x_{n-i} + \sum_{i=0}^{n+1} (n+1-2i)\bar{x}_i \otimes \bar{x}_{n-i+1},$$

$$\Delta_J(\bar{x}_n) = \sum_{i=0}^n (\bar{x}_i \otimes x_{n-i} + x_i \otimes \bar{x}_{n-i}),$$

for all $n \geq 0$.

Theorem 6. *The Jordan supercoalgebra $(J(C, d), \Delta_J)$ is simple.*

Proof. Let B be a nonzero subcoalgebra of $(J(C, d), \Delta_J)$. Notice that if $x_0 \in B$ or $\bar{x}_0 \in B$ then $B = J(C, d)$. In fact, $\Delta_J(\bar{x}_n) \in B \otimes B$ implies that $x_0, \dots, x_n, \bar{x}_0, \dots, \bar{x}_n \in B$ and $\Delta_J(x_n) \in B \otimes B$ implies that $x_0, \dots, x_n, \bar{x}_0, \dots, \bar{x}_{n+1} \in B$ by Lemma 2.

Suppose that

$$f = x_n + \sum_{i < n} \alpha_i x_i + \bar{c} \in B,$$

where $\bar{c} \in \bar{C}$. Set

$$\Delta_J(f) = \sum_i a_i \otimes x_i + \sum_i b_i \otimes \bar{x}_i \in B \otimes B.$$

Notice that

$$b_{n+1} = (n+1)\bar{x}_0 + \sum_j \gamma_j x_j.$$

By Lemma 2, $b_{n+1} \in B$. If $\gamma_j = 0$ for all j then $\bar{x}_0 \in B$. This gives $B = J(C, d)$ as noticed above. Otherwise we get an element of the form

$$f = x_n + \sum_{i < n} \alpha_i x_i + \gamma \bar{x}_0$$

that belongs to B . Repeating the same discussions with this f we get $\bar{x}_0 \in B$.

If

$$f = \bar{x}_n + \sum_{i < n} \alpha_i \bar{x}_i \in B,$$

then $\Delta_J(f) \in B \otimes B$ implies that $x_0, \bar{x}_0 \in B$ by Lemma 2. Consequently, $B = J(C, d)$. \square

Notice that an example of a Jordan super-coalgebra without finite dimensional sub-coalgebras was constructed in [23]

6. Non locally finite right alternative coalgebra

In this section we give a corrected version of the example of a non locally finite right-alternative coalgebra from [9]. This example was constructed on the base of the example of a finitely generated metabelian right-alternative algebra that is not residually finite [19].

An algebra A is called *right-alternative* if it satisfies the following identities:

$$(yx)x = yx^2, \tag{6}$$

$$((xy)z)y = x((yz)y). \tag{7}$$

The identity (6) is called *right-alternativity* and can be written in terms of the associators as

$$(y, x, x) = 0. \tag{8}$$

Over fields of characteristic $\neq 2$ this identity also implies the *Moufang identity* (7) (see [25]).

Proposition 6. *Let (A, Δ) be a coalgebra over a field F of characteristic $\neq 2$. The coalgebra (A, Δ) is right-alternative if and only if the following identity holds:*

$$(\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta + (\text{id} \otimes \tau)(\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta = 0. \tag{9}$$

Proof. Let (C, Δ) be a coalgebra and let $\alpha, \beta, \gamma \in C^*$ and $c \in C$. Then

$$(\alpha, \beta, \gamma)(c) = (\alpha \otimes \beta \otimes \gamma)((\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta(c))$$

and

$$(\alpha, \gamma, \beta)(c) = (\alpha \otimes \beta \otimes \gamma)((\text{id} \otimes \tau)(\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta(c)).$$

Consequently,

$$\begin{aligned}
 & [(\alpha, \beta, \gamma) + (\alpha, \gamma, \beta)](c) \\
 & = (\alpha \otimes \beta \otimes \gamma)[(\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta + (\text{id} \otimes \tau)(\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta](c).
 \end{aligned}$$

This implies that the identity (9) in C is equivalent to linearized version of the identity (8) in C^* . Consequently, (9) is equivalent to (8) over fields of characteristic $\neq 2$. \square

Example 9. Let A be a vector space with a linear basis

$$e_1, e_2, f_1, f_2, \dots, f_n, \dots$$

Define a comultiplication Δ on the vector space A by

$$\begin{aligned}
 \Delta(e_1) &= 0, \quad \Delta(e_2) = 0, \\
 \Delta(f_{3n-2}) &= e_1 \otimes f_{3n}, \quad \Delta(f_{3n-1}) = e_2 \otimes f_{3n}, \\
 \Delta(f_{3n}) &= e_2 \otimes f_{3n+1} - f_{3n+1} \otimes e_2 - e_1 \otimes f_{3n+2} + f_{3n+2} \otimes e_1, \quad n \geq 1.
 \end{aligned}$$

Theorem 7. *The coalgebra (A, Δ) is a right-alternative non locally finite coalgebra. Moreover, the dual algebra A^* satisfies the identities $(xy)(zt) = ((xy)z)t = 0$.*

Proof. Let $\Delta_{Ass} = (\Delta \otimes \text{id} - \text{id} \otimes \Delta)\Delta$. If $f, g, h \in A^*$, then

$$(f, g, h)(a) = (f \otimes g \otimes h)(\Delta_{ass}(a))$$

for all $a \in A$. We have $(f, g, g)(e_i) = 0$ since $\Delta(e_i) = 0$ for $i = 1, 2$.

Direct calculations give that

$$\begin{aligned}
 \Delta_{Ass}(f_{3n-2}) &= (\Delta \otimes \text{id} - \text{id} \otimes \Delta)(e_1 \otimes f_{3n}) = -e_1 \otimes \Delta(f_{3n}) \\
 &= -e_1 \otimes e_2 \otimes f_{3n+1} + e_1 \otimes f_{3n+1} \otimes e_2 + e_1 \otimes e_1 \otimes f_{3n+2} - e_1 \otimes f_{3n+2} \otimes e_1, \\
 \Delta_{Ass}(f_{3n-1}) &= -e_2 \otimes \Delta(f_{3n}) = -e_2 \otimes e_2 \otimes f_{3n+1} \\
 &+ e_2 \otimes f_{3n+1} \otimes e_2 + e_2 \otimes e_1 \otimes f_{3n+2} - e_2 \otimes f_{3n+2} \otimes e_1,
 \end{aligned}$$

and

$$\begin{aligned}
 \Delta_{Ass}(f_{3n}) &= (\Delta \otimes \text{id} - \text{id} \otimes \Delta)(e_2 \otimes f_{3n+1} - f_{3n+1} \otimes e_2 - e_1 \otimes f_{3n+2} + f_{3n+2} \otimes e_1) \\
 &= -\Delta(f_{3n+1}) \otimes e_2 + \Delta(f_{3n+2}) \otimes e_1 - e_2 \otimes \Delta(f_{3n+1}) + e_1 \otimes \Delta(f_{3n+2}) \\
 &= -e_1 \otimes f_{3(n+1)} \otimes e_2 + e_2 \otimes f_{3(n+1)} \otimes e_1 - e_2 \otimes e_1 \otimes f_{3(n+1)} + e_1 \otimes e_2 \otimes f_{3(n+1)}.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 (f, g, g)(f_{3n-2}) &= -f(e_1)g(e_2)g(f_{3n+1}) + f(e_1)g(f_{3n+1})g(e_2) \\
 &+ f(e_1)g(e_1)g(f_{3n+2}) - f(e_1)g(f_{3n+2})g(e_1) = 0,
 \end{aligned}$$

$$(f, g, g)(f_{3n-1}) = -f(e_2)g(e_2)g(f_{3n+1}) + f(e_2)g(f_{3n+1})g(e_2) + f(e_2)g(e_1)g(f_{3n+2}) - f(e_2)g(f_{3n+2})g(e_1) = 0,$$

and

$$(f, g, g)(f_{3n}) = -f(e_1)g(f_{3(n+1)})g(e_2) + f(e_2)g(f_{3(n+1)})g(e_1) - f(e_2)g(e_1)g(f_{3(n+1)}) + f(e_1)g(e_2)g(f_{3(n+1)}) = 0.$$

Consequently, $(f, g, g)(a) = 0$ for all $f, g \in A^*$ and $a \in A$, i.e., $(f, g, g) = 0$. This means that A^* satisfies the identity (8).

Set $\phi = (\Delta \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})\Delta$. Direct calculations give that

$$\phi(e_1) = \phi(e_2) = 0,$$

$$\phi(f_{3n-2}) = (\Delta \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})(e_1 \otimes f_{3n}) = (\Delta \otimes \text{id} \otimes \text{id})(\Delta(e_1) \otimes f_{3n}) = 0,$$

$$\phi(f_{3n-1}) = (\Delta \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})(e_2 \otimes f_{3n}) = (\Delta \otimes \text{id} \otimes \text{id})(\Delta(e_2) \otimes f_{3n}) = 0,$$

and

$$\begin{aligned} \phi(f_{3n}) &= (\Delta \otimes \text{id} \otimes \text{id})(-\Delta(f_{3n+1}) \otimes e_2 + \Delta(f_{3n+2}) \otimes e_1) = \\ &(\Delta \otimes \text{id} \otimes \text{id})(-e_1 \otimes f_{3(n+1)} \otimes e_2 + e_2 \otimes f_{3(n+1)} \otimes e_1) = 0 \end{aligned}$$

for all $n \geq 1$.

Consequently, $((fgh)e)(a) = 0$ for all $f, g, h, e \in A^*$ and $a \in A$, i.e., $((fgh)e) = 0$. This means that A^* satisfies the identity $((xy)z)t = 0$.

Now set $\psi = (\text{id} \otimes \Delta \otimes \text{id})(\text{id} \otimes \Delta)\Delta$. Then

$$\psi(e_1) = \psi(e_2) = 0,$$

$$\psi(f_{3n-2}) = (\text{id} \otimes \Delta \otimes \text{id})(\text{id} \otimes \Delta)(e_1 \otimes f_{3n}) = (\text{id} \otimes \Delta \otimes \text{id})(e_1 \otimes \Delta(f_{3n}))$$

$$= -e_1 \otimes \Delta(f_{3n+1}) \otimes e_2 + e_1 \otimes \Delta(f_{3n+2}) \otimes e_1$$

$$= -e_1 \otimes e_1 \otimes f_{3(n+1)} \otimes e_2 + e_1 \otimes e_2 \otimes f_{3(n+1)} \otimes e_1,$$

$$\psi(f_{3n-1}) = (\text{id} \otimes \Delta \otimes \text{id})(\text{id} \otimes \Delta)(e_2 \otimes f_{3n}) = (\text{id} \otimes \Delta \otimes \text{id})(e_2 \otimes \Delta(f_{3n}))$$

$$= -e_2 \otimes \Delta(f_{3n+1}) \otimes e_2 + e_2 \otimes \Delta(f_{3n+2}) \otimes e_1$$

$$= -e_2 \otimes e_1 \otimes f_{3(n+1)} \otimes e_2 + e_2 \otimes e_2 \otimes f_{3(n+1)} \otimes e_1,$$

and

$$\psi(f_{3n}) = (\text{id} \otimes \Delta \otimes \text{id})(\text{id} \otimes \Delta)(e_2 \otimes f_{3n+1} - f_{3n+1} \otimes e_2 - e_1 \otimes f_{3n+2} + f_{3n+2} \otimes e_1)$$

$$= (\text{id} \otimes \Delta \otimes \text{id})(e_2 \otimes \Delta(f_{3n+1}) - e_1 \otimes \Delta(f_{3n+2}))$$

$$= (\text{id} \otimes \Delta \otimes \text{id})(e_2 \otimes e_1 \otimes f_{3(n+1)} - e_1 \otimes e_2 \otimes f_{3(n+1)})$$

$$= e_2 \otimes \Delta(e_1) \otimes f_{3(n+1)} - e_1 \otimes \Delta(e_2) \otimes f_{3(n+1)} = 0$$

for all $n \geq 1$.

Therefore,

$$\begin{aligned} (f((gh)g))(f_{3n-2}) &= -f(e_1)g(e_1)h(f_{3(n+1)})g(e_2) + f(e_1)g(e_2)h(f_{3(n+1)})g(e_1) = 0, \\ (f((gh)g))(f_{3n-1}) &= 0 \end{aligned}$$

for all $f, g, h \in A^*$.

This means $f((gh)g) = 0$ for all $f, g, h \in A^*$. Together with the identity $((xy)z)t = 0$, this proves that the Moufang identity (7) holds in A^* .

It is easy to check that $(\Delta \otimes \Delta)\Delta(a) = 0$ for all $a \in A$. Therefore A^* satisfies the identity $(xy)(zt) = 0$.

Now we show that (A, Δ) is not locally finite. Let B a subalgebra of (A, Δ) generated by f_1, f_2 .

Suppose that $f_i \in B$. If $i = 3n - 1$ or $i = 3n - 2$, then $\Delta(f_i) \in B \otimes B$ implies that $e_1, e_2, f_{3n} \in B$ by Lemma 2. If $i = 3n$, then we get $e_1, e_2, f_{3n+1}, f_{3n+2} \in B$. This implies that $B = C$ is infinite-dimensional. \square

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Declaration of competing interest

There is no competing interest.

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