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On submajorisation of the Rotfeld's inequality

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Abstract. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra, $L_0(\mathcal{M})$ be the set of all τ-measurable operators, $\mu_t(x)$ be the generalized singular number of $x \in L_0(\mathcal{M})$. We proved that if $g : [0, \infty) \to [0, \infty)$ is an increasing continuous function, then for any x, y in $L_0(\mathcal{M})$,

$$\mu_t(g(|x + y|)) \leq \mu_t(g(\frac{1}{2}\begin{pmatrix} |x| + |y| & x^* + y^* \\ x + y & |x^*| + |y^*| \end{pmatrix})), \qquad 0 < t < \tau(1).$$

We also obtained that if $f : [0, \infty) \to [0, \infty)$ is a concave function, then $\mu(f(\frac{1}{2}\begin{pmatrix} |x| + |y| & x^* + y^* \\ x + y & |x^*| + |y^*| \end{pmatrix}))$ is submajorized by $\mu(f(|x|)) + \mu(f(|y|))$.

1. Introduction

We denote the set of all $n \times n$ complex matrices by $\mathbb{M}_n$ and the Schatten p-norm $(0 < p \leq \infty)$ by $\| \cdot \|_p$. Rotfel'd [13] proved that if $x,\ y \in \mathbb{M}_n$ and $f : [0, \infty) \to [0, \infty)$ is a concave function, then

$$\|f(|x + y|)\|_1 \leq \|f(|x|) + f(|y|)\|_1. \tag{1}$$

In the case $x,\ y \in \mathbb{M}_n$ are positive semidefinite and $f(t) = t^p$ $(0 < p < 1)$, the above inequality follows from a trace inequality in [12]. Ando and Zhan [2] extended (1) as following if x, y are positive semidefinite matrices, $\| \cdot \|$ is a symmetric norm and $f : [0, \infty) \to [0, \infty)$ is an operator concave functions, then

$$\|f(x + y)\| \leq \|f(x) + f(y)\| \tag{2}$$

holds. Bourin and Uchiyama proved that for concave function $f : [0, \infty) \to [0, \infty)$, (2) remain holds (see [3]). In [7], Dodds and Sukochev showed that this inequality continue to hold in the more general setting of measurable operators affiliated with a semi-finite von Neumann algebra.

Uchiyama [15] extended (1) that for any unitarily invariant norm $\| \cdot \|$ and concave function $f : [0, \infty) \to [0, \infty)$,

$$\|f(|x + y|)\| \leq \|f(|x|)\| + \|f(|y|)\|, \qquad x, y \in \mathbb{M}_n. \tag{3}$$

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In [16], the author interpolated (3) by proving that for all unitarily invariant norms $\|\cdot\|$ and concave functions $f : [0,\infty) \to [0,\infty)$,

$$\|f(|x+y|)\| \le \|f(\frac{1}{2}\begin{pmatrix} |x|+|y| & x^*+y^* \\ x+y & |x^*|+|y^*| \end{pmatrix})\| \le \|f(|x|)\| + \|f(|y|)\|, \qquad x,y \in \mathbb{M}_n. \tag{4}$$

In [10], (1) is extended by Lee to the whole class of unitarily invariant norms $\|\cdot\|$ and concave functions $f : [0,\infty) \to [0,\infty)$,

$$\|f(\begin{pmatrix} x & z^* \\ z & y \end{pmatrix})\| \le \|f(|x|)\| + \|f(|x|)\|, \qquad 0 \le \begin{pmatrix} x & z^* \\ z & y \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n). \tag{5}$$

In this paper, we prove that if $g : [0,\infty) \to [0,\infty)$ is an increasing continuous function, then for any τ-measurable operators x, y,

$$\mu_t(g(|x+y|)) \le \mu_t(g(\frac{1}{2}\begin{pmatrix} |x|+|y| & x^*+y^* \\ x+y & |x^*|+|y^*| \end{pmatrix})) \qquad 0 < t < \tau(1).$$

As application, we extend (4) to the τ-measurable operators case. We also proved the τ-measurable operators analogue of (5).

2. Preliminaries

We denote by $L_0(0,\alpha)$ the space of Lebesque measurable real-valued functions f on $(0,\alpha)$ and define the decreasing rearrangement function $f^* : (0,\alpha) \mapsto (0,\alpha)$ for $f \in L_0(0,\alpha)$ by

$$f^*(t) = \inf\{s > 0 : \ \mu(\{\omega \in (0,\alpha) : \ |f(\omega)| > s\}) \le t\}, \qquad t \ge 0.$$

If $f, g \in L_0(0,\alpha)$ satisfy that $\int_0^t f^*(s)ds \le \int_0^t g^*(s)ds$ for all $t \ge 0$, we say that f is *submajorized* by g, denote by $f \preccurlyeq g$. Let E be a Banach subspace of $L_0(0,\alpha)$, simply called a Banach function space on $(0,\alpha)$ in the sequel. E is called to be *symmetric* if, for $f \in E$ and $g \in L_0(0,\alpha)$ such that $g^*(t) \le f^*(t)$ for all $t \ge 0$, one has $g \in E$ and $\|g\|_E \le \|f\|_E$; E is *fully symmetric* if, for $f \in L_0(0,\alpha)$ and $g \in E$ such that $f \preccurlyeq g$, we have $f \in E$ and $\|f\|_E \le \|g\|_E$.

Throughout this paper, $\mathcal{M}$ always denotes a semi-finite von Neumann algebra with a faithful normal semi-finite trace τ. The set of all τ-measurable operators is denoted by $L_0(\mathcal{M})$. For $x \in L_0(\mathcal{M})$, we define the *distribution function* $\lambda(x)$ of x by $\lambda_t(x) = \tau(e_{(t,\infty)}(|x|))$ for $t > 0$, where $e_{(t,\infty)}(|x|)$ is the spectral projection of $|x|$ in the interval (t,∞), and define the generalized singular numbers $\mu(x)$ of x by $\mu_t(x) = \inf\{s > 0 : \ \lambda_s(x) \le t\}$ for $t > 0$. It is easy to check that $\mu_t(x) = 0$, for all $t \ge \tau(1)$. For further information about elementary properties of the generalized singular numbers, we refer the reader to [8].

Given a symmetric Banach function space E on $(0,\alpha)$ $(\tau(1) = \alpha)$, the space

$$E(\mathcal{M},\tau) = \{x \in L_0(\mathcal{M}) : \ \|\mu(x)\|_E < \infty\}$$

is a Banach space under the norm $\|x\|_E = \|\mu(x)\|_E$, denoted by $E(\mathcal{M})$ for convenience. It is called non-commutative symmetric space. If $1 \le p \le \infty$ and $E = L_p(0,\alpha)$, then $E(\mathcal{M}) = L_p(\mathcal{M})$, which are the usual noncommutative L_p-spaces associated with $(\mathcal{M},\tau)$. For more details on noncommutative symmetric spaces, see [6, 11]

We remark that if $\mathcal{M} = \mathbb{M}_n$ and τ is the standard trace, then

$$\mu_t(x) = s_j(x), \quad t \in [j-1, j), \quad j = 1, 2, \cdots.$$

Hence, if $x, y \in \mathbb{M}_n$, then $\mu(y) \preccurlyeq \mu(x)$ is equivalent to

$$\sum_{j=1}^{k} s_j(y) \le \sum_{j=1}^{k} s_j(x), \quad 1 \le k \le n.$$

Let x, y be self-adjoint elements of $\mathcal{M}$, we say that x spectrally dominates y, denoted by $y \preceq x$, if $e_{(\alpha,\infty)}(y)$ is equivalent, in the sense of Murray-von Neumann, to a subprojection of $e_{(\alpha,\infty)}(x)$ for every real number α (see [5]).

We will use the following result (see [9, Lemma 2.2]).

Lemma 2.1. *Let* $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ *be a matrix with entries in* $\mathcal{M}$. *Then* $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geq 0$ *if and only if* $x \geq 0,\ y \geq 0$ *and there exists a contraction a such that* $z = x^{\frac{1}{2}} a y^{\frac{1}{2}}$.

3. Main results

We denote by $\mathbb{M}_2(\mathcal{M})$ the semifinite von Neumann algebra

$$\mathbb{M}_2(\mathcal{M}) = \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix},\ x_{i,j} \in \mathcal{M},\ i, j = 1, 2 \right\}$$

on Hilbert space $\mathcal{H} \oplus \mathcal{H}$ with trace $Tr \otimes \tau$.

Lemma 3.1. *Let* $f : [0, \infty) \to [0, \infty)$ *be a convex function. If* $x, y \in L_0(\mathcal{M})$, *then* $\mu(f(|x + y|) \preccurlyeq \frac{1}{2}\mu(f(|x| + |y|)) + \frac{1}{2}\mu(f(|x^*| + |y^*|))$.

Proof. Suppose that $x, y \in \mathcal{M}$. Let $x = u|x|$ be the polar decomposition of x. Then $x = |x^*|^{\frac{1}{2}} u |x|^{\frac{1}{2}}$. Using Lemma 2.1, we obtain that $\begin{pmatrix} |x^*| & x \\ x^* & |x| \end{pmatrix} \geq 0$. Similarly, $\begin{pmatrix} |y^*| & y \\ y^* & |y| \end{pmatrix} \geq 0$, and so $\begin{pmatrix} |x^*| + |y^*| & x + y \\ x^* + y^* & |x| + |y| \end{pmatrix} \geq 0$. Also by Lemma 2.1, there is a contraction b such that $x + y = (|x^*| + |y^*|)^{\frac{1}{2}} b(|x| + |y|)^{\frac{1}{2}}$. We use [8, Lemma 2.5, Theorem 4.2(iii)] to obtain that

$$\begin{aligned}
\int_0^t \mu_s(f(|x + y|))ds &= \int_0^t f(\mu_s(|x + y|))ds \\
&= \int_0^t f(\mu_s(|(|x^*| + |y^*|)^{\frac{1}{2}} b(|x| + |y|)^{\frac{1}{2}}|))ds \\
&\leq \int_0^t f(\mu_s((|x^*| + |y^*|)^{\frac{1}{2}})\mu_s(b(|x| + |y|)^{\frac{1}{2}}))ds \\
&\leq \int_0^t f(\mu_s(|x^*| + |y^*|)^{\frac{1}{2}} \mu_s(|x| + |y|)^{\frac{1}{2}})ds \\
&\leq \int_0^t f(\frac{1}{2}\mu_s(|x^*| + |y^*|) + \frac{1}{2}\mu_s(|x| + |y|))ds \\
&\leq \int_0^t [\frac{1}{2} f(\mu_s(|x^*| + |y^*|)) + \frac{1}{2} f(\mu_s(|x| + |y|))]ds \\
&\leq \int_0^t [\frac{1}{2}\mu_s(f(|x^*| + |y^*|)) + \frac{1}{2}\mu_s(f(|x| + |y|))]ds.
\end{aligned}$$

Now assume that $x, y \in L_0(\mathcal{M})$. Set $x_n = xe_{[0,n]}(|x|)$ and $y_n = ye_{[0,n]}(|y|)$ for $n \in \mathbb{N}$. Then $x_n \to x$ and $y_n \to y$ in measure (see [8]). It follows that

$$|x_n| \to |x|, \qquad |y_n| \to |y|$$

in measure (see [14]). On the other hand,

$$|x_n| = |x|e_{[0,n]}(|x|) \leq |x|, \quad |(x_n)^*| \leq |x^*|, \qquad |y_n| = |y|e_{[0,n]}(|y|) \leq |y|, \quad |(y_n)^*| \leq |y^*|. \tag{6}$$

Applying [8, Lemma 3.4], usual Fatou's lemma, the first case and (6), we deduce that

$$\begin{aligned}\int_0^t \mu_s(f(|x+y|))ds &\leq \int_0^t \liminf_{n\to\infty} \mu_s(f(|x_n+y_n|))ds\\ &\leq \liminf_{n\to\infty}\int_0^t \mu_s(f(|x_n+y_n|))ds\\ &\leq \liminf_{n\to\infty}\int_0^t [\frac{1}{2}\mu_s(f(|x_n^*|+|y_n^*|))+\frac{1}{2}\mu_s(f(|x_n|+|y_n|))]ds\\ &\leq \int_0^t [\frac{1}{2}\mu_s(f(|x^*|+|y^*|))+\frac{1}{2}\mu_s(f(|x|+|y|))]ds.\end{aligned}$$

□

Theorem 3.2. *Let E be a fully symmetric Banach function space on* $(0,\alpha)$. *If* $f:[0,\infty)\to[0,\infty)$ *is a convex function, then*

$$\|f(|x+y|)\|_E \leq \max\{\|f(|x|+|y|)\|_E,\ \|f(|x^*|+|y^*|)\|_E\},\qquad x,y\in E(\mathcal{M}).$$

Proof. By Lemma 3.1, we obtain that

$$\begin{aligned}\|\|f(|x+y|)\|_E &= \|\mu(f(|x+y|))\|_E\\ &\leq \|\frac{1}{2}\mu(f(|x|+|y|))+\frac{1}{2}\mu(f(|x^*|+|y^*|))\|_E\\ &\leq \frac{1}{2}\|\mu(f(|x|+|y|))\|_E+\frac{1}{2}\|\mu(f(|x^*|+|y^*|))\|_E\\ &= \frac{1}{2}\|f(|x|+|y|)\|_E+\frac{1}{2}\|f(|x^*|+|y^*|)\|_E\\ &\leq \max\{\|f(|x|+|y|)\|_E,\ \|f(|x^*|+|y^*|)\|_E\}.\end{aligned}$$

□

Lemma 3.3. *If* $x\in L_0(\mathcal{M})$, *then*

$$\begin{pmatrix}0 & x^*\\ x & 0\end{pmatrix}_+ = \frac{1}{2}\begin{pmatrix}|x| & x^*\\ x & |x^*|\end{pmatrix},\qquad \begin{pmatrix}0 & x^*\\ x & 0\end{pmatrix}_- = \frac{1}{2}\begin{pmatrix}|x| & -x^*\\ -x & |x^*|\end{pmatrix}.$$

Proof. Since $\begin{pmatrix}0 & x^*\\ x & 0\end{pmatrix}$ is Hermitian operator and $\left|\begin{pmatrix}0 & x^*\\ x & 0\end{pmatrix}\right| = \begin{pmatrix}|x| & 0\\ 0 & |x^*|\end{pmatrix}$,

$$\begin{pmatrix}0 & x^*\\ x & 0\end{pmatrix}_+ = \frac{1}{2}\left[\begin{pmatrix}0 & x^*\\ x & 0\end{pmatrix}+\begin{pmatrix}|x| & 0\\ 0 & |x^*|\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}|x| & x^*\\ x & |x^*|\end{pmatrix}.$$

The second result follows analogously. □

Lemma 3.4. *If* $x,y\in L_0(\mathcal{M})$, *then*

$$\begin{pmatrix}|x+y| & x^*+y^*\\ x+y & |x^*+y^*|\end{pmatrix} \leq \begin{pmatrix}|x|+|y| & x^*+y^*\\ x+y & |x^*|+|y^*|\end{pmatrix}.$$

Consequently,

$$\mu_t(\begin{pmatrix}|x+y| & x^*+y^*\\ x+y & |x^*+y^*|\end{pmatrix}) \leq \mu_t(\begin{pmatrix}|x|+|y| & x^*+y^*\\ x+y & |x^*|+|y^*|\end{pmatrix}),\qquad t>0.$$

Proof. Since $\begin{pmatrix} 0 & x^* + y^* \\ x+y & 0 \end{pmatrix}$ is Hermitian operator and

$$\begin{aligned}
\begin{pmatrix} 0 & x^* + y^* \\ x+y & 0 \end{pmatrix} &= \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix} \\
&= \left[\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}_+ - \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}_-\right] + \left[\begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix}_+ - \begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix}_-\right] \\
&= \left[\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}_+ + \begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix}_+\right] - \left[\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}_- + \begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix}_-\right].
\end{aligned}$$

By minimality of the Jordan decomposition (see [5, Lemma 6])), we get that

$$\begin{pmatrix} 0 & x^* + y^* \\ x+y & 0 \end{pmatrix}_+ \le \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}_+ + \begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix}_+ .$$

Using Lemma 3.3, we obtain the desired result. □

Theorem 3.5. *Let $x, y \in L_0(\mathcal{M})$. If $g : [0, \infty) \to [0, \infty)$ is an increasing continuous function, then*

$$\mu_t(g(|x+y|)) \le \mu_t(g(\frac{1}{2}\begin{pmatrix} |x| + |y| & x^* + y^* \\ x+y & |x^*| + |y^*| \end{pmatrix})), \qquad 0 < t < \tau(1).$$

Proof. Using[8, Lemma 2.5] and [1, Lemma 2.1] , we get that for $0 < t < \tau(1)$,

$$\begin{aligned}
\mu_t(g(|x+y|)) &= g(\mu_t(|x+y|)) = g(\mu_t(\begin{pmatrix} |x+y| & 0 \\ 0 & 0 \end{pmatrix})) \\
&= g(\tfrac{1}{2}\mu_t(\left|\begin{pmatrix} |x+y|^{\frac{1}{2}} & 0 \\ |x+y|^{\frac{1}{2}} & 0 \end{pmatrix}\right|)^2) \\
&= g(\tfrac{1}{2}\mu_t(\left|\begin{pmatrix} |x+y|^{\frac{1}{2}} & |x+y|^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}\right|)^2) \\
&= g(\tfrac{1}{2}\mu_t(\begin{pmatrix} |x+y| & |x+y| \\ |x+y| & |x+y| \end{pmatrix}))
\end{aligned}$$

Let $x + y = u|x + y|$ be the polar decomposition of $x + y$. Then

$$\begin{pmatrix} |x+y| & |x+y| \\ |x+y| & |x+y| \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u^* \end{pmatrix}\begin{pmatrix} |x+y| & x^* + y^* \\ x+y & |x^* + y^*| \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}.$$

Hence, by [8, Lemma 2.5] and Lemma 3.4,

$$\begin{aligned}
\mu_t(g(|x+y|)) &= g(\tfrac{1}{2}\mu_t(\begin{pmatrix} |x+y| & |x+y| \\ |x+y| & |x+y| \end{pmatrix})) \\
&\le g(\tfrac{1}{2}\mu_t(\begin{pmatrix} |x+y| & x^* + y^* \\ x+y & |x^* + y^*| \end{pmatrix})) \\
&\le g(\tfrac{1}{2}\mu_t(\begin{pmatrix} |x| + |y| & x^* + y^* \\ x+y & |x^*| + |y^*| \end{pmatrix})) \\
&\le \mu_t(g(\tfrac{1}{2}\begin{pmatrix} |x| + |y| & x^* + y^* \\ x+y & |x^*| + |y^*| \end{pmatrix})).
\end{aligned}$$

□

Theorem 3.6. *Let $x, y \in L_0(\mathcal{M})$. If $f : [0, \infty) \to [0, \infty)$ is a concave function, then*

$$\mu(f(|x+y|)) \preccurlyeq \mu(f(\frac{1}{2}\begin{pmatrix} |x| + |y| & x^* + y^* \\ x+y & |x^*| + |y^*| \end{pmatrix})) \preccurlyeq \mu(f(|x|)) + \mu(f(|y|)).$$

Proof. We use Theorem 3.5 and [7, Theorem 5.3], we obtain that for $0 < t < \tau(1)$,

$$\begin{aligned}
\int_0^t \mu_s(f(|x+y|))ds &\le \int_0^t \mu_s(f(\tfrac{1}{2}\begin{pmatrix} |x|+|y| & x^*+y^* \\ x+y & |x^*|+|y^*| \end{pmatrix}))ds \\
&= \int_0^t \mu_s(f(\tfrac{1}{2}\begin{pmatrix} |x| & x^* \\ x & |x^*| \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} |y| & y^* \\ y & |y^*| \end{pmatrix}))ds \\
&\le \int_0^t \mu_s(f(\tfrac{1}{2}\begin{pmatrix} |x| & x^* \\ x & |x^*| \end{pmatrix}) + f(\tfrac{1}{2}\begin{pmatrix} |y| & y^* \\ y & |y^*| \end{pmatrix}))ds \\
&\le \int_0^t \mu_s(f(\tfrac{1}{2}\begin{pmatrix} |x| & x^* \\ x & |x^*| \end{pmatrix}))ds + \int_0^t \mu_s(f(\tfrac{1}{2}\begin{pmatrix} |y| & y^* \\ y & |y^*| \end{pmatrix}))ds.
\end{aligned}$$

Let $x = u|x|$ and $y = v|y|$ be the polar decompositions of x and y. Then

$$\begin{aligned}
\int_0^t \mu_s(f(|x+y|))ds &\le \int_0^t \mu_s(f(\tfrac{1}{2}\begin{pmatrix} |x| & x^* \\ x & |x^*| \end{pmatrix}))ds + \int_0^t \mu_s(f(\tfrac{1}{2}\begin{pmatrix} |y| & y^* \\ y & |y^*| \end{pmatrix}))ds \\
&= \int_0^t f(\tfrac{1}{2}\mu_s(\begin{pmatrix} |x| & x^* \\ x & |x^*| \end{pmatrix}))ds + \int_0^t f(\tfrac{1}{2}\mu_s(\begin{pmatrix} |y| & y^* \\ y & |y^*| \end{pmatrix}))ds \\
&= \int_0^t f(\tfrac{1}{2}\mu_s(\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} |x| & |x| \\ |x| & |x| \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u^* \end{pmatrix}))ds \\
&\quad + \int_0^t f(\tfrac{1}{2}\mu_s(\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} |y| & |y| \\ |y| & |y| \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix}))ds \\
&\le \int_0^t f(\tfrac{1}{2}\mu_s(\begin{pmatrix} |x| & |x| \\ |x| & |x| \end{pmatrix}))ds + \int_0^t f(\tfrac{1}{2}\mu_s(\begin{pmatrix} |y| & |y| \\ |y| & |y| \end{pmatrix}))ds \\
&= \int_0^t \mu_s(f(\begin{pmatrix} |x| & 0 \\ 0 & 0 \end{pmatrix}))ds + \int_0^t \mu_s(f(\begin{pmatrix} |y| & 0 \\ 0 & 0 \end{pmatrix}))ds \\
&= \int_0^t [\mu_s(f(|x|)) + \mu_s(f(|y|))]ds.
\end{aligned}$$

□

Corollary 3.7. *Let E be a fully symmetric Banach function space on $(0,\alpha)$. If $f : [0,\infty) \to [0,\infty)$ is a concave function, then*

$$\|f(|x+y|)\|_E \le \|f(\frac{1}{2}\begin{pmatrix} |x|+|y| & x^*+y^* \\ x+y & |x^*|+|y^*| \end{pmatrix})\|_E \le \|f(|x|)\|_E + \|f(|y|)\|_E, \qquad x, y \in E(\mathcal{M}).$$

We use the method in the proof of [4, Lemma 3.4] to obtain the following result.

Proposition 3.8. *Let $f : [0,\infty) \to [0,\infty)$ be a concave function. If $\begin{pmatrix} x & z^* \\ z & y \end{pmatrix}$ is positive operator in $L_0(\mathbb{M}_2(\mathcal{M}))$, then*

$$\mu(f(\begin{pmatrix} x & z^* \\ z & y \end{pmatrix})) \preccurlyeq \mu(f(x)) + \mu(f(y)).$$

Proof. Since $\begin{pmatrix} x & z^* \\ z & y \end{pmatrix}$ is positive operator in $L_0(\mathbb{M}_2(\mathcal{M}))$, there is a positive $\begin{pmatrix} a & c^* \\ c & b \end{pmatrix} \in L_0(\mathbb{M}_2(\mathcal{M}))$ such that

$$\begin{pmatrix} x & z^* \\ z & y \end{pmatrix} = \begin{pmatrix} a & c^* \\ c & b \end{pmatrix}\begin{pmatrix} a & c^* \\ c & b \end{pmatrix}. \tag{7}$$

Hence,

$$\begin{aligned}
\begin{pmatrix} x & z^* \\ z & y \end{pmatrix} &= \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} a & c^* \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c^* \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 0 \\ c & b \end{pmatrix} \\
&= \left|\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}^*\right|^2 + \left|\begin{pmatrix} 0 & 0 \\ c & b \end{pmatrix}\right|^2.
\end{aligned}$$

By [7, Theorem 5.3] and (7), we get

$$\begin{aligned}
\int_0^t \mu_s(f(\begin{pmatrix} x & z^* \\ z & y \end{pmatrix}))ds &= \int_0^t \mu_s(f(\left|\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}^*\right|^2 + \left|\begin{pmatrix} 0 & 0 \\ c & b \end{pmatrix}\right|^2))ds \\
&\leq \int_0^t \mu_s(f(\left|\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}^*\right|^2) + f(\left|\begin{pmatrix} 0 & 0 \\ c & b \end{pmatrix}\right|^2))ds \\
&\leq \int_0^t [\mu_s(f(\left|\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}^*\right|^2)) + \mu_s(f(\left|\begin{pmatrix} 0 & 0 \\ c & b \end{pmatrix}\right|^2))]ds \\
&\leq \int_0^t [f(\mu_s(\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}^*)^2) + f(\mu_s(\begin{pmatrix} 0 & 0 \\ c & b \end{pmatrix})^2)]ds \\
&\leq \int_0^t [f(\mu_s(\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix})^2) + f(\mu_s(\begin{pmatrix} 0 & 0 \\ c & b \end{pmatrix}^*)^2)]ds \\
&= \int_0^t [f(\mu_s(\begin{pmatrix} a^2 + c^*c & 0 \\ 0 & 0 \end{pmatrix})) + f(\mu_s(\begin{pmatrix} 0 & 0 \\ 0 & cc^* + b^2 \end{pmatrix}))]ds \\
&= \int_0^t [f(\mu_s(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix})) + f(\mu_s(\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}))]ds \\
&= \int_0^t [\mu_s(f(x)) + \mu_s(f(y))]ds
\end{aligned}$$

□

Theorem 3.9. *Let $x, y \in L_0(\mathcal{M})$. If $f : [0, \infty) \to [0, \infty)$ is a concave function, then*

$$\mu(f(|x + y|)) \preccurlyeq \mu(f(\frac{|x| + |y|}{2})) + \mu(f(\frac{|x^*| + |y^*|}{2})).$$

Proof. Since $\begin{pmatrix} \frac{|x|+|y|}{2} & \frac{x^*+y^*}{2} \\ \frac{x+y}{2} & \frac{|x^*|+|y^*|}{2} \end{pmatrix} \geq 0$, using Theorem 3.5 and 3.8, we obtain the desired result. □

Corollary 3.10. *Let E be a fully symmetric Banach function space on $(0, \alpha)$. If $f : [0, \infty) \to [0, \infty)$ is a concave function, then*

$$\|f(|x + y|)\|_E \leq \|f(\frac{|x| + |y|}{2})\|_E + \|f(\frac{|x^*| + |y^*|}{2})\|_E, \qquad x, y \in E(\mathcal{M}).$$

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