Hindawi
Abstract and Applied Analysis
Volume 2023, Article ID 2488165, 8 pages
https://doi.org/10.1155/2023/2488165

Research Article

On the Heat and Wave Equations with the Sturm-Liouville Operator in Quantum Calculus

Serikbol Shaimardan,[1] **Lars-Erik Persson**,[2,3] **and Nariman Tokmagambetov**[4]

[1]*L. N. Gumilyov Eurasian National University, Astana, Kazakhstan*
[2]*Department of Mathematics and Computer Science, Karlstad University, Karlstad, Sweden*
[3]*Department of Computer Science and Computational Engineering, Campus Narvik, The Arctic University of Norway, Narvik, Norway*
[4]*Karagandy University of the Name of Academician E.A. Buketov, Karaganda, Kazakhstan*

Correspondence should be addressed to Nariman Tokmagambetov; nariman.tokmagambetov@gmail.com

Received 2 October 2022; Revised 29 November 2022; Accepted 12 December 2022; Published 19 January 2023

Academic Editor: Douglas R. Anderson

In this paper, we explore a generalised solution of the Cauchy problems for the q-heat and q-wave equations which are generated by Jackson's and the q-Sturm-Liouville operators with respect to t and x, respectively. For this, we use a new method, where a crucial tool is used to represent functions in the Fourier series expansions in a Hilbert space on quantum calculus. We show that these solutions can be represented by explicit formulas generated by the q-Mittag-Leffler function. Moreover, we prove the unique existence and stability of the weak solutions.

1. Introduction

In the last decade, the theory of quantum groups and q-deformed algebras have been the subject of intense investigation. Many physical applications have been investigated on the basis of the q-deformation of the Heisenberg algebra (see [1, 2]). For instance, the q-deformed Schrödinger equations have been proposed in [3, 4], and applications to the study of q-deformed version of the hydrogen atom and of the quantum harmonic oscillator have been presented (see [5]). Fractional calculus and the q-deformed Lie algebras are closely related. A new class of fractional q-deformed Lie algebras is proposed, which for the first time allows a smooth transition between the different Lie algebras (see [6]).

The origin of the q-difference calculus can be traced back to the works by Jackson (see [7, 8]) and Carmichael (see [9]) from the beginning of the twentieth century, while basic definitions and properties can be found, e.g., in the monographs [10, 11] and the PhD thesis [12]. Recently, the fractional q-difference calculus has been proposed by Al-salam (see [13]) and Agarwal (see [14]). We can also mention papers [15, 16], where the authors investigated the explicit solutions to linear fractional q-differential equations with the q-fractional derivative, and in [17], the q-analogue nonhomogeneous wave equations were studied.

A motivation behind this work is to state some new results about the q-heat and q-wave equations associated to the q-Sturm-Liouville operator (see (10)). We attempt to extend the heat representation theory studied in some cases (see [18–20], etc.). We define a generalised solution of the Cauchy problem for these equations generated by the q-Mittag-Leffler function and the q-associated functions of a biorthogonal system (see (13)). We investigate the well-posedness of the Cauchy problem for the q-heat and q-wave equations for operators with a discrete nonnegative spectrum acting on $L_q^2[0,1]$. In particular, we prove both unique existence and stability of the corresponding the generalised solution.

The paper is organized as follows: the main results are presented and proved in Section 3 and Section 4. In order to not disturb these presentations, we include in Section 2 some necessary Preliminaries.

2. Preliminaries

In this section, we recall some notations and basic facts in q-calculus. We will always assume that $0<q<1$. The q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}.$$

The q-shifted factorial is defined by

$$(a;q)_n = \begin{cases} 1, & n=0, \\ (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right), & n\in\mathbb{N}. \end{cases}$$

Moreover, their natural expansions to the reals are

$$(a-b)_q^\alpha = a^\alpha \frac{(b/a;q)_\infty}{(q^\alpha b/a;q)_\infty}, (a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, (a;q)_\infty = \prod_{i=0}^{\infty}\left(1-aq^i\right). \tag{1}$$

The Jackson's q-difference operator $D_q f(x)$ is (see, [8, 12] Section 2.1])

$$D_q f(x) = \frac{f(x)-f(qx)}{x(1-q)}. \tag{2}$$

The q-derivative D_q of a product of the functions f and g as defined by

$$D_q(fg)(x) = f(qx)D_q(g)(x) + D_q(f)(x)g(x). \tag{3}$$

As given in [10], two q-analogues of the exponential functions are defined by

$$e_q^x = \frac{1}{((1-q)x;q)_\infty}, E_q^x = (-(1-q)x;q)_\infty. \tag{4}$$

Moreover, we have that

$$D_q e_q^x = e_q^x, D_q E_q^{-x} = E_q^{-qx}, e_q^x E_q^{-x} = 1. \tag{5}$$

Due to the various types of q-differences introduced in quantum calculus, trigonometric functions have various q-analogues (see, [21] Section 2 [10], Section 10 and [12], Section 2.12). The following definition of cosine and sine will be useful in this investigation (see [20]):

$$\cos\left(z;q^2\right) = \sum_{k=0}^{\infty}\frac{(-1)^k q^{k^2} z^{2k}}{[2k]_q!}, \sin\left(z;q^2\right) = \sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k+1)} z^{2k+1}}{[2k+1]_q!}, \tag{6}$$

where the q-analogue of the binomial coefficients $[n]_q!$ is defined by

$$[n]_q! = \begin{cases} 1, & \text{if } n=0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q, & \text{if } n \in N. \end{cases}$$

The q-integral (or Jackson's integral) is defined by (see [8])

$$\int_0^x f(t)d_q t = (1-q)x\sum_{m=0}^{\infty} q^m f(xq^m), \tag{7}$$

and a more general form is given by

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x,$$

for $0<a<b$.

The q-version of integration by parts reads

$$\int_a^b f(x)D_q g(x)d_q x = [fg]_a^b - \int_a^b g(qx)D_q f(x)d_q x, \tag{8}$$

and if $f\equiv 1$, then we get that

$$\int_a^b D_q g(x)d_q x = g(b)-g(a). \tag{9}$$

The q -Sturm-Liouville Problem. Let $L_q^2[0,1]$ be the space of all real-valued functions defined on $[0,1]$ such that

$$\|f\|_{L_q^2[0,1]} := \left(\int_0^1 |f(x)|^2 d_q x\right)^{1/2} < \infty.$$

The space $L_q^2[0,1]$ is a separable Hilbert space with the inner product:

$$\langle f,g\rangle := \int_0^1 f(x)g(x)d_q x, f,g\in L_q^2[0,1].$$

Now, we shortly describe the study introduced by Annaby and Mansour in of a basic q-Sturm-Liouville eigenvalue problem in a Hilbert space (see [21], Chapter 3). In particular, they investigated the basic q-Sturm-Liouville equation:

$$-\frac{1}{q}D_{q^{-1}}D_q y(x) + v(x)y(x) = \lambda y(x), (0\le x\le 1;\lambda\in\mathbb{C}),$$

where $v(\cdot)$ is defined on $[0,1]$ and continuous at zero. Let $C_{q,0}^2[0,1]$ denotes the space of all functions $y(\cdot)$ such that y

and $D_q y$ are continuous at zero. If $v \equiv 0$, then we get the operator $\mathscr{L}$ in the following form:

$$\mathscr{L} := \begin{cases} -\frac{1}{q} D_{q^{-1}} D_q y(x) = \lambda y(x), \\ y(0) = y(1) = 0, \end{cases} \tag{10}$$

for $0 \le x \le 1$ and $\lambda \in \mathbb{R}$. The operator $\mathscr{L}$ is self adjoint on $C^2_{q,0}[0,1] \cap L^2_q[0,1]$ (see [21], Theorem 3.4.). A fundamental set of solutions of (10) are $\cos(\sqrt{\lambda}; q^2)$ and $\sin(\sqrt{\lambda}; q^2)/\sqrt{\lambda}$. Moreover, the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ are the zeros of $\sin(\sqrt{\lambda_k}; q^2)$, where

$$\lambda_k = (1-q)^{-2} q^{-2k+2\mu_k^{-1/2}}, k = 0, 1, \cdots, \tag{11}$$

and $\sum_{k=1}^{\infty} \mu_k < \infty$, $0 \le \mu_k \le 1$, and

$$\lambda_0 := (1-q)^{-2} q \le \lambda_k, k = 1, 2, 3, \cdots. \tag{12}$$

Additionally, the corresponding set of eigenfunctions $\{\sin(\sqrt{\lambda_k}; q^2)/\sqrt{\lambda_k}\}_{k=1}^{\infty}$ is an orthogonal basis in $L^2_q(0,1)$. Thus, we can identify $f \in L^2_q[0,1]$ with its Fourier series:

$$f(x) := \sum_{k=1}^{\infty} \langle f, \phi_k \rangle \phi_k(x),$$

where

$$\phi_k(x) = \frac{\sin\left(\sqrt{\lambda_k} x; q^2\right)}{\sqrt{\lambda_k}}. \tag{13}$$

The Sobolev Space Associated with $\mathscr{L}$. The next step is to recall the essential elements of the Fourier analysis presented in [22–24], as well as its applications to the spectral properties of $\mathscr{L}$. The space $C^{\infty}_{\mathscr{L}}[0,1] := \bigcap_{m=1}^{\infty} \mathrm{Dom}(\mathscr{L}^m)$ is called the space of test functions for $\mathscr{L}$, where

$$\mathrm{Dom}(\mathscr{L}^m) := \left\{ f \in L^2_q[0,1] : \mathscr{L}^j f \in \mathrm{Dom}(\mathscr{L}), j = 0, 1, 2, \cdots, m-1 \right\}.$$

For $g \in C^{\infty}_{\mathscr{L}}[0,1]$, we introduce the Fréchet topology of $C^{\infty}_{\mathscr{L}}[0,1]$ by the family of norms:

$$\|g\|_{C^m_{\mathscr{L}}[0,1]} := \max_{i \le m} \left\| \mathscr{L}^i g \right\|_{L^2_q[0,1]}.$$

The space of $\mathscr{L}$-distributions $\mathscr{D}'_{\mathscr{L}}[0,1] := L(C^{\infty}_{\mathscr{L}}[0,1], \mathbb{R})$ is the space of all linear continuous functionals on $C^{\infty}_{\mathscr{L}}[0,1]$.

Thus, for $s \in \mathbb{R}$, we can also define the Sobolev spaces $W^s_{q,\mathscr{L}}$ associated to $\mathscr{L}$ in the following form:

$$W^s_{q,\mathscr{L}} := \left\{ f \in \mathscr{D}'_{\mathscr{L}}[0,1] : \mathscr{L}^{s/2} f \in L^2_q[0,1] \right\},$$

with the norm $\|f\|_{W^s_{q,\mathscr{L}}} := \|\mathscr{L}^{s/2} f\|_{L^2_q[0,1]}$.

For $m \in \mathbb{N}_0$, we introduce the space $C^m_q([0,1]; W^s_{q,\mathscr{L}}[0,1])$ defined by the norms

$$\|u\|_{C^m_q\left([0,T]; W^s_{q,\mathscr{L}}[0,1]\right)} := \sum_{n=0}^{m} \max_{0 \le t \le T} \left\| D^n_{q,t} u(t,.) \right\|_{W^s_{q,\mathscr{L}}[0,1]}, 0 < T < \infty,$$

where the q-partial differential operator $D_{q,t} u(t,x)$ with respect to t has the following form:

$$D_{q,t} u(t,x) = \frac{u(t,x) - u(qt,x)}{(1-q)t}.$$

Notation: the symbol $M \lesssim K$ means that there exists $\gamma > 0$ such that $M \le \gamma K$, where γ is a constant.

3. The q-Heat Equation

We start with a study of the following Cauchy problem:

$$D_{q,t} u(t,x) + \mathscr{L} u(t,x) = f(t,x), x \in [0,1], t > 0, \tag{14}$$

with the initial condition

$$u(0,x) = \varphi(x), x \in [0,1]. \tag{15}$$

We say a generalised solution of the problem (14)-(15) is a function $u(t,x)$ such that they satisfy equation (14) and condition (15).

Theorem 1. *We assume that $0 < T < \infty$. Let $\varphi \in W^2_{q,\mathscr{L}}[0,1]$ and $f \in C([0,T]; W^2_{q,\mathscr{L}}[0,1])$. Then, there exists the generalised solution of u to problem (14)-(15), and*

$$u \in C^1_q\left([0,T]; L^2_q[0,1]\right) \cap C\left([0,T]; W^2_{q,\mathscr{L}}[0,1]\right). \tag{16}$$

Moreover, this solution can be written in the following explicit form

$$u(t,x) = \sum_{k \in \mathbb{N}} \left[e_q^{-\lambda_k t} \varphi_k + e_q^{-\lambda_k t} \int_0^t E_q^{\lambda_k q s} f_k(s,\cdot) d_q s \right] \phi_k(x). \tag{17}$$

Proof. Existence. Since the system of eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ is a basis in $L^2_q[0,1]$ (see (11)), we seek for a function $u(t,x)$ in the form

$$u(t,x) = \sum_{k \in \mathbb{N}} u_k(t) \phi_k(x), \tag{18}$$

for each fixed $0 < t < T < \infty$. *The coefficients will then be given by the Fourier coefficients formula* $u_k(t) = \langle u(t, \cdot)\phi_k \rangle$.

We can similarly expand the source function,

$$f(t, x) = \sum_{k\in\mathbb{N}} f_k(t)\phi_k(x), f_k(t) = \langle f(t, \cdot), \phi_k \rangle. \quad (19)$$

From (11) and (18), we have that

$$\mathscr{L}\phi_k(x) = \lambda_k \phi_k(x), k \in \mathbb{N}.$$

Hence,

$$\mathscr{L}u(t, x) = \sum_{k\in\mathbb{N}_0} u_k(t)\lambda_k \phi_k(x), \quad (20)$$

and

$$D_{q,t}u(t, x) = \sum_{k\in\mathbb{N}} D_q u_k(t)\phi_k(x). \quad (21)$$

Substituting (20) and (21) into the equation (14), we find that

$$\sum_{k\in\mathbb{N}} \left[D_q u_k(t) + \lambda_k u_k(t)\right]\phi_k(x) = \sum_{k\in\mathbb{N}} f_k(t)\phi_k(x). \quad (22)$$

But then, due to the completeness,

$$D_q u_k(t) + \lambda_k u_k(t) = f_k(t), k \in \mathbb{N}, \quad (23)$$

which are ODEs for the coefficients $u_k(t)$ *of the series (18). Using the integrating factor* $E_q^{\lambda_k q t}$ *and (2) and (3), we can rewrite the ODE as*

$$\begin{aligned} E_q^{\lambda_k q t} f_k(t) &= E_q^{\lambda_k q t} D_q u_k(t) + E_q^{\lambda_k q t} \lambda_k u_k(t) \\ &= E_q^{\lambda_k q t} D_q u_k(t) + D_q\left[E_q^{\lambda_k t}\right] u_k(t) \\ &= D_q\left[E_q^{\lambda_k t} u_k(t)\right]. \end{aligned} \quad (24)$$

Form (3), (5), and (24), we get that

$$\int_0^t D_q\left[E_q^{\lambda_k t} u_k(t)\right] d_q s = \int_0^t E_q^{\lambda_k q s} f_k(t) f_k(s) d_q s,$$

so that

$$E_q^{\lambda_k t} u_k(t) = u_k(0) + \int_0^t E_q^{\lambda_k q t} f_k(s) d_q s,$$

which, in its turn, implies that

$$u_k(t) = \frac{u_k(0)}{E_q^{\lambda_k t}} + \frac{1}{E_q^{\lambda_k t}} \int_0^t E_q^{\lambda_k q s} f_k(s) d_q s,$$

and we conclude that

$$u_k(t) = e_q^{-\lambda_k t} u_k(0) + e_q^{-\lambda_k t} \int_0^t E_q^{\lambda_k q s} f_k(s) d_q s.$$

But the initial conditions (16) and (22) imply that $u_k(0) = \varphi_k$. *Thus,*

$$u_k(t) = e_q^{-\lambda_k t} \varphi_k + e_q^{-\lambda_k t} \int_0^t E_q^{\lambda_k q s} f_k(s, \cdot) d_q s. \quad (25)$$

Therefore, the solution $u(t, x)$ *can be written in the series form as*

$$u(t, x) = \sum_{k\in\mathbb{N}} \left[e_q^{-\lambda_k t} \varphi_k + e_q^{-\lambda_k t} \int_0^t E_q^{\lambda_k q s} f_k(s, \cdot) d_q s\right] \phi_k(x),$$

so, also (17) is proved.

Convergence. From (1), (4), and (5), we have that

$$e_q^{-x} = \frac{1}{(-(1-q)x ; q)_\infty} \leq \frac{1}{1 + (1-q)x} \leq 1, E_q^{qx} \leq E_q^x,$$

for $x \in [0, 1]$. *Hence, using for* $0 < t < T < \infty$, *(5), (23), and (25), we get that*

$$\begin{aligned} |u_k(t)| &\overset{(25)}{\leq} e_q^{-\lambda_k t} |\varphi_k| + \int_0^t \frac{E_q^{\lambda_k q s}}{e_q^{\lambda_k t}} |f_k(s)| d_q s \leq |\langle \varphi, \phi_k \rangle| \\ &+ \int_0^t |\langle f(s, \cdot), \phi_k \rangle| d_q s \leq |\langle \varphi, \phi_k \rangle| \\ &+ T \max_{0\leq s\leq T} |\langle f(s, \cdot), \phi_k \rangle| \leq \max\{1, T\} \\ &\cdot \left[|\langle \varphi, \phi_k \rangle| + \max_{0\leq s\leq T} |\langle f(s, \cdot), \phi_k \rangle|\right] \lesssim |\langle \varphi, \phi_k \rangle| \\ &+ \max_{0\leq s\leq T} |\langle f(s, \cdot), \phi_k \rangle|, \end{aligned} \quad (26)$$

and

$$\begin{aligned} \left|D_q u_k(t)\right| &\overset{(23)}{\leq} \lambda_k |u_k(t)| + |f_k(t)| \overset{(26)}{\lesssim} |\langle \lambda_k \varphi, \phi_k \rangle| \\ &+ |\langle \lambda_k f_k(t, \cdot), \phi_k \rangle| + \lambda_k^{-1} |\langle \lambda_k f_k(t, \cdot), \phi_k \rangle| \leq |\langle \lambda_k \varphi, \phi_k \rangle| \\ &+ (1 + \lambda_0) |\langle \lambda_k f_k(t, \cdot), \phi_k \rangle| \lesssim |\langle \mathscr{L}\varphi, \phi_k \rangle| \\ &+ \max_{0\leq t\leq T} |\langle \mathscr{L} f_k(t, \cdot), \phi_k \rangle|. \end{aligned} \quad (27)$$

Hence,

$$\begin{aligned} |\mathscr{L}u(t, \cdot)| &= |\langle \lambda_k u_k(t), \phi_k \rangle| \overset{(26)}{\lesssim} |\langle \lambda_k \varphi, \phi_k \rangle| \\ &+ \max_{0\leq s\leq T} |\langle \lambda_k f(s, \cdot), \phi_k \rangle| = |\langle \mathscr{L}\varphi, \phi_k \rangle| \\ &+ \max_{0\leq s\leq T} |\langle \mathscr{L} f(s, \cdot), \phi_k \rangle|. \end{aligned} \quad (28)$$

Since $\varphi \in W^2_{q,\mathscr{L}}$, $f \in C([0, 1] ; W^2_{q,\mathscr{L}})$, *and, hence, by using the Plancherel identity and (27) and (28), we can conclude* *that*

$$\|u(t,\cdot)\|^2_{L^2_q[0,1]} = \sum_{k\in\mathbb{N}} |u_k(t)|^2 \overset{(3.13)}{\lesssim} \sum_{k\in\mathbb{N}} |\langle \varphi, \phi_k\rangle|^2 + \max_{0\le s\le T} \sum_{k\in\mathbb{N}} |\langle f(s,\cdot), \phi_k\rangle|^2 = \|\varphi\|^2_{W^2_{q,\mathscr{L}}[0,1]} + \|f\|^2_{C\left([0,T];W^2_{q,\mathscr{L}}[0,1]\right)} < \infty,$$

and

$$\left\|D_q u(t,\cdot)\right\|^2_{L^2_q[0,1]} = \sum_{k\in\mathbb{N}} \left|D_q u_k(t,\cdot)\right|^2 \overset{(27)}{\lesssim} \sum_{k\in\mathbb{N}} |\langle \mathscr{L}\varphi, \phi_k\rangle|^2 + \max_{0\le s\le T} \sum_{k\in\mathbb{N}} |\langle \mathscr{L}f(s,\cdot), \phi_k\rangle|^2 = \|\varphi\|^2_{W^2_{q,\mathscr{L}}[0,1]} + \|f\|^2_{C\left([0,T];W^2_{q,\mathscr{L}}[0,1]\right)} < \infty,$$

and

$$\|\mathscr{L}u(t,\cdot)\|^2_{L^2_q[0,1]} \overset{(28)}{\lesssim} \|\varphi\|^2_{W^2_{q,\mathscr{L}}[0,1]} + \|f\|^2_{C\left([0,T];W^2_{q,\mathscr{L}}[0,1]\right)} < \infty,$$

which mean that $u \in C^1_q([0,T]; L^2_q[0,1]) \cap C([0,T]; W^2_{q,\mathscr{L}}[0,1])$.

Uniqueness. It only remains to prove the uniqueness of the solution. We assume the opposite; namely, that there exist functions $u(t,x)$ *and* $v(t,x)$*, which are two different solutions of problem (14)-(15). Let* $0 < t < T < \infty$*. Then, we have that*

$$\begin{cases} D_{q,t}u(t,x) + \mathscr{L}u(t,x) = f(t,x), & 0 < x < 1, \\ u(0,x) = \varphi(x), & 0 \le x \le 1, \end{cases}$$

$$\begin{cases} D_{q,t}v(t,x) + \mathscr{L}v(t,x) = f(t,x), & 0 < x < 1, \\ v(0,x) = \varphi(x), & 0 \le x \le 1. \end{cases}$$

We define $W(t,x) = u(t,x) - v(t,x)$*. Then, the function* $W(t,x)$ *is a solution of the following problem*

$$\begin{cases} D_{q,t}w(t,x) + \mathscr{L}w(t,x) = 0, & 0 < x < 1, \\ w(0,x) = 0, & 0 \le x \le 1. \end{cases}$$

From (18), it follows that $W(t,x) \equiv 0$*, that is,* $u(x,t) \equiv v(x,t)$*, and this contradiction to our assumption proves the uniqueness of the solution. The proof is complete.*

4. The q-Wave Equation

In this section, we will seek for a generalised function $u(t,x)$, which satisfies the following q-wave equation

$$D^2_{q,t}u(t,x) + \mathscr{L}u(t,x) = f(t,x), 0 < x < 1, \tag{29}$$

for $0 < t < T < \infty$ with the initial conditions

$$u(0,x) = \psi(x), D_{q,t}u(0,x) = \eta(x), 0 < x \le 1. \tag{30}$$

Theorem 2. *We assume that* $0 < T < \infty$*. Let* $\psi, \eta \in W^2_{q,\mathscr{L}}[0,1]$ *and* $f \in C^1_q([0,T]; W^2_{q,\mathscr{L}}[0,1])$*. Then, there exists the generalised solution of problem (29)-(30):*

$$u \in C^2_q\left([0,1]; L^2_q[0,T]\right) \cap C\left([0,T]; W^2_{q,\mathscr{L}}[0,1]\right).$$

Moreover, this solution can be written in the following explicit form:

$$u(t,x) = \sum_{k\in\mathbb{N}_0} \left(\psi_k e_{2,1}\left(-\lambda_k t^2; q\right) + t\eta_k e_{2,2}\left(\lambda_k t^2; q\right) - \frac{1}{\lambda_k} f_k(0) e_{2,1}\left(-\lambda_k t^2; q\right) - \frac{1}{\lambda_k}\int_0^t e_{2,1} \cdot \left(\lambda_k\left(t - q^3 s\right)^2_q; q\right) D_{q,s} f_k(s) d_q s\right)\phi_k, \tag{31}$$

where the q-Mittag-Leffler function $e_{\alpha,\beta}(\lambda_k(t-q^2s)^\alpha_q; q)$ *is given by (see [25] and [26], Section 7):*

$$e_{\alpha,\beta}\left(\lambda_k (t-qs)^\alpha_q; q\right) = \sum_{m=0}^{\infty} \frac{\lambda_k^m (t-qs)^{m\alpha}_q}{\Gamma_q(m\alpha+\beta)}, \tag{32}$$

for $\alpha, \beta \in \mathbb{R}$ *and* $0 < s \le t < \infty$*, where the gamma function* $\Gamma_q(x)$ *is defined by*

$$\Gamma_q(x) = \frac{(q,q)^\infty_q}{(q^x,q)^\infty_q}(1-q)^{1-x}, \Gamma_q(n+1) = [n]_q!, n \in \mathbb{N}. \tag{33}$$

Proof. Existence. By repeating the arguments in the proof of Theorem 1., we have the Cauchy type problem:

$$D^2_q u_k(t) + \lambda_k u_k(t) = f_k(t), k \in \mathbb{N}_0, \tag{34}$$

with the initial conditions

$$u_k(0) = \psi_k, D_q u_k(0) = \eta_k, k \in \mathbb{N}_0, \tag{35}$$

where $f_k(t) = \langle f(t,\cdot)\phi_k\rangle$*,* $\psi_k = \langle \psi(\cdot)\phi_k\rangle$ *and* $\eta_k = \langle \eta(\cdot)\phi_k\rangle$*.*

Then, the solution to this Cauchy type in problem (29)-(30) is given (see [25], Example 6)

$$u_k(t) = \psi_k e_{2,1}\left(-\lambda_k t^2; q\right) + t\eta_k e_{2,2}\left(-\lambda_k t^2; q\right) + \int_0^t (t-qs) e_{2,2}\left(-\lambda_k\left(t-q^2s\right)^2_q; q\right) f_k(s) d_q s. \tag{36}$$

By using (2) and we find that

$$\begin{aligned} D_{q,s}\left[e_{2,1}\left(-\lambda_k\left(t-q^2s\right)_q^2;q\right)\right] &= -\sum_{k\in\mathbb{N}}\frac{(-\lambda_k)^m}{\Gamma_q(2m+1)}[2m]_q\left(t-q^3s\right)_q^{2m-1} \\ &= \lambda_k(t-qs)\sum_{k\in\mathbb{N}}\frac{(-\lambda_k)^{m-1}}{\Gamma_q(2m)}\left(t-q^2s\right)_q^{2m-2} \\ &= \lambda_k(t-qs)e_{2,2}\left(\lambda_k\left(t-q^2s\right)_q^2;q\right). \end{aligned} \tag{37}$$

By applying (8) and using (36) and (37), we get that

$$\begin{aligned} u_k(t) &= \psi_k e_{2,1}\left(-\lambda_k t^2;q\right) + t\eta_k e_{2,2}\left(-\lambda_k t^2;q\right) \\ &\quad + \frac{1}{\lambda_k}\int_0^t D_{q,s}\left[e_{2,1}\left(-\lambda_k\left(t-q^2s\right)_q^2;q\right)\right]f_k(s)d_qs \\ &= \psi_k e_{2,1}\left(-\lambda_k t^2;q\right) + t\eta_k e_{2,2}\left(-\lambda_k t^2;q\right) \\ &\quad + \frac{1}{\lambda_k}f_k(t)e_{2,1}\left(-\lambda_k\left(t-q^2\right)_q^2;q\right) \\ &\quad - \frac{1}{\lambda_k}f_k(0)e_{2,1}\left(-\lambda_k t^2;q\right) \\ &\quad - \frac{1}{\lambda_k}\int_0^t e_{2,1}\left(-\lambda_k\left(t-q^3s\right)_q^2;q\right)D_{q,s}f_k(s)d_qs. \end{aligned} \tag{38}$$

Since $e_{2,1}(-\lambda_k t^2(q^2;q)_2)\equiv 0$ (see [21], Theorem 7.12]), by using (18) and (38), it follows that solution exists and can be written as

$$\begin{aligned} u(t,x) = \sum_{k\in\mathbb{N}}\Bigg(&\psi_k e_{2,1}\left(-\lambda_k t^2;q\right) + t\eta_k e_{2,2}\left(\lambda_k t^2;q\right) \\ &- \frac{1}{\lambda_k}f_k(0)e_{2,1}\left(-\lambda_k t^2;q\right) - \frac{1}{\lambda_k}\int_0^t e_{2,1} \\ &\cdot\left(-\lambda_k\left(t-q^3s\right)_q^2;q\right)D_{q,s}f_k(s)d_qs\Bigg)\phi_k, \end{aligned}$$

i.e., on the explicit form (34).

Convergence. Firstly, using the results in [27], Lemma 6 and in [17], Lemma 1 for the q-trigonometric functions in (6), we see that $e_{2,2}(-\lambda_k t^2;q)$ and $e_{2,1}(-\lambda_k t^2;q)$ are also bounded with $t>0$. Then, forms (4), (12), and (32) follow that

$$\begin{aligned} \left|e_{2,2}\left(-\lambda_k t^2;q\right)\right| &= \left|\sum_{m=0}^{\infty}\frac{(-\lambda_k)^m t^{2m}}{[2m+1]_q!}\right| \\ &\le \sum_{m=0}^{\infty}\frac{\lambda_k^m T^{2m}}{[2m+1]_q!} \\ &= \frac{\sin\left(\sqrt{\lambda_k}T;q^2\right)}{2T\sqrt{\lambda_k}} \\ &\le \frac{C_{1,q}}{2T\sqrt{\lambda_k}}, \end{aligned} \tag{39}$$

and

$$\begin{aligned} \left|e_{2,1}\left(-\lambda_k t^2;q\right)\right| &= \left|\sum_{m=0}^{\infty}\frac{(-1)^m\left(T\sqrt{\lambda_k}\right)^{2m}}{[2m]_q!}\right| \\ &\le \cos\left(\sqrt{\lambda_k}T;q^2\right) \le C_{2,q}, \end{aligned} \tag{40}$$

where $C_{1,q}$, $C_{2,q}$ are any constant which only depends on q.

Next, by using (38), (39), and (40), we obtain that

$$\begin{aligned} |u_k(t)| &\overset{(40)}{\lesssim} |\langle\psi,\phi_k\rangle| + |\langle\eta,\phi_k\rangle| + \frac{|\langle f(0,\cdot),\phi_k\rangle|}{\lambda_k} \\ &\quad + \frac{1}{\lambda_k}\int_0^t\left|\left\langle D_qf(s,\cdot),\phi_k\right\rangle\right|d_qs \le |\langle\psi,\phi_k\rangle| + |\langle\eta,\phi_k\rangle| \\ &\quad + \frac{|\langle f(0,\cdot),\phi_k\rangle|}{\lambda_k} + \frac{T}{\lambda_k}\max_{0\le s\le T}\left|\left\langle D_qf(s,\cdot),\phi_k\right\rangle\right| \le C_{\lambda_0} \\ &\quad \cdot\left[|\langle\psi,\phi_k\rangle| + |\langle\eta,\phi_k\rangle| + \sum_{m=0}^{1}\max_{0\le t\le T}\left|\left\langle D_q^mf(t,\cdot),\phi_k\right\rangle\right|\right], \end{aligned} \tag{41}$$

where $C_{\lambda_0} := \max\{1,(1/\lambda_0),(T/\lambda_0)\}$.

Therefore, by using (7), (8), (30), (34), (33), and (41), we have that

$$\begin{aligned} \left|D_qu_k(t)\right| &= \left|-D_qu_k(0) + \int_0^t D_q^2u_k(s)d_qs\right| \overset{(30)(34)}{\le} |\langle\eta,\phi_k\rangle| \\ &\quad + \int_0^t|\langle f(s,\cdot),\phi_k\rangle|d_qs \\ &\quad + \lambda_k\int_0^t|u_k(s)|d_qs \overset{(41)}{\lesssim} \frac{1}{\lambda_k}|\langle\lambda_k\eta,\phi_k\rangle| \\ &\quad + \frac{T}{\lambda_k}\max_{0\le s\le T}|\langle\lambda_kf(s,\cdot),\phi_k\rangle| + TC_{\lambda_0} \\ &\quad \cdot\left[|\langle\lambda_k\psi,\phi_k\rangle| + |\langle\lambda_k\eta,\phi_k\rangle| + \sum_{m=0}^{1}\max_{0\le t\le T}\left|\left\langle\lambda_kD_q^mf(t,\cdot),\phi_k\right\rangle\right|\right] \\ &\lesssim |\langle\mathcal{L}\psi,\phi_k\rangle| + |\langle\mathcal{L}\eta,\phi_k\rangle| + \sum_{m=0}^{1}\max_{0\le t\le T}\left|\left\langle\mathcal{L}D_q^mf(t,\cdot),\phi_k\right\rangle\right|, \end{aligned} \tag{42}$$

and

$$\begin{aligned} \left|D_a^2u_k(t)\right| &\overset{(34)}{\lesssim} \lambda_k|u_k(t)| + |f_k(t)| \overset{(41)}{\lesssim} |\langle\lambda_k\psi,\phi_k\rangle| + |\langle\lambda_k\eta,\phi_k\rangle| \\ &\quad + \sum_{m=0}^{1}\max_{0\le t\le T}\left|\left\langle\lambda_kD_q^mf(t,\cdot),\phi_k\right\rangle\right| = |\langle\mathcal{L}\psi,\phi_k\rangle| + |\langle\mathcal{L}\eta,\phi_k\rangle| \\ &\quad + \sum_{m=0}^{1}\max_{0\le t\le T}\left|\left\langle\mathcal{L}D_q^mf(t,\cdot),\phi_k\right\rangle\right|. \end{aligned} \tag{43}$$

Thus,

$$
\begin{aligned}
\|u(t)\|^2_{L^2_q[0,1]} &= \sum_{k\in\mathbb{N}} |u_k(t)|^2 \overset{(41)}{\lesssim} \sum_{k\in\mathbb{N}} |\langle \psi, \phi_k\rangle|^2 + \sum_{k\in\mathbb{N}} |\langle \eta, \phi_k\rangle|^2 \\
&\quad + \sum_{m=0}^{1} \max_{0\le t\le T} \sum_{k\in\mathbb{N}} \left|\left\langle D_q^m f(t,\cdot), \phi_k\right\rangle\right|^2 \\
&= \|\eta\|^2_{W^2_{q,\mathscr{L}}[0,1]} + \|\psi\|^2_{W^2_{q,\mathscr{L}}[0,1]} \\
&\quad + \|f\|^2_{C^1_q\left([0,T];W^2_{q,\mathscr{L}}[0,1]\right)} < \infty,
\end{aligned}
$$

$$
\begin{aligned}
\left\|D_q u(t)\right\|^2_{L^2_q[0,1]} &= \sum_{k\in\mathbb{N}} \left|D_q u_k(t)\right|^2 \lesssim \sum_{k\in\mathbb{N}} |\langle \mathscr{L}\psi, \phi_k\rangle|^2 \\
&\quad + \sum_{k\in\mathbb{N}} |\langle \mathscr{L}\eta, \phi_k\rangle|^2 \\
&\quad + \sum_{m=0}^{1} \max_{0\le t\le T} \left|\sum_{k\in\mathbb{N}} \left\langle \mathscr{L}D_q^m f(t,\cdot), \phi_k\right\rangle\right|^2 \\
&\le \|\eta\|^2_{W^2_{q,\mathscr{L}}[0,1]} + \|\psi\|^2_{W^2_{q,\mathscr{L}}[0,1]} \\
&\quad + \|f\|^2_{C^1_q\left([0,T];W^2_{q,\mathscr{L}}[0,1]\right)} < \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|D_q^2 u(t)\right\|^2_{L^2_q[0,1]} &= \sum_{k\in\mathbb{N}} \left|D_q^2 u_k(t)\right|^2 \lesssim \|\eta\|^2_{W^2_{q,\mathscr{L}}[0,1]} \\
&\quad + \|\psi\|^2_{W^2_{q,\mathscr{L}}[0,1]} + \|f\|^2_{C^1_q\left([0,T];W^2_{q,\mathscr{L}}[0,1]\right)} < \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathscr{L}u(t)\|^2_H &= \sum_{k\in I} \left|\langle \mathscr{L}u(t), \phi_k\rangle_H\right|^2 \\
&= \sum_{k\in I} [\lambda_k |u_k(t)|]^2 \lesssim \|\eta\|^2_{W^2_{q,\mathscr{L}}[0,1]} \\
&\quad + \|\psi\|^2_{W^2_{q,\mathscr{L}}[0,1]} \\
&\quad + \|f\|^2_{C^1_q\left([0,T];W^2_{q,\mathscr{L}}[0,1]\right)},
\end{aligned}
$$

which is means that $u \in C_q^2([0,1]; L_q^2[0,T]) \cap C_q([0,T]; W^2_{q,\mathscr{L}}[0,1])$.

Uniqueness. This part can be proved completely similar as the proof of Theorem 1.. So we omit the details.

Data Availability

Data supporting this manuscript are available from Scopus, Web of Science, and Google Scholar.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

SS and NT were supported in parts by the MESRK (Ministry of Education and Science of the Republic of Kazakhstan) grant AP08052208.

References

[1] V. Bardek and S. Meljanac, "Deformed Heisenberg algebras, a Fock-space representation and the Calogero model," *The European Physical Journal C-Particles and Fields*, vol. 17, no. 3, pp. 539–547, 2000.

[2] R. Hinterding and J. Wess, "q-deformed Hermite polynomials in q-quantum mechanics," *The European Physical Journal C-Particles and Fields*, vol. 17, no. 6, pp. 183–186, 1999.

[3] A. Lavagno, "Basic-deformed quantum mechanics," *Reports on Mathematical Physics*, vol. 64, no. 1–2, pp. 79–91, 2009.

[4] M. Micu, "A q-deformed Schrodinger equation," *Journal of Physics A*, vol. 32, no. 44, p. 77, 1999.

[5] V. V. Eremin and A. A. Meldianov, "The q-deformed harmonic oscillator, coherent states, and the uncertainty relation," *Theoretical and Mathematical Physics*, vol. 147, no. 2, pp. 709–715, 2006.

[6] R. Herrmann, "Common aspects of q-deformed Lie algebras and fractional calculus," *Physica A*, vol. 389, no. 21, pp. 4613–4622, 2010.

[7] F. H. Jackson, "On q-functions and a certain difference operator," *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, vol. 46, pp. 253–281, 1909.

[8] F. H. Jackson, "On a q-definite integrals," *The Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.

[9] R. D. Carmichael, "The general theory of linear q-difference equations," *American journal of mathematics*, vol. 34, no. 2, pp. 147–168, 1912.

[10] P. Cheung and V. Kac, *Quantum Calculus*, Edwards Brothers, Inc., Ann Arbor, MI, USA, 2000.

[11] T. Ernst, *A comprehensive treatment of q-calculus*, Birkhäuser/Springer Basel AG, Basel, 2012.

[12] T. Ernst, "A new method of q-calculus," Uppsala university, 2002, Doctoral thesis,.

[13] W. Al-Salam, "Some fractional q-integrals and q-derivatives," *Proceedings of the Edinburgh Mathematical Society*, vol. 15, no. 2, pp. 135–140, 1966.

[14] R. P. Agarwal, "Certain fractional q-integrals and q-derivatives," *Proceedings of the Cambridge Philosophical Society*, vol. 66, no. 2, pp. 365–370, 1969.

[15] S. Shaimardan and N. S. Tokmagambetov, "The Schrädinger equations generated by q-Bessel operator in quantum calculus," *The Karaganda University*, vol. 104, no. 1, pp. 102–108, 2022.

[16] S. Shaimardan, N. S. Tokmagambetov, and A. M. Temirkhanova, "The Cauchy problems for q-difference equations with the Caputo fractional derivatives," *Journal of Mathematics, Mechanics, Computer Science*, vol. 113, no. 1, pp. 43–57, 2022.

[17] R. L. Rubin, "Duhamel solutions of non-homogeneous q^2-analogue wave equations," *Proceedings of the American Mathematical Society*, vol. 135, no. 3, pp. 777–785, 2007.

[18] D. T. Haimo, "Expansion of generalized heat polynomials and their Appell transform," *Journal of Applied Mathematics and Mechanics*, vol. 15, pp. 735–758, 1966.

[19] A. Fitouhi, "Heat "polynomials" for a singular differential operator on $(0, \infty)$," *Constructive Approximation*, vol. 5, no. 1, pp. 241–270, 1989.

[20] A. Fitouhi and F. Bouzeffour, "q-cosine Fourier transform and q-heat equation," *The Ramanujan Journal*, vol. 28, no. 3, pp. 443–461, 2012.

[21] M. H. Annaby and Z. S. Mansour, "Basic Sturm Liouville problems," *Journal of Physics A*, vol. 38, no. 17, pp. 3775–3797, 2005.

[22] M. Ruzhansky and N. Tokmagambetov, "Nonharmonic analysis of boundary value problems," *International Mathematics Research Notices*, vol. 2016, no. 12, pp. 3548–3615, 2016.

[23] M. Ruzhansky and N. Tokmagambetov, "Convolution, Fourier analysis, and distributions generated by Riesz bases," *Monatshefte für Mathematik*, vol. 187, no. 1, pp. 147–170, 2018.

[24] M. Ruzhansky, N. Tokmagambetov, and B. T. Torebek, "Inverse source problems for positive operators. I: hypoelliptic diffusion and subdiffusion equations," *Journal of Inverse and Ill-Posed Problems*, vol. 27, no. 6, pp. 891–911, 2019.

[25] S. Shaimardan and N. S. Tokmagambetov, "On the solutions of some fractional q-differential equations with the Riemann-Liouville fractional q-derivative," *Bulletin of the Karaganda university*, vol. 104, no. 4, pp. 130–141, 2021.

[26] M. H. Annaby and Z. S. Mansour, "q-fractional calculus and equations," Springer, Heidelberg, 2012.

[27] R. L. Rubin, "Aq^2-Analogue Operator for q^2-Analogue Fourier Analysis," *Journal of Mathematical Analysis and Applications*, vol. 212, no. 2, pp. 571–582, 1997.