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ATTRACTORS OF THE NAVIER–STOKES EQUATIONS IN A TWO-DIMENSIONAL POROUS MEDIUM

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In a perforated domain, we consider the two-dimensional system of Navier–Stokes equations with rapidly oscillating terms in the equations and boundary conditions. We prove that the trajectory attractors of this system converge in some weak topology to trajectory attractors of the homogenized Navier–Stokes equations with an additional potential. Bibliography: 11 *titles. Illustrations*: 1 *figure.*

In this paper, we study the asymptotic behavior of attractors of initial-boundary-value problems for two-dimensional systems of Navier–Stokes equations in perforated domains (cf. Figure below) in the case where a potential appears in the limit equation. We study the weak convergence and limit behavior of attractors as the small parameter converges to zero. We mention recent works [1]–[3] devoted to homogenization of attractors. We prove that the trajectory attractors $\mathfrak{A}_\varepsilon$ of the two-dimensional system of Navier–Stokes equations in a perforated domain weakly converge as $\varepsilon \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized system of equations in the

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corresponding function space. The small parameter ε characterizes the cavity diameter, as well as the distance between cavities in the perforated medium.

In Section 1, we formulate the main notions and the theorem on trajectory attractors of autonomous evolution equations. In Section 2, we describe the geometric structure of a perforated domain, formulate the problem under consideration, and introduce some function spaces. Section 3 is devoted to homogenization of attractors of an autonomous two-dimensional system of Navier–Stokes equations with rapidly oscillating terms in a perforated domain.

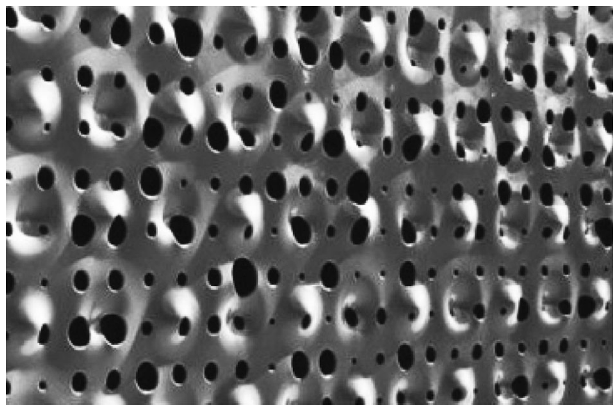

Figure. Two-dimensional perforated domain.

1 Trajectory Attractors of Evolution Equations

We describe a general scheme of constructing trajectory attractors of autonomous evolution equations. This scheme will be used in Section 2 to study trajectory attractors of a two-dimensional system of Navier–Stokes equations in a perforated domain with rapidly oscillating terms in equations and boundary conditions and the corresponding homogenized equation.

We consider the abstract autonomous evolution equation

$$\frac{\partial u}{\partial t} = A(u), \quad t \geqslant 0, \tag{1.1}$$

where $A(\cdot) : E_1 \to E_0$ is a given nonlinear operator, E_1 and E_0 are Banach spaces such that $E_1 \subseteq E_0$. For example,

$$A(u) = \nu\Delta u - (u, \nabla u) + g(\cdot).$$

We will study a solution $u(s)$ to Equation (1.1) globally, as a function of variable $s \in \mathbb{R}_+$. Here, $s \equiv t$ denotes the time-variable. The set of solutions to Equation (1.1) is called the *trajectory space* of Equation (1.1) and is denoted by $\mathscr{K}^+$. We describe the trajectory space $\mathscr{K}^+$ in detail.

First of all, we consider the solution $u(s)$ to Equation (1.1) defined on a fixed time-segment $[t_1, t_2]$ in $\mathbb{R}$. We study solutions to Equation (1.1) in the Banach space $\mathscr{F}_{t_1,t_2}$ which depends on t_1 and t_2. The space $\mathscr{F}_{t_1,t_2}$ consists of functions $f(s)$, $s \in [t_1, t_2]$, such that $f(s) \in E$ for almost all $s \in [t_1, t_2]$, where E is a Banach space. It is assumed that $E_1 \subseteq E \subseteq E_0$.

For example, for $\mathscr{F}_{t_1,t_2}$ we can take the space $C([t_1, t_2]; E)$, the space $L_p(t_1, t_2; E)$, $p \in [1, \infty]$, or the intersection of such spaces. We assume that $\Pi_{t_1,t_2}\mathscr{F}_{\tau_1,\tau_2} \subseteq \mathscr{F}_{t_1,t_2}$ and

$$\|\Pi_{t_1,t_2} f\|_{\mathscr{F}_{t_1,t_2}} \leqslant C(t_1, t_2, \tau_1, \tau_2)\|f\|_{\mathscr{F}_{\tau_1,\tau_2}} \quad \forall f \in \mathscr{F}_{\tau_1,\tau_2}, \tag{1.2}$$

where $[t_1, t_2] \subseteq [\tau_1, \tau_2]$, Π_{t_1,t_2} is the restriction operator on $[t_1, t_2]$, and $C(t_1, t_2, \tau_1, \tau_2)$ is independent of f. Usually, one consider the homogeneous case of the space where

$$C(t_1, t_2, \tau_1, \tau_2) = C(t_2 - t_1, \tau_2 - \tau_1).$$

Let $S(h)$, $h \in \mathbb{R}$, denote the translation operator

$$S(h)f(s) = f(h + s).$$

It is obvious that if the variable s of $f(\cdot)$ belongs to $[t_1, t_2]$, then the variable s of $S(h)f(\cdot)$ belongs to $[t_1 - h, t_2 - h]$ for $h \in \mathbb{R}$. We assume that the mapping $S(h)$ is an isomorphism from F_{t_1,t_2} to F_{t_1-h,t_2-h} and

$$\|S(h)f\|_{\mathscr{F}_{t_1-h,t_2-h}} = \|f\|_{\mathscr{F}t_1,t_2} \quad \forall f \in \mathscr{F}_{t_1,t_2}. \tag{1.3}$$

This assumption is natural, for example, for the homogeneous space.

We assume that if $f(s) \in \mathscr{F}_{t_1,t_2}$, then $A(f(s)) \in \mathscr{D}_{t_1,t_2}$, where the Banach space $\mathscr{D}_{t_1,t_2}$ is wider than $\mathscr{F}_{t_1,t_2}$, $\mathscr{F}_{t_1,t_2} \subseteq \mathscr{D}_{t_1,t_2}$. The derivative $\dfrac{\partial f(t)}{\partial t}$ is a generalized function taking the values in E_0; $\dfrac{\partial f}{\partial t} \in D'((t_1, t_2); E_0)$. We assume that $\mathscr{D}_{t_1,t_2} \subseteq D'((t_1, t_2); E_0)$ for all $(t_1, t_2) \subset \mathbb{R}$. A function $u(s) \in \mathscr{F}_{t_1,t_2}$ is called a *solution* to Equation (1.1) in the space $\mathscr{F}_{t_1,t_2}$ (on the interval (t_1, t_2)) if $\dfrac{\partial u}{\partial t}(s) = A(u(s))$ in the sense of distributions in $D'((t_1, t_2); E_0)$.

We also introduce the space

$$\mathscr{F}_+^{\mathrm{loc}} = \{f(s),\, s \in \mathbb{R}_+ \mid \Pi_{t_1,t_2} f(s) \in \mathscr{F}_{t_1,t_2} \quad \forall\, [t_1, t_2] \subset \mathbb{R}_+\}. \tag{1.4}$$

For example, $\mathscr{F}_{t_1,t_2} = C([t_1, t_2]; E)$ implies $\mathscr{F}_+^{\mathrm{loc}} = C(\mathbb{R}_+; E)$ and $\mathscr{F}_{t_1,t_2} = L_p(t_1, t_2; E)$ implies $\mathscr{F}_+^{\mathrm{loc}} = L_p^{\mathrm{loc}}(\mathbb{R}_+; E)$.

A function $u(s) \in \mathscr{F}_+^{\mathrm{loc}}$ is called a *solution* to Equation (1.1) in $\mathscr{F}_+^{\mathrm{loc}}$ if $\Pi_{t_1,t_2} u(s) \in \mathscr{F}_{t_1,t_2}$ and the function $\Pi_{t_1,t_2} u(s)$ is a solution to Equation (1.1) for any time-segment $[t_1, t_2] \subset \mathbb{R}_+$.

Let $\mathscr{K}^+$ be a set of solutions to Equation (1.1) om the space $\mathscr{F}_+^{\mathrm{loc}}$, but $\mathscr{K}^+$ does not necessarily coincides with the set of all solutions to Equation (1.1) in $\mathscr{F}_+^{\mathrm{loc}}$. Elements of $\mathscr{K}^+$ are called *trajectories*, and $\mathscr{K}^+$ is said to be the *trajectory space* of Equation (1.1).

We assume that the trajectory space $\mathscr{K}^+$ is *translation invariant* in the following sense: if $u(s) \in \mathscr{K}^+$, then $u(h + s) \in \mathscr{K}^+$ for any $h \geqslant 0$. This condition is natural for solutions to autonomous equations in homogeneous spaces.

We consider the translation operators $S(h)$ in $\mathscr{F}_+^{\mathrm{loc}}$, i.e., $S(h)f(s) = f(s + h)$ for $h \geqslant 0$. It is clear that $\{S(h), h \geqslant 0\}$ is a semigroup in $\mathscr{F}_+^{\mathrm{loc}}$: $S(h_1)S(h_2) = S(h_1 + h_2)$ for $h_1, h_2 \geqslant 0$ and $S(0) = I$ is the identity mapping. We replace the variable h with the time-variable t. The semigroup $\{S(t), t \geqslant 0\}$ is called the *translation semigroup*. By assumption, the translation semigroup maps the trajectory space $\mathscr{K}^+$ onto itself:

$$S(t)\mathscr{K}^+ \subseteq \mathscr{K}^+ \quad \forall t \geqslant 0. \tag{1.5}$$

In what follows, we study the attraction property of the translation semigroup $\{S(t)\}$ acting on the trajectory space $\mathscr{K}^+ \subset \mathscr{F}_+^{\mathrm{loc}}$. We introduce a topology in $\mathscr{F}_+^{\mathrm{loc}}$. Let $\rho_{t_1,t_2}(\cdot, \cdot)$ be a group defined on the space $\mathscr{F}_{t_1,t_2}$ for all segments $[t_1, t_2] \subset \mathbb{R}$. As in (1.2) and (1.3), we assume that

$$\rho_{t_1,t_2}(\Pi_{t_1,t_2} f, \Pi_{t_1,t_2} g) \leqslant D(t_1, t_2, \tau_1, \tau_2)\rho_{\tau_1,\tau_2}(f, g) \quad \forall f, g \in \mathscr{F}_{\tau_1,\tau_2},\ [t_1, t_2] \subseteq [\tau_1, \tau_2],$$

$$\rho_{t_1-h,t_2-h}(S(h)f, S(h)g) = \rho_{t_1,t_2}(f, g) \quad \forall f, g \in \mathscr{F}_{t_1,t_2},\ [t_1, t_2] \subset \mathbb{R},\ h \in \mathbb{R}.$$

(For a homogeneous space $D(t_1, t_2, \tau_1, \tau_2) = D(t_2 - t_2, \tau_2 - \tau_1)$.)

We denote by Θ_{t_1,t_2} the corresponding metric space on $\mathscr{F}_{t_1,t_2}$. For example, ρ_{t_1,t_2} can be the metric generated by the norm $\|\cdot\|_{\mathscr{F}_{t_1,t_2}}$ in the Banach space $\mathscr{F}_{t_1,t_2}$. In applications, it can happen that the metric ρ_{t_1,t_2} generates a weaker topology in Θ_{t_1,t_2} than the strong convergence topology in the Banach space $\mathscr{F}_{t_1,t_2}$.

We denote by Θ_+^{loc} the space $\mathscr{F}_+^{\mathrm{loc}}$ equipped with the local convergence topology in Θ_{t_1,t_2} for any $[t_1, t_2] \subset \mathbb{R}_+$. More exactly, by definition, a sequence of functions $\{f_k(s)\} \subset \mathscr{F}_+^{\mathrm{loc}}$ converges to a function $f(s) \in \mathscr{F}_+^{\mathrm{loc}}$ in Θ_+^{loc} as $k \to \infty$ if $\rho_{t_1,t_2}(\Pi_{t_1,t_2} f_k, \Pi_{t_1,t_2} f) \to 0$ as $k \to \infty$ for any $[t_1, t_2] \subset \mathbb{R}_+$. It is easy to prove that the topology in Θ_+^{loc} is metrizable by using the Frechet metric

$$\rho_+(f_1, f_2) := \sum_{m \in \mathbb{N}} 2^{-m} \frac{\rho_{0,m}(f_1, f_2)}{1 + \rho_{0,m}(f_1, f_2)}. \tag{1.6}$$

If all metric spaces Θ_{t_1,t_2} are complete, then the metric space Θ_+^{loc} is also complete.

We note that the translation semigroup $\{S(t)\}$ is continuous in the topology of the space Θ_+^{loc}. This fact directly follows from the definition of the topological space Θ_+^{loc}.

We define the Banach space

$$\mathscr{F}_+^b := \{f(s) \in \mathscr{F}_+^{\mathrm{loc}} \mid \|f\|_{\mathscr{F}_+^b} < +\infty\} \tag{1.7}$$

equipped with the norm

$$\|f\|_{\mathscr{F}_+^b} := \sup_{h \geqslant 0} \|\Pi_{0,1} f(h+s)\|_{\mathscr{F}_{0,1}}. \tag{1.8}$$

For example, if $\mathscr{F}_+^{\mathrm{loc}} = C(\mathbb{R}_+; E)$, then $\mathscr{F}_+^b = C^b(\mathbb{R}_+; E)$ is equipped with the norm $\|f\|_{\mathscr{F}_+^b} = \sup\limits_{h \geqslant 0} \|f(h)\|_E$ and, if $\mathscr{F}_+^{\mathrm{loc}} = L_p^{\mathrm{loc}}(\mathbb{R}_+; E)$, then $\mathscr{F}_+^b = L_p^b(\mathbb{R}_+; E)$ is equipped with the norm

$$\|f\|_{\mathscr{F}_+^b} = \left(\sup_{h \geqslant 0} \int_h^{h+1} \|f(s)\|_E^p ds \right)^{1/p}.$$

We note that $\mathscr{F}_+^b \subseteq \Theta_+^{\mathrm{loc}}$. The Banach space $\mathscr{F}_+^b$ is necessary to introduce bounded sets in the trajectory space $\mathscr{K}^+$. To construct a trajectory attractor in $\mathscr{K}^+$, we use the weaker local convergence topology in Θ_+^{loc} instead of the uniform convergences in the topology of the space $\mathscr{F}_+^b$.

We assume that $\mathscr{K}^+ \subseteq \mathscr{F}_+^b$, i.e., any trajectory $u(s) \in \mathscr{K}^+$ of Equation (1.1) has finite norm (1.8). We define an attracting set and a trajectory attractor of the translation semigroup $\{S(t)\}$ acting on $\mathscr{K}^+$.

Definition 1.1. A set $\mathscr{P} \subseteq \Theta_+^{\mathrm{loc}}$ is called an *attracting set* of the translation semigroup $\{S(t)\}$ acting on $\mathscr{K}^+$ in the topology of Θ_+^{loc} if for any bounded set $\mathscr{B} \subseteq \mathscr{K}^+$ in $\mathscr{F}_+^b$ the set $\mathscr{P}$ attracts $S(t)\mathscr{B}$ in the topology of Θ_+^{loc} as $t \to +\infty$, i.e., for any ε-neighborhood $O_\varepsilon(\mathscr{P})$ in Θ_+^{loc} there exists $t_1 \geqslant 0$ such that $S(t)\mathscr{B} \subseteq O_\varepsilon(\mathscr{P})$ for any $t \geqslant t_1$.

The attraction property of $\mathscr{P}$ can be formulated in the equivalent form: for any bounded set $\mathscr{B} \subseteq \mathscr{K}^+$ in $\mathscr{F}_+^b$ and any $M > 0$

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathscr{B}, \Pi_{0,M}\mathscr{P}) \to 0, \quad t \to +\infty,$$

where

$$\operatorname{dist}_{\mathscr{M}}(X,Y) := \sup_{x\in X} \operatorname{dist}_{\mathscr{M}}(x,Y) = \sup_{x\in X} \inf_{y\in Y} \rho_{\mathscr{M}}(x,y)$$

denotes the Hausdorff semi-distance between sets X and Y in the metric space $\mathscr{M}$.

Definition 1.2 (cf. [4]). A set $\mathfrak{A} \subseteq \mathscr{K}^+$ is called a *trajectory attractor* of the translation semigroup $\{S(t)\}$ on $\mathscr{K}^+$ in the topology of Θ_+^{loc} if the following conditions are satisfied:

(i) $\mathfrak{A}$ is bounded in $\mathscr{F}_+^b$ and compact in Θ_+^{loc},

(ii) $\mathfrak{A}$ is strictly invariant under the translation semigroup: $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geqslant 0$,

(iii) $\mathfrak{A}$ is an attracting set of the translation semigroup $\{S(t)\}$ for $\mathscr{K}^+$ in the topology of Θ_+^{loc}, i.e., for any $M > 0$

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathscr{B}, \Pi_{0,M}\mathfrak{A}) \to 0, \quad t \to +\infty.$$

Remark 1.1. Using the terminology of [5], we can say that a trajectory attractor $\mathfrak{A}$ is *global* $(\mathscr{F}_+^b, \Theta_+^{\mathrm{loc}})$*-attractor* of the translation semigroup $\{S(t)\}$ acting on $\mathscr{K}^+$, i.e., $\mathfrak{A}$ attracts $S(t)\,\mathscr{B}$ in the topology of Θ_+^{loc} as $t \to +\infty$, where $\mathscr{B}$ is any bounded (in $\mathscr{F}_+^b$) set in $\mathscr{K}^+$:

$$\operatorname{dist}_{\Theta_+^{\mathrm{loc}}}(S(t)\,\mathscr{B}, \mathfrak{A}) \to 0, \quad t \to +\infty.$$

We formulate the main result concerning the existence and structure of a trajectory attractor of Equation (1.1).

Theorem 1.1 (cf. [4]–[6]). *Let the trajectory space $\mathscr{K}^+$ corresponding to Equation* (1.1) *be closed in $\mathscr{F}_+^b$ and satisfy the condition* (1.5). *Let the translation semigroup $\{S(t)\}$ have an attracting set $\mathscr{P} \subseteq \mathscr{K}^+$ that is bounded in $\mathscr{F}_+^b$ and compact in Θ_+^{loc}. Then the translation semigroup $\{S(t), t \geqslant 0\}$ acting on $\mathscr{K}^+$ has a trajectory attractor $\mathfrak{A} \subseteq \mathscr{P}$. The set $\mathfrak{A}$ is bounded in $\mathscr{F}_+^b$ and compact in Θ_+^{loc}.*

We describe the structure of trajectory attractors $\mathfrak{A}$ of Equation (1.1) in terms of complete trajectories of this equation. We consider Equation (1.1) on the whole time-axis

$$\frac{\partial u}{\partial t} = A(u), \quad t \in \mathbb{R}. \tag{1.9}$$

Now, we extend the notion of the trajectory space $\mathscr{K}^+$ of Equation (1.9) introduced on $\mathbb{R}_+$ to the case of the whole axis $\mathbb{R}$. If a function $f(s)$, $s \in \mathbb{R}$, is given on the whole time-axis, then the translations $S(h)f(s) = f(s+h)$ are also defined for negative h. A function $u(s)$, $s \in \mathbb{R}$, is called a *complete trajectory* of Equation (1.9) if $\Pi_+ u(s+h) \in \mathscr{K}^+$ for any $h \in \mathbb{R}$. Here, $\Pi_+ = \Pi_{0,\infty}$ denotes the operator of restriction onto the half-axis $\mathbb{R}_+$.

We introduced the spaces $\mathscr{F}_+^{\mathrm{loc}}, \mathscr{F}_+^b$, and Θ_+^{loc}. Now, we can introduce the space $\mathscr{F}^{\mathrm{loc}}, \mathscr{F}^b$, and Θ^{loc} as follows:

$$\mathscr{F}^{\mathrm{loc}} := \{f(s), s \in \mathbb{R} \mid \Pi_{t_1,t_2} f(s) \in \mathscr{F}_{t_1,t_2} \ \forall\, [t_1,t_2] \subseteq \mathbb{R}\},$$

$$\mathscr{F}^b := \{f(s) \in \mathscr{F}^{\mathrm{loc}} \mid \|f\|_{\mathscr{F}^b} < +\infty\},$$

where

$$\|f\|_{\mathscr{F}^b} := \sup_{h\in\mathbb{R}} \|\Pi_{0,1} f(h+s)\|_{\mathscr{F}_{0,1}}. \tag{1.10}$$

The topological space Θ^{loc} coincides (as a set) with $\mathscr{F}^{\mathrm{loc}}$. By definition, $f_k(s) \to f(s)$ in Θ^{loc} as $k \to \infty$ if $\Pi_{t_1,t_2} f_k(s) \to \Pi_{t_1,t_2} f(s)$ in Θ_{t_1,t_2} as $k \to \infty$ for any $[t_1, t_2] \subseteq \mathbb{R}$. It is clear that Θ^{loc} is a metric space, as well as Θ_+^{loc}.

Definition 1.3. The *kernel* $\mathscr{K}$ in the space $\mathscr{F}^b$ of Equation (1.9) is the union of all complete trajectories $u(s)$, $s \in \mathbb{R}$, of Equation (1.9) that are bounded in $\mathscr{F}^b$ in the norm (1.10):

$$\|\Pi_{0,1} u(h+s)\|_{\mathscr{F}_{0,1}} \leqslant C_u \quad \forall h \in \mathbb{R}.$$

Theorem 1.2 (cf. [6, 4])**.** *Let the assumptions of Theorem* 1.1 *hold. Then*

$$\mathfrak{A} = \Pi_+ \mathscr{K}.$$

The set $\mathscr{K}$ is compact in Θ^{loc} and bounded in $\mathscr{F}^b$.

To prove that some ball in $\mathscr{F}_+^b$ is compact in Θ_+^{loc}, we use the following lemma. Let E_0 and E_1 be Banach spaces such that $E_1 \subset E_0$. We consider the Banach spaces

$$W_{p_1,p_0}(0, M; E_1, E_0) = \{\psi(s), s \in 0, M \mid \psi(\cdot) \in L_{p_1}(0, M; E_1),\ \psi'(\cdot) \in L_{p_0}(0, M; E_0)\},$$

$$W_{\infty,p_0}(0, M; E_1, E_0) = \{\psi(s), s \in 0, M \mid \psi(\cdot) \in L_{\infty}(0, M; E_1),\ \psi'(\cdot) \in L_{p_0}(0, M; E_0)\},$$

where $p_1 \geqslant 1$ and $p_0 > 1$, with the norms

$$\|\psi\|_{W_{p_1,p_0}} := \left(\int_0^M \|\psi(s)\|_{E_1}^{p_1} ds \right)^{1/p_1} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds \right)^{1/p_0},$$

$$\|\psi\|_{W_{\infty,p_0}} := \operatorname{ess\,sup}\{\|\psi(s)\|_{E_1} \mid s \in [0, M]\} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds \right)^{1/p_0}.$$

Lemma 1.1 (cf. [7])**.** *Let $E_1 \Subset E \subset E_0$. Then the following embeddings are compact*:

$$W_{p_1,p_0}(0, T; E_1, E_0) \Subset L_{p_1}(0, T; E), \tag{1.11}$$

$$W_{\infty,p_0}(0, T; E_1, E_0) \Subset C([0, T]; E). \tag{1.12}$$

Definition 1.4. We say that trajectory attractors $\mathfrak{A}_\varepsilon$ *converge* to a trajectory attractor $\overline{\mathfrak{A}}$ in the topological space Θ_+^{loc} as $\varepsilon \to 0$ if for any neighborhood $\mathscr{O}(\mathfrak{A})$ in Θ_+^{loc} there is $\varepsilon_1 \geqslant 0$ such that $\mathfrak{A}_\varepsilon \subseteq \mathscr{O}(\overline{\mathfrak{A}})$ for any $\varepsilon < \varepsilon_1$, i.e., for any $M > 0$

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M} \mathfrak{A}_\varepsilon, \Pi_{0,M} \overline{\mathfrak{A}}) \to 0, \quad \varepsilon \to 0.$$

2 Notation and Statement of the Problem

Let Ω be a bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. Let G_0 be a domain in $Y = (-1/2, 1/2)^2$ such that $\overline{G}_0$ is a compact set diffeomorphic to a ball.

We assume that $\delta > 0$ and M is a set. We introduce the notation $\delta M = \{x : \delta^{-1}x \in M\}$. For $j \in \mathbb{Z}^2$ we define

$$P_\varepsilon^j = \varepsilon j, \quad Y_\varepsilon^j = P_\varepsilon^j + \varepsilon Y, \quad G_\varepsilon^j = P_\varepsilon^j + \varepsilon G_0.$$

Further, we introduce the domain $\widetilde{\Omega}_\varepsilon = \{x \in \Omega : \rho(x, \partial\Omega) > \sqrt{2}\,\varepsilon\}$ and the set of admissible indices $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : G_\varepsilon^j \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \varnothing\}$. We note that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-2}$, where $d > 0$ is a constant. We consider the domain $\Omega_\varepsilon = \Omega \setminus \overline{G}_\varepsilon$, where

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j.$$

We set $Q_\varepsilon = \Omega_\varepsilon \times (0, +\infty)$ and $Q = \Omega \times (0, +\infty)$. We introduce the function spaces: $\mathbf{H} := [L_2(\Omega)]^2$, $\mathbf{H}_\varepsilon := [L_2(\Omega_\varepsilon)]^2$, $\mathbf{V} := [H_0^1(\Omega)]^2$, and $\mathbf{V}_\varepsilon := [H^1(\Omega_\varepsilon; \partial\Omega)]^2$ is the set of vector-valued functions in $[H^1(\Omega_\varepsilon)]^2$ with zero trace on $\partial\Omega$. The norms in these spaces are defined by

$$\|v\|^2 := \int_\Omega \sum_{i=1}^2 |v^i(x)|^2 dx, \quad \|v\|_\varepsilon^2 := \int_{\Omega_\varepsilon} \sum_{i=1}^2 |v^i(x)|^2 dx,$$

$$\|v\|_1^2 := \int_\Omega \sum_{i=1}^2 |\nabla v^i(x)|^2 dx, \quad \|v\|_{1\varepsilon}^2 := \int_{\Omega_\varepsilon} \sum_{i=1}^2 |\nabla v^i(x)|^2 dx.$$

We study the asymptotic behavior of trajectory attractors of the following initial-boundary-value problem for autonomous two-dimensional system of Navier–Stokes equations:

$$\begin{aligned}
&\frac{\partial u_\varepsilon}{\partial t} - \nu \Delta u_\varepsilon + (u_\varepsilon, \nabla) u_\varepsilon = g_0\Big(x, \frac{x}{\varepsilon}\Big), \quad x \in \Omega_\varepsilon, \\
&(\nabla, u_\varepsilon) = 0, \quad x \in \Omega_\varepsilon, \\
&\nu \frac{\partial u_\varepsilon}{\partial n} + B\Big(\frac{x}{\varepsilon}\Big) u_\varepsilon = 0, \quad x \in \partial G_\varepsilon,\ t \in (0, +\infty), \\
&u_\varepsilon = 0, \quad x \in \partial\Omega, \\
&u_\varepsilon = U(x), \quad x \in \Omega_\varepsilon,\ t = 0.
\end{aligned} \tag{2.1}$$

Here, $u_\varepsilon = u_\varepsilon(x,t) = (u_\varepsilon^1, u_\varepsilon^2)$, $g = g(x,y) = (g^1, g^2) \in \mathbf{H}$, n is the outward normal vector to the boundary, and $\nu > 0$. Further,

$$B(s) = \begin{pmatrix} b^1(s) & 0 \\ 0 & b^2(s) \end{pmatrix},$$

where $b^k(s) \in C(\mathbb{R}^2)$ are 1-periodic in each variable on $\mathbb{R}^2$ and

$$\int_{\partial G_0} b^k(s)\, d\sigma = 0;$$

here, σ is the length element of the curve ∂G_0, $k = 1, 2$. For a vector-valued function $g(x, y)$ we assume that $g(x, x/\varepsilon) \in \mathbf{H}$ for any $\varepsilon > 0$ and has the mean $\overline{g}(x)$ in the space $\mathbf{H}$ as $\varepsilon \to 0+$, i.e.,

$$g\Big(x, \frac{x}{\varepsilon}\Big) \rightharpoonup \overline{g}(x) \quad \text{weakly in } \mathbf{H}, \quad \varepsilon \to 0+,$$

or

$$\int\limits_{\Omega} g\Big(x, \frac{x}{\varepsilon}\Big)\varphi(x)dx \to \int\limits_{\Omega} \overline{g}(x)\varphi(x)dx, \quad \varepsilon \to 0+, \tag{2.2}$$

for any function $\varphi \in \mathbf{H}$.

By the absolute continuity of the Lebesgue integral, from (2.2) we find

$$\int\limits_{\Omega_\varepsilon} g\Big(x, \frac{x}{\varepsilon}\Big)\varphi(x)dx \to \int\limits_{\Omega} \overline{g}(x)\varphi(x)dx \quad \forall\, \varphi \in \mathbf{H}, \quad \varepsilon \to 0+\,. \tag{2.3}$$

It is known (cf., for example, [6, 8]) that for $U \in \mathbf{H}$ there exists a weak solution $u(s)$ to the problem (2.1) in the space $\mathbf{L}^{\mathrm{loc}}_{2,w}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}^{\mathrm{loc}}_{\infty,*w}(\mathbb{R}_+; \mathbf{H}_\varepsilon)$ such that $u(0) = U$. Moreover, $\dfrac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}^{\mathrm{loc}}_{2,w}(\mathbb{R}_+; \mathbf{H}_\varepsilon)$. We consider weak solutions to the problem (2.1), i.e.,

$$u_\varepsilon(x, s) \in \mathbf{L}^{\mathrm{loc}}_{2,w}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}^{\mathrm{loc}}_{\infty,*w}(\mathbb{R}_+; \mathbf{H}_\varepsilon) \cap \Big\{v : \frac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}^{\mathrm{loc}}_{2,w}(\mathbb{R}_+; \mathbf{H}_\varepsilon)\Big\}$$

that satisfy the problem (2.1) in the sense of distributions, i.e.,

$$\int\limits_{Q_\varepsilon} \frac{\partial u_\varepsilon}{\partial t} \cdot \psi \; dxdt + \nu \int\limits_{Q_\varepsilon} \nabla u_\varepsilon \cdot \nabla \psi \; dxdt + \int\limits_{Q_\varepsilon} (u_\varepsilon, \nabla) u_\varepsilon \psi \; dxdt$$

$$+ \sum_{j \in \Upsilon_\varepsilon} \int\limits_0^{+\infty} \int\limits_{\partial G^j_\varepsilon} B\Big(\frac{x}{\varepsilon}\Big) u_\varepsilon \cdot \psi \; d\sigma dt = \int\limits_{Q_\varepsilon} g_\varepsilon(x) \cdot \psi \; dxdt \quad \forall\, \psi \in \mathbf{C}^\infty_0(\mathbb{R}_+; \mathbf{H}_\varepsilon). \tag{2.4}$$

Here, $y_1 \cdot y_2$ denotes the inner product vectors $y_1, y_2 \in \mathbb{R}^N$.

To describe the trajectory space $\mathscr{K}^+_\varepsilon$ of the problem (2.1), we follow the general scheme of Section 1 and, on every segment $[t_1, t_2] \in \mathbb{R}$, introduce the Banach space

$$\mathscr{F}_{t_1,t_2} := \mathbf{L}^{\mathrm{loc}}_{2,w}(t_1, t_2; \mathbf{V}_\varepsilon) \cap \mathbf{L}^{\mathrm{loc}}_{\infty,*w}(t_1, t_2; \mathbf{H}_\varepsilon) \cap \Big\{v : \frac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}^{\mathrm{loc}}_{2,w}(t_1, t_2; \mathbf{H}_\varepsilon)\Big\} \tag{2.5}$$

equipped with the norm

$$\|v\|_{\mathscr{F}_{t_1,t_2}} := \|v\|_{\mathbf{L}_2(t_1,t_2;\mathbf{V})} + \|v\|_{\mathbf{L}_\infty(t_1,t_2;\mathbf{H})} + \Big\|\frac{\partial v}{\partial t}\Big\|_{\mathbf{L}_2(t_1,t_2;\mathbf{H})}. \tag{2.6}$$

It is obvious that the condition (1.2) holds for the norm (2.6) and the translation semigroup $\{S(h)\}$ satisfies (1.3).

Setting $\mathscr{D}_{t_1,t_2} = \mathbf{L_2}(t_1, t_2; \mathbf{V})$, we find that $\mathscr{F}_{t_1,t_2} \subseteq \mathscr{D}_{t_1,t_2}$. If $u(s) \in \mathscr{F}_{t_1,t_2}$, then $A(u(s)) \in \mathscr{D}_{t_1,t_2}$. Further, we can consider a weak solution to the problem (2.1) as a solution to the system of equations in accordance with the general scheme of Section 1. Introducing the space (1.4), we find

$$\mathscr{F}_+^{\mathrm{loc}} = \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}_\infty^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}) \cap \Big\{ v \Big| \frac{\partial v}{\partial t} \in \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}) \Big\},$$

$$\mathscr{F}_{\varepsilon,+}^{\mathrm{loc}} = \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \Big\{ v \Big| \frac{\partial v}{\partial t} \in \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \Big\}.$$

We denote by $\mathscr{K}_\varepsilon^+$ the set of all weak solutions to the problem (2.1). We recall that for any function $U \in \mathbf{H}$ there exists at least one trajectory $u(\cdot) \in \mathscr{K}_\varepsilon^+$ such that $u(0) = U(x)$. Consequently, the trajectory space $\mathscr{K}_\varepsilon^+$ of the problem (2.1) is not empty.

It is clear that $\mathscr{K}_\varepsilon^+ \subset \mathscr{F}_+^{\mathrm{loc}}$ and the trajectory space $\mathscr{K}_\varepsilon^+$ is translation invariant, i.e., if $u(s) \in \mathscr{K}_\varepsilon^+$, then and $u(h+s) \in \mathscr{K}_\varepsilon^+$ for any $h \geqslant 0$. Consequently,

$$S(h)\mathscr{K}_\varepsilon^+ \subseteq \mathscr{K}_\varepsilon^+ \quad \forall h \geqslant 0.$$

Further, using the $\mathbf{L}_2(t_1,t_2;\mathbf{V})$-norms, we introduce the metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ in the spaces $\mathscr{F}_{t_1,t_2}$ as follows:

$$\rho_{0,M}(u,v) = \bigg(\int\limits_0^M \|u(s) - v(s)\|^2 ds \bigg)^{1/2} \quad \forall\, u(\cdot), v(\cdot) \in \mathscr{F}_{0,M}.$$

These metrics generate the topology of Θ_+^{loc} in the space $\mathscr{F}_+^{\mathrm{loc}}$ (respectively $\Theta_{\varepsilon,+}^{\mathrm{loc}}$ in $\mathscr{F}_{\varepsilon,+}^{\mathrm{loc}}$). We recall that a sequence $\{v_k\} \subset \mathscr{F}_+^{\mathrm{loc}}$ converges to a function $v \in \mathscr{F}_+^{\mathrm{loc}}$ in Θ_+^{loc} as $k \to \infty$ if $\|v_k(\cdot) - v(\cdot)\|_{\mathbf{L}_2(0,M;\mathbf{H})} \to 0$ $(k \to \infty)$ for any $M > 0$. The topology of Θ_+^{loc} is metrizable (cf. (1.6)) and the corresponding metric space is complete. We consider the topology in the trajectory space $\mathscr{K}_\varepsilon^+$ of the problem (2.1). The translation semigroup $\{S(t)\}$ acting on $\mathscr{K}_\varepsilon^+$ is continuous in the topology of the space Θ_+^{loc}.

Following the general scheme of Section 1, we consider the bounded set in $\mathscr{K}_\varepsilon^+$ by using the Banach space $\mathscr{F}_+^b$ (cf. (1.7)). It is clear that

$$\mathscr{F}_+^b = \mathbf{L}_2^b(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}_\infty(\mathbb{R}_+;\mathbf{H}) \cap \Big\{ v \Big| \frac{\partial v}{\partial t} \in \mathbf{L}_2^b(\mathbb{R}_+;\mathbf{H}) \Big\} \tag{2.7}$$

and $\mathscr{F}_+^b$ is a subspace of the space $\mathscr{F}_+^{\mathrm{loc}}$.

We consider the translation semigroup $\{S(t)\}$ on $\mathscr{K}_\varepsilon^+$, $S(t) : \mathscr{K}_\varepsilon^+ \to \mathscr{K}_\varepsilon^+,\ t \geqslant 0$.

Let $\mathscr{K}_\varepsilon$ denote the kernel of the problem (2.1) consisting of all weak solutions $u(s)$, $s \in \mathbb{R}$, bounded in the space

$$\mathscr{F}^b = \mathbf{L}_2^b(\mathbb{R};\mathbf{V}) \cap \mathbf{L}_\infty(\mathbb{R};\mathbf{H}) \cap \Big\{ v \Big| \frac{\partial v}{\partial t} \in \mathbf{L}_2^b(\mathbb{R};\mathbf{H}) \Big\}$$

Proposition 2.1. *The problem* (2.1) *has trajectory attractors* $\mathfrak{A}_\varepsilon$ *in the topological space* Θ_+^{loc}. *The set* $\mathfrak{A}_\varepsilon$ *is uniformly* (*with respect to* $\varepsilon \in (0,1)$) *bounded in* $\mathscr{F}_+^b$ *and compact in* Θ_+^{loc}. *Furthermore,* $\mathfrak{A}_\varepsilon = \Pi_+\mathscr{K}_\varepsilon$, *the kernel* $\mathscr{K}_\varepsilon$ *is nonempty and uniformly* (*with respect to* $\varepsilon \in (0,1)$) *bounded in* $\mathscr{F}^b$. *We recall that the spaces* $\mathscr{F}_+^b$ *and* Θ_+^{loc} *depend on* ε.

The proof of Proposition 2.1 is similar to the proof in [4] given in a particular case..

3 Homogenization of Attractors

3.1. The main assertion. In this subsection, we study the limit behavior of attractors $\mathfrak{A}_\varepsilon$ of the Navier–Stokes equations (2.1) as $\varepsilon \to 0+$ and their convergence to a trajectory attractor of the corresponding homogenized equation. The homogenized (limit) problem has the form

$$\begin{aligned}
&\frac{\partial u_0}{\partial t} - \nu \sum_{i,l=1}^{2} \widehat{a}_{il} \frac{\partial^2 u_0}{\partial x_i \partial x_l} + (u_0, \nabla)u_0 + V u_0 = \overline{g}(x), \quad x \in \Omega,\\
&(\nabla, u_0) = 0, \quad x \in \Omega, \qquad\qquad (3.1)\\
&u_0 = 0, \quad x \in \partial\Omega,\\
&u_0 = U(x), \quad x \in \Omega, \quad t = 0,
\end{aligned}$$

where

$$\widehat{a}_{il} = \int\limits_{Y\setminus G_0} \Big(\frac{\partial N_l(\xi)}{\partial \xi_i} + \delta_{il}\Big) d\xi, \quad \overline{g}(x) = \int\limits_{Y\setminus G_0} g(x,\xi)\, d\xi,$$

$$m_k = -\int\limits_{\partial G_0} b^k(\xi) M^k(\xi)\, d\sigma, \quad V = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix};$$

here, $M^k(\xi)$ and $N_l(\xi)$ are 1-periodic functions of ξ satisfying the problems

$$\begin{cases} \Delta M^k = 0 \quad \text{in } Y \setminus G_0, \\ \dfrac{\partial M^k}{\partial \nu} = -b^k(\xi) \quad \text{on } \partial G_0, \end{cases}$$

$$\begin{cases} \Delta N_l = 0 \quad \text{in } Y \setminus G_0, \\ \dfrac{\partial N_l}{\partial \nu} = -\nu_l \quad \text{on } \partial G_0 \end{cases}$$

and having zero mean over the periodicity cell.

We consider the weak solution to the problem (3.1), i.e., a function

$$u_0(x,s) \in \mathbf{L}^{\text{loc}}_{2,w}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}^{\text{loc}}_{\infty,*w}(\mathbb{R}_+; \mathbf{H}) \cap \Big\{ v : \frac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}^{\text{loc}}_{2,w}(\mathbb{R}_+; \mathbf{H}) \Big\}$$

satisfying the problem (2.1) in the sense of distributions:

$$\begin{aligned}
&\int\limits_Q \frac{\partial u_0}{\partial t} \cdot \psi \, dxdt + \nu \int\limits_Q \sum_{i,l=1}^{2} \widehat{a}_{il} \frac{\partial u_0}{\partial x_i} \frac{\partial \psi}{\partial x_l}\, dxdt + \int\limits_Q (u_0, \nabla) u_0 \psi \, dxdt \\
&+ \int\limits_Q V u_0 \cdot \psi \, dxdt = \int\limits_Q \overline{g}(x) \cdot \psi \, dxdt \quad \forall\, \psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{H}). \qquad (3.2)
\end{aligned}$$

Remark 3.1. The coercivity of the limit operator (3.1) is a delicate problem since the constants m^j are always positive. In particular, the well-posedness of the problem (3.1) connected with the coercivity of the operator is guaranteed by the inequalities

$$\lambda_0 > \max\{m^1, m^2\}, \tag{3.3}$$

where λ_0 is the first eigenvalue of the operator $\nu \sum_{i,l=1}^{2} \widehat{a}_{il} \frac{\partial^2}{\partial x_i \partial x_l}$ in the space $H^1(\Omega)$. The proof of this assertion can be found in [9].

Under the condition (3.3), the problem (3.1) has a trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathscr{K}}^+$ of the problem (3.1); moreover, $\overline{\mathfrak{A}} = \Pi_+ \overline{\mathscr{K}}$ where $\overline{\mathscr{K}}$ is the kernel of the problem (3.1) in $\mathscr{F}^b$.

We formulate the main theorem on homogenization of attractors of the system of Navier–Stokes equations.

Theorem 3.1. *Let* $\lambda_0 > \max\{m^1, m^2\}$. *Then*

$$\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}}, \quad \text{in } \Theta_+^{\mathrm{loc}}, \quad \varepsilon \to 0+, \tag{3.4}$$

$$\mathscr{K}_\varepsilon \to \overline{\mathscr{K}}, \quad \text{in } \Theta^{\mathrm{loc}}. \quad \varepsilon \to 0+. \tag{3.5}$$

Remark 3.2. We recall that the spaces in Theorem 3.1 depend on ε. We assume that all functions under consideration can be extended over the holes with preserving the norms.

3.2. Auxiliaries. We formulate some results of [9] which will be used below.

We consider the auxiliary problem

$$\begin{aligned} & -\nu \Delta_{xx} u_\varepsilon^k = g^k\Big(x, \frac{x}{\varepsilon}\Big), \quad x \in \Omega_\varepsilon, \\ & \nu \frac{\partial u_\varepsilon^k}{\partial n} + b^k\Big(\frac{x}{\varepsilon}\Big) u_\varepsilon^k = 0, \quad x \in \partial G_\varepsilon, \quad k = 1, 2, \\ & u_\varepsilon^k = 0, \quad x \in \partial \Omega. \end{aligned} \tag{3.6}$$

We also require that

$$\int_{\partial G_0} b^k(x) d\sigma = 0. \tag{3.7}$$

We look for a solution in the form of a series

$$u_\varepsilon^k = u_0^k(x) + \varepsilon u_1^k(x, \xi) + \varepsilon^2 u_2^k(x, \xi) + \ldots, \qquad \xi = \frac{x}{\varepsilon}. \tag{3.8}$$

Substituting the series (3.8) into (3.6) and collecting terms with ε of the same order in the equation and boundary conditions, we find a recurrent sequence of problems such that the first one has the form

$$\begin{aligned} & -\nu \Delta_{\xi\xi} u_1^k + \frac{\partial^2 u_0^k}{\partial \xi_1 \partial x_1} + \frac{\partial^2 u_0^k}{\partial \xi_2 \partial x_2} = 0, \quad x \in Y \setminus G_0, \\ & \frac{\partial u_1^k}{\partial n_\xi} + \frac{\partial u_0^k}{\partial n_x} + b^k(\xi) u_0^k = 0, \quad x \in \partial G_0. \end{aligned} \tag{3.9}$$

The integral identity for the problem (3.9) is as follows:

$$\iint\limits_{Y\backslash G_0} \Big(\frac{\partial u_1^k}{\partial \xi_1}\frac{\partial v}{\partial \xi_1} + \frac{\partial u_1^k}{\partial \xi_2}\frac{\partial v}{\partial \xi_2}\Big) d\xi_1 d\xi_2 + \iint\limits_{Y\backslash G_0} \left(\frac{\partial u_0^k}{\partial x_1}\frac{\partial v}{\partial \xi_1} + \frac{\partial u_0^k}{\partial x_2}\frac{\partial v}{\partial \xi_2}\right) d\xi_1 d\xi_2$$
$$+ \int\limits_{\partial G_0} b^k(\xi) u_0^k d\sigma = 0, \tag{3.10}$$

where $v \in H^1_{per}(Y\backslash G_0)$. From the form of the integral identity we can propose that the functions $u_1^k(x,\xi)$ have the following structure:

$$u_1^k(x,\xi) = M^k(\xi)u_0^k(x) + N_1(\xi)\frac{\partial u_0^k}{\partial x_1} + N_2(\xi)\frac{\partial u_0^k}{\partial x_2}. \tag{3.11}$$

Substituting the last expression into (3.10) and collecting the corresponding terms, we obtain the following problem for the functions $N_l(\xi)$ and $M^k(\xi)$:

$$\iint\limits_{Y\backslash G_0} \Big(\frac{\partial N_l}{\partial \xi_1}\frac{\partial v}{\partial \xi_1} + \frac{\partial N_l}{\partial \xi_2}\frac{\partial v}{\partial \xi_2}\Big) d\xi_1 d\xi_2 + \iint\limits_{Y\backslash G_0} \frac{\partial v}{\partial \xi_l} d\xi_1 d\xi_2 = 0 \tag{3.12}$$

or, in the classical form:

$$\begin{cases} \Delta_{\xi\xi}(N_l + \xi_l) = 0, \quad x \in Y \setminus G_0, \\ \dfrac{\partial N_l}{\partial n_\xi} = n_l, \quad x \in \partial G_0, \end{cases}$$

$$\iint\limits_{Y\backslash G_0} \Big(\frac{\partial M^k}{\partial \xi_1}\frac{\partial v}{\partial \xi_1} + \frac{\partial M^k}{\partial \xi_2}\frac{\partial v}{\partial \xi_2}\Big) d\xi_1 d\xi_2 + \int\limits_{\partial G_0} b^j(\xi) v d\sigma = 0 \tag{3.13}$$

or

$$\Delta_{\xi\xi} M^k = 0, \quad x \in Y \setminus G_0,$$
$$\frac{\partial M^k}{\partial n_\xi} + b^k\big(\xi\big) = 0, \quad x \in \partial G_0.$$

The compatibility condition in the problem (3.12) can be easily verified by integrating by parts and using (3.7) in the problem (3.13). We note that the functions $M^k(\xi)$ and $N_l(\xi)$ are defined up to an additive constant and the natural normalization conditions are the following:

$$\iint\limits_{Y\backslash G_0} M^k(\xi) d\xi = \iint\limits_{Y\backslash G_0} N_l(\xi) d\xi = 0.$$

In what follows, we assume that these conditions are satisfied.

The next power of ε yields the problem for $u_2^k(x,\xi)$:

$$\begin{aligned} &\Delta_{\xi\xi} u_2^k + 2\Big(\frac{\partial^2 u_1^k}{\partial \xi_1 \partial x_1} + \frac{\partial^2 u_1^k}{\partial \xi_2 \partial x_2}\Big) + \Delta_{xx} u_0^k = -g^k, \quad x \in Y \setminus G_0, \\ &\frac{\partial u_2^k}{\partial n_\xi} + \frac{\partial u_1^k}{\partial n_x} + b^k\Big(\frac{x}{\varepsilon}\Big) u_1^k = 0, \quad x \in \partial G_0. \end{aligned} \tag{3.14}$$

Lemma 3.1. *The functions $M^k(\xi)$ and $N_l(\xi)$ are connected by the integral identity*

$$\frac{\partial u_0^k(x)}{\partial x_l}\left(\iint\limits_{Y\setminus G_0}\frac{\partial M^k}{\partial \xi_l}d\xi_1 d\xi_2-\int\limits_{\partial G_0}b^k N_l d\sigma\right)=0.$$

We also need the integral identity corresponding to the problem (3.14)

$$\begin{aligned}
&\iint\limits_{Y\setminus G_0}\Big(\frac{\partial u_2^k}{\partial \xi_1}\frac{\partial v}{\partial \xi_1}+\frac{\partial u_2^k}{\partial \xi_2}\frac{\partial v}{\partial \xi_2}\Big)d\xi_1 d\xi_2+\iint\limits_{Y\setminus G_0}\Big(\frac{\partial u_1^k}{\partial x_1}\frac{\partial v}{\partial \xi_1}+\frac{\partial u_1^k}{\partial x_2}\frac{\partial v}{\partial \xi_2}\Big)d\xi_1 d\xi_2\\
&+\int\limits_{\partial G_0}b^k(\xi)u_1^k v d\sigma-\iint\limits_{Y\setminus G_0}\frac{\partial M^k}{\partial \xi_1}v d\xi_1 d\xi_2\cdot\frac{\partial u_0^k}{\partial x_1}-\iint\limits_{Y\setminus G_0}\frac{\partial M^k}{\partial \xi_2}v d\xi_1 d\xi_2\cdot\frac{\partial u_0^k}{\partial x_2}\\
&-\iint\limits_{Y\setminus G_0}\Big(\frac{\partial N_1}{\partial \xi_1}+1\Big)v d\xi_1 d\xi_2\cdot\frac{\partial^2 u_0^k}{\partial x_1^2}-\iint\limits_{Y\setminus G_0}\Big(\frac{\partial N_1}{\partial \xi_2}+\frac{\partial N_2}{\partial \xi_1}\Big)v d\xi_1 d\xi_2\cdot\frac{\partial^2 u_0^k}{\partial x_1\partial x_2}\\
&-\iint\limits_{Y\setminus G_0}\Big(\frac{\partial N_2}{\partial \xi_2}+1\Big)v d\xi_1 d\xi_2\cdot\frac{\partial^2 u_0^k}{\partial x_2^2}+\overline{g}^k=0,
\end{aligned}$$

where

$$\overline{g}^k(x)=\iint\limits_{Y\setminus G_0}g^k(x,\xi)d\xi_1 d\xi_2.$$

The solvability condition for the problem (3.14) leads to the equation for $u_0^k(x)$ which is the required formal homogenized equations. Applying Lemma 3.1, we can write it in the form

$$\nu\sum_{i,l=1}^{2}\widehat{a}_{il}\frac{\partial^2 u_0^k}{\partial x_i\partial x_l}-u_0^k\int\limits_{\partial G_0}b^k(\xi)M^k(\xi)d\sigma=\overline{g}^k(x), \tag{3.15}$$

where

$$\widehat{a}_{il}=\iint\limits_{Y\setminus G_0}\Big(\frac{\partial N_l}{\partial \xi_i}+\delta_{il}\Big)d\xi_1 d\xi_2$$

and δ_{il} is the Kronecker symbol. Thus, the homogenized problem can be written as

$$\begin{aligned}
&\nu\sum_{i,l=1}^{2}\widehat{a}_{il}\frac{\partial^2 u_0^k(x)}{\partial x_i\partial x_l}-m^k u_0^k(x)=\overline{g}^k(x),\quad x\in\Omega,\\
&u_0^k(x)=0,\quad x\in\partial\Omega,
\end{aligned}\tag{3.16}$$

where

$$m^k=\int\limits_{\partial G_0}b^k(\xi)M^k(\xi)d\sigma,\quad k=1,2.$$

Lemma 3.2 (cf. [9]). *If u_ε is a solution to the problem* (3.6) *and u_0 is a solution to the problem* (3.16), *then*

$$\nu \int\limits_{Q_\varepsilon} \nabla u_\varepsilon \cdot \nabla \psi \; dxdt + \sum_{j \in \Upsilon_\varepsilon} \int\limits_0^{+\infty} \int\limits_{\partial G_\varepsilon^j} B\Big(\frac{x}{\varepsilon}\Big) u_\varepsilon \cdot \psi \; d\sigma dt$$

$$\longrightarrow \nu \int\limits_Q \widehat{a} \nabla u_0 \cdot \nabla \psi \; dxdt + \int\limits_Q V u_0 \cdot \psi \; dxdt, \quad \varepsilon \to 0. \tag{3.17}$$

Following [10] and taking into account Remark 3.2, we show that

$$(u_\varepsilon, \nabla) u_\varepsilon \longrightarrow (u, \nabla) u \quad \text{strongly in } L_2(Q). \tag{3.18}$$

For this purpose we use the estimate

$$\|(u_\varepsilon, \nabla) u_\varepsilon - (u, \nabla) u\|_{L_2(Q)} \leqslant \|(u_\varepsilon - u, \nabla) u_\varepsilon\|_{L_2(Q)} + \|(u, \nabla)(u_\varepsilon - u)\|_{L_2(Q)}$$

$$\leqslant C \bigg(\int\limits_Q |u_\varepsilon - u|^2 |\nabla u_\varepsilon|^2 dxds \bigg)^{\frac{1}{2}} + C \bigg(\int\limits_Q |u|^2 |\nabla (u_\varepsilon - u)|^2 dxds \bigg)^{\frac{1}{2}}$$

$$\leqslant C_1 \bigg(\int\limits_Q |\nabla u_\varepsilon|^3 dxds \bigg)^{\frac{1}{3}} \bigg(\int\limits_Q |u_\varepsilon - u|^6 dxds \bigg)^{\frac{1}{6}}$$

$$+ C_1 \bigg(\int\limits_Q |u|^6 dxds \bigg)^{\frac{1}{6}} \bigg(\int\limits_Q |\nabla (u_\varepsilon - u)|^3 dxds \bigg)^{\frac{1}{3}}. \tag{3.19}$$

As proved in [4], the trajectory attractors $\mathfrak{A}_\varepsilon$ and $\overline{\mathfrak{A}}$ of Equations (2.1) and (3.1) exist in the following space with a stronger topology:

$$H_w^{(2,2,1)}(Q) = L_{2,w}(\mathbb{R}_+; [W_2^2(\Omega)]^2) \cap \Big\{ v : \frac{\partial v}{\partial t} \in L_{2,w}(\mathbb{R}_+; \mathbf{H}) \Big\}.$$

We set $H_{3,w}^{(1,1,0)}(Q) = L_{3,w}(\mathbb{R}_+; [W_3^1(\Omega)]^2)$. Since $H^{(2,2,1)}(Q) \Subset H_3^{(1,1,0)}(Q)$ and $H^{(2,2,1)}(Q) \Subset L_6(Q)$, we find

$$\int\limits_Q |u_\varepsilon - u|^6 dxds \to 0, \quad \int\limits_Q |\nabla (u_\varepsilon - u)|^3 dxds \to 0, \quad \varepsilon \to 0. \tag{3.20}$$

Here, we used the uniform boundedness of the integral

$$\int\limits_Q |\nabla u_\varepsilon|^3 dxds \leqslant M.$$

Thus, we have proved the convergence (3.18).

3.3. Proof of Theorem 3.1. It is clear that (3.5) implies (3.4). Therefore, it suffices to prove (3.5), i.e., for any neighborhood $\mathscr{O}(\overline{\mathscr{K}})$ in Θ^{loc} there is $\varepsilon_1 = \varepsilon_1(\mathscr{O}) > 0$ such that

$$\mathscr{K}_\varepsilon \subset \mathscr{O}(\overline{\mathscr{K}}) \quad \forall\, \varepsilon < \varepsilon_1. \tag{3.21}$$

If (3.21) fails, then there exists a neighborhood $\mathscr{O}'(\overline{\mathscr{K}})$ in Θ^{loc}, a sequence $\varepsilon_k \to 0+$ $(k \to \infty)$, and a sequence $u_{\varepsilon_k}(\cdot) = u_{\varepsilon_k}(s) \in \mathscr{K}_{\varepsilon_k}$ such that

$$u_{\varepsilon_k} \notin \mathscr{O}'(\overline{\mathscr{K}}) \quad \forall\, k \in \mathbb{N}. \tag{3.22}$$

By (2.2), the sequence $\{g(x, x/\varepsilon_n)\}$ is bounded in $\mathbf{H}$. Consequently, using the integral identity and the Cauchy–Bunyakowsky inequality, we conclude that the sequence of solutions $\{u_{\varepsilon_n}\}$ is bounded in $\mathscr{F}^b$. Passing to a subsequence, we can assume that $u_{\varepsilon_n} \to u_0$ in Θ^{loc} as $n \to \infty$. We assert that $u_0 \in \overline{\mathscr{K}}$. The functions $u_{\varepsilon_n}(x,s)$ satisfy the equation

$$\frac{\partial u_{\varepsilon_n}}{\partial t} - \nu \Delta u_{\varepsilon_n} + (u_{\varepsilon_n}, \nabla) u_{\varepsilon_n} = g_0\Big(x, \frac{x}{\varepsilon_n}\Big), \quad t \in \mathbb{R}, \tag{3.23}$$

the condition

$$\nu \frac{\partial u_{\varepsilon_n}}{\partial n} + B\Big(\frac{x}{\varepsilon_n}\Big) u_{\varepsilon_n} = 0, \quad x \in \partial G_{\varepsilon_n},$$

and the energy identity

$$-\frac{1}{2} \int\limits_{-M}^{M} \|u_{\varepsilon_n}(s)\|_{\mathbf{H}}^2 \psi'(s)ds + \nu \int\limits_{-M}^{M} \|u_{\varepsilon_n}(s)\|_{\mathbf{V}}^2 \psi(s)ds$$

$$+ \sum_{j \in \Upsilon_\varepsilon} \int\limits_{-M}^{M} \int\limits_{\partial G_\varepsilon^j} B(\frac{x}{\varepsilon}) u_{\varepsilon_n}^2(x,s) \cdot \psi(s) d\sigma ds = \int\limits_{-M}^{M} \Big(g\Big(x, \frac{x}{\varepsilon_n}\Big), u_{\varepsilon_n}(s)\Big) \psi(s) ds \tag{3.24}$$

for any $M > 0$ and any function $\psi \in C_0^\infty(]-M, M[)$, $\psi \geqslant 0$. Furthermore,

$$u_{\varepsilon_n}(s) \rightharpoonup u_0(s) \quad \text{weakly in } \mathbf{L}_2(-M, M; \mathbf{V}) \text{ and } *\text{-weakly in } \mathbf{L}_\infty(-M, M; \mathbf{H}), \quad n \to \infty;$$

moreover,

$$\frac{\partial u_{\varepsilon_n}(s)}{\partial t} \rightharpoonup \frac{\partial u_0(s)}{\partial t} \quad \text{weakly in } \mathbf{L}_2(-M, M; \mathbf{H}), \quad n \to \infty.$$

By the known compactness theorem [8], we can assume that

$$u_{\varepsilon_n}(s) \to u_0(s) \quad \text{strongly in } \mathbf{L}_2(-M, M; \mathbf{H}), \quad n \to \infty,$$

$$u_{\varepsilon_n}(x,s) \to u_0(x,s) \quad \text{for almost all } (x,s) \in D \times (-M, M), \quad n \to \infty.$$

In particular,

$$u_{\varepsilon_n}(s) \to u_0(s) \quad \text{strongly in } \Theta_+^{\mathrm{loc}} = \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}; \mathbf{H}), \quad n \to \infty.$$

By (2.2),

$$g\Big(x, \frac{x}{\varepsilon_n}\Big) \rightharpoonup \overline{g}(x) \quad \text{in } \mathbf{H}_w \text{ and weakly in } \mathbf{L}_2(-M, M; \mathbf{H}), \quad n \to \infty.$$

Now, taking into account Lemma 3.2 and the convergence (3.18), we pass to the limit in (3.23) and (3.24) as $\varepsilon \to 0$, based on a standard argument in [8] (cf. also [6, 4, 11]). Consequently,

$u_0 \in \overline{\mathscr{K}}$, i.e., u_0 is a solution to the problem (3.1) satisfying the corresponding identity (3.24) with the exterior force $\overline{g}(x)$. At the same time, we have established that

$$u_{\varepsilon_n}(s) \to u_0(s) \quad \text{in } \Theta_+^{\text{loc}}, \quad n \to \infty,$$

and, consequently, $u_{\varepsilon_n}(s) \in \mathscr{O}'(u_0(s)) \subset \mathscr{O}'(\overline{\mathscr{K}})$ for $\varepsilon_n \ll 1$. Thus, we arrive at a contradiction with (3.22). The theorem is proved.

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