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Compactness of Commutators for Riesz Potential on Generalized Morrey Spaces

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Abstract: In this paper, we give the sufficient conditions for the compactness of sets in generalized Morrey spaces $M_p^{w(\cdot)}$. This result is an analogue of the well-known Fréchet–Kolmogorov theorem on the compactness of a set in Lebesgue spaces $L_p, p > 0$. As an application, we prove the compactness of the commutator of the Riesz potential $[b, I_\alpha]$ in generalized Morrey spaces, where $b \in VMO$ ($VMO(\mathbb{R}^n)$ denote the BMO-closure of $C_0^\infty(\mathbb{R}^n)$). We prove auxiliary statements regarding the connection between the norm of average functions and the norm of the difference of functions in the generalized Morrey spaces. Such results are also of independent interest.

Keywords: commutator; Riesz potential; compactness; generalized Morrey space; VMO

MSC: 42B20; 42B25

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1. Introduction

Morrey spaces M_p^λ, named after C. Morrey, were introduced by him in 1938 in [1] and defined as follows: For $1 \le p \le \infty$, $n \ge 1$, $0 < \lambda < n$, $f \in M_p^\lambda$ if $f \in L_p^{loc}$ and

$$\|f\|_{M_p^\lambda} \equiv \|f\|_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \left(r^{-\lambda} \|f\|_{L_p(B(x,r))} \right) < \infty,$$

where $B(x,r)$ is a ball with center at the point x and of radius $r > 0$.

For $\lambda = 0$ and $\lambda = n$, the Morrey spaces $M_p^0(\mathbb{R}^n)$ and $M_p^n(\mathbb{R}^n)$ coincide (with equality of norms) with the spaces $L_p(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$, respectively.

Later, the Morrey spaces were found to have many important applications to the Navier–Stokes equations (see [2,3]), the Shrodinger equations (see [4,5]) and the potential analysis (see [6,7]).

Generalized Morrey spaces $M_p^{w(\cdot)}$ were first considered by T. Mizuhara [8], E. Nakai [9] and V.S. Guliyev [10].

Let $1 \le p \le \infty$ and let w be a measurable non-negative function on $(0,\infty)$ that is not equivalent to zero. The generalized Morrey space $M_p^{w(\cdot)} \equiv M_p^{w(\cdot)}(\mathbb{R}^n)$ is defined as the set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with $\|f\|_{M_p^{w(\cdot)}} < \infty$, where

$$\|f\|_{M_p^{w(\cdot)}} = \sup_{x\in\mathbb{R}^n, \quad r>0} \Big(w(r) \|f\|_{L_p(B(x,r))} \Big).$$

The space $M_p^{w(\cdot)}$ coincides with the Morrey space M_p^λ if $w(r) = r^{-\lambda}$, where $0 \le \lambda \le \frac{n}{p}$.

By $\Omega_{p\infty}$ we denote the set of all non-negative, measurable on $(0,\infty)$ functions, not equivalent to 0 and such that for some $t>0$,

$$\|w(r)r^{\frac{n}{p}}\|_{L_\infty(0,t)}<\infty, \quad \|w(r)\|_{L_\infty(t,\infty)}<\infty.$$

The space $M_p^{w(\cdot)}$ is non-trivial if and only if $w\in\Omega_{p\infty}$ [11,12].

The Riesz potential I_α of order $\alpha(0<\alpha<n)$ is defined by

$$I_\alpha f(x)=\int\limits_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

For the function $b\in L_{loc}(\mathbb{R}^n)$, let M_b denote the multiplication operator $M_b f=bf$, where f is a measurable function. Then, the commutator for the Riesz potential I_α and the operator M_b is defined by

$$[b,I_\alpha](f)(x)=M_b(I_\alpha(f(x)))-I_\alpha(M_b f)(x)=\int\limits_{\mathbb{R}^n}\frac{[b(x)-b(y)]f(y)}{|x-y|^{n-\alpha}}dy.$$

The function $b\in L_\infty(\mathbb{R}^n)$ is said to belong to the space $BMO(\mathbb{R}^n)$ if

$$\|b\|_*=\sup_{Q\subset\mathbb{R}^n}\frac{1}{|Q|}\int\limits_Q|b(x)-b_Q|dx<\infty,$$

where Q is a ball in $\mathbb{R}^n$ and $b_Q=\frac{1}{|Q|}\int\limits_{\mathbb{R}^n}b(y)dy$.

By $VMO(\mathbb{R}^n)$, we denote the BMO-closure of the space $C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ is the set of all functions from $C^\infty(\mathbb{R}^n)$ with compact support.

The boundedness of the Riesz potential on the Morrey spaces was investigated by S. Spanne, J. Peetre [13] and D. Adams. [14]. T. Mizuhara [8], E. Nakai [9] and V.S. Guliyev [10] generalized the results of D. Adams and obtained sufficient conditions for the boundedness of I_α on the generalized Morrey spaces. Boundedness of the commutator for the Riesz potential on the Morrey spaces and on the generalized Morrey spaces was considered in [15,16], respectively. The compactness of the commutator for the Riesz potential on the Morrey spaces and on the Morrey spaces with non-doubling measures was considered in [17,18], respectively. The pre-compactness of sets on the Morrey spaces and on variable exponent Morrey spaces was considered in [17,19,20]. The compactness of the commutator for the Riesz potential $[b,I_\alpha]$ on the Morrey-type spaces was also considered in [21,22].

The boundedness and compactness of integral operators and their commutators on various function spaces play an important role in harmonic analysis, in potential theory and PDE [23,24] and in some important physical properties and physical structures [25,26]. Moreover, the interest in the compactness of operator $[b,T]$, where T is the classical Calderón–Zygmund singular integral operator, in complex analysis is from the connection between the commutators and the Hankel-type operators. The compactness of $[b,T]$ attracted attention among researchers in PDEs. For example, with the aid of the compactness of $[b,T]$, one easily derives a Fredholm alternative for equations with VMO coefficients in all L_p spaces for $1<p<\infty$ (see [27]). Hence, it is possible that the compactness of $[b,I_\alpha]$ on generalized Morrey spaces will be applied to discuss some local problems of PDEs with VMO coefficients (see also [28]).

The main goal of this paper is to find the conditions for the pre-compactness of sets on generalized Morrey spaces and to find sufficient conditions for the compactness of the commutator of the Riesz potential $[b,I_\alpha]$ on the generalized Morrey spaces $M_p^{w(\cdot)}(\mathbb{R}^n)$, namely, to find conditions for parameters p,q,α and functions w_1 and w_2 ensuring the compactness of operators $[b,I_\alpha]$ from $M_p^{w_1(\cdot)}$ to $M_q^{w_2(\cdot)}$.

This paper is organized as follows: In Section 2, we present results on the pre-compactness of a set in generalized Morrey spaces. To do this, we will establish some auxiliary lemmas. In Section 3, we give sufficient conditions for the compactness of the commutator for the Riesz potential $[b, I_\alpha]$ on the generalized Morrey space $M_p^{w(\cdot)}(\mathbb{R}^n)$. We will also recall some theorems and establish some auxiliary lemmas. Finally, we draw conclusions in Section 4.

We make some conventions on notation. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, are dependent on the subscript p. We denote $f \lesssim g$ if $f \leq Cg$. By $C(\mathbb{R})$, we denote the set of all continuous bounded functions on $\mathbb{R}$ with the uniform norm, by χ_A we denote the characteristic function of the set $A \subset \mathbb{R}^n$ and by ${}^c A$ we denote the complement of A.

2. On the Pre-Compactness of a Set in Generalized Morrey Spaces

In this section, we give sufficient conditions for the pre-compactness of sets in generalized Morrey spaces.

Theorem 1. *Let $1 \leq p < \infty$ and $w \in \Omega_{p\infty}$. Suppose that the set $S \subset M_p^{w(\cdot)}$ satisfies the following conditions:*

$$\sup_{f \in S} \|f\|_{M_p^{w(\cdot)}} < \infty, \tag{1}$$

$$\lim_{u \to 0} \sup_{f \in S} \|f(\cdot + u) - f(\cdot)\|_{M_p^{w(\cdot)}} = 0, \tag{2}$$

$$\lim_{r \to \infty} \sup_{f \in S} \|f \chi_{{}^c B(0,r)}\|_{M_p^{w(\cdot)}} = 0. \tag{3}$$

Then S is a pre-compact set in $M_p^{w(\cdot)}$.

For the Morrey space M_p^λ, an analogue of Theorem 1 was proved in [17,19]. If $\lambda = 0$, it coincides with the well-known Fréchet–Kolmogorov theorem (see [29]). Theorem 1 is formulated in terms of the difference of a function (see condition (2)). The conditions for the pre-compactness of sets in the global and local Morrey-type spaces were given in terms of the average functions

$$(M_r f)(x) = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)dy, \quad f \in M_p^{w(\cdot)},$$

in [30–32]. Here, $|\ A\ |$ is the Lebesgue measure of the set $A \subset \mathbb{R}^n$.

To prove Theorem 1, we will need the following auxiliary statements.

Lemma 1. *Let $1 \leq p < \infty$ and $w \in \Omega_{p\infty}$. Then, for all $f \in M_p^{w(\cdot)}$ and $r > 0$*

$$\|M_r f - f\|_{M_p^{w(\cdot)}} \leq \sup_{u \in B(0,r)} \|f(\cdot + u) - f(\cdot)\|_{M_p^{w(\cdot)}}. \tag{4}$$

Proof. Let $z \in \mathbb{R}^n$ and $\rho > 0$. Using the Hölder inequality, we have

$$\|M_r f - f\|_{L_p(B(z,\rho))} =$$

$$= \left(\int\limits_{B(z,\rho)} \left| \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)dy - f(x) \right|^p dx \right)^{\frac{1}{p}}$$

$$= \left(\int_{B(z,\rho)} \left| \frac{1}{|B(x,r)|} \int_{B(x,r)} (f(y) - f(x))dy \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq \left(\int_{B(z,\rho)} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|^p dy \right) dx \right)^{\frac{1}{p}}.$$

Next, using the change of variables $y = x + u$ and the Fubini theorem, we obtain

$$\|M_r f - f\|_{L_p(B(z,\rho))} \leq \left(\int_{B(z,\rho)} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x+u) - f(x)|^p du \right) dx \right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} \left(\int_{B(z,\rho)} |f(x+u) - f(x)|^p dx \right) du \right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} \|f(\cdot + u) - f(\cdot)\|^p_{L_p(B(z,\rho))} du \right)^{\frac{1}{p}}.$$

Hence,

$$\|M_r f - f\|_{M_p^{w(\cdot)}} = \sup_{z \in \mathbb{R}^n,\, \rho > 0} w(\rho) \|M_r f - f\|_{L_p(B(z,\rho))}$$

$$\leq \sup_{z \in \mathbb{R}^n,\, \rho > 0} w(\rho) \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} \|f(\cdot + u) - f(\cdot)\|^p_{L_p(B(z,\rho))} du \right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} \sup_{z \in \mathbb{R}^n,\, \rho > 0} w(\rho) \|f(\cdot + u) - f(\cdot)\|^p_{L_p(B(z,\rho))} du \right)^{\frac{1}{p}}.$$

$$= \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} \|f(\cdot + u) - f(\cdot)\|^p_{M_p^{w(\cdot)}} du \right)^{\frac{1}{p}}$$

$$\leq \sup_{u \in B(0,r)} \|f(\cdot + u) - f(\cdot)\|_{M_p^{w(\cdot)}}.$$

Lemma 1 is proved. □

Lemma 2. *Let* $1 \leq p < \infty$, $w \in \Omega_{p\infty}$. *Then, for all* $f \in M_p^{w(\cdot)}$ *and* $r > 0$

$$\|M_r f\|_{M_p^{w(\cdot)}} \leq \|f\|_{M_p^{w(\cdot)}}. \tag{5}$$

Proof. Using the change of variables $y = x + u$, the Hölder inequality and the Fubini theorem, we obtain

$$\|M_r f\|_{L_p(B(z,\rho))} = \left(\int\limits_{B(z,\rho)} \left| \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)dy \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq \left(\int\limits_{B(z,\rho)} \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|^p dy \right) dx \right)^{\frac{1}{p}}$$

$$= \left(\int\limits_{B(z,\rho)} \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} |f(x+u)|^p du \right) dx \right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_{B(z,\rho)} |f(x+u)|^p dx \right) du \right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_{B(z+u,\rho)} |f(v)|^p dv \right) du \right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \|f\|^p_{L_p(B(z+u,\rho))} du \right)^{\frac{1}{p}}.$$

Therefore,

$$\|M_r f\|_{M_p^{w(\cdot)}} = \sup_{z\in\mathbb{R}^n,\ \rho>0} \Big(w(\rho) \|M_r f\|_{L_p(B(z,\rho))} \Big)$$

$$\leq \sup_{z\in\mathbb{R}^n,\ \rho>0} \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \Big(w(\rho) \|f\|_{L_p(B(z+u,\rho))} \Big)^p du \right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\sup_{z\in\mathbb{R}^n,\ \rho>0} w(\rho) \|f\|_{L_p(B(z+u,\rho))} \right)^p du \right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\sup_{x\in\mathbb{R}^n,\ \rho>0} w(\rho) \|f\|_{L_p(B(x,\rho))} \right)^p du \right)^{\frac{1}{p}} = \|f\|_{M_p^{w(\cdot)}}.$$

Lemma 2 is proved. □

Lemma 3. *Let $1 \leq p < \infty$, $w \in \Omega_{p\infty}$. Then, there exists $r_0 > 0$ and for any $0 < r \leq r_0$ there is $C_1 > 0$, depending only on r, n, p, w, such that*

(1) for any $f \in M_p^{w(\cdot)}$

$$\|M_r f\|_{C(\mathbb{R}^n)} \leq C_1 \|f\|_{M_p^{w(\cdot)}(\mathbb{R}^n)}. \tag{6}$$

(2) for any $\delta > 0$

$$\sup_{u \in B(0,\delta)} \|M_r f(\cdot + u) - M_r f(\cdot)\|_{C(\mathbb{R}^n)} \le C_1 \sup_{u \in B(0,\delta)} \|f(\cdot + u) - f(\cdot)\|_{M_p^{w(\cdot)}(\mathbb{R}^n)}. \tag{7}$$

Proof. (1) Since the function $w \in \Omega_{p\infty}$ is not equivalent to 0, then there exists $r_0 > 0$ such that $\sup\limits_{r_0<\rho<\infty} w(\rho) > 0$. Let $0 < r \le r_0$. Using the Hölder inequality, for any $x \in \mathbb{R}^n$, we have

$$|M_r f(x)| \le \frac{1}{|B(x,r)|^{\frac{1}{p}}} \|f\|_{L_p(B(x,r))}.$$

Hence,

$$|M_r f(x)| w(\rho) \le \frac{1}{(v_n r^n)^{\frac{1}{p}}} \Big(w(\rho) \|f\|_{L_p(B(x,r))} \Big),$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$, and

$$|M_r f(x)| \sup_{r<\rho<\infty} w(\rho) \le \frac{1}{(v_n r^n)^{\frac{1}{p}}} \left(\sup_{r<\rho<\infty} w(\rho) \|f\|_{L_p(B(x,r))} \right)$$

$$\le \frac{1}{(v_n r^n)^{\frac{1}{p}}} \left(\sup_{r<\rho<\infty} w(\rho) \|f\|_{L_p(B(x,\rho))} \right) \le \frac{1}{(v_n r^n)^{\frac{1}{p}}} \left(\sup_{\rho>0} w(\rho) \|f\|_{L_p(B(x,\rho))} \right).$$

Therefore, for any $x \in \mathbb{R}^n$

$$|M_r f(x)| \le C_1 \|f\|_{M_p^{w(\cdot)}}, \tag{8}$$

where $C_1 = \left(\big(\sup\limits_{r<\rho<\infty} w(\rho) \big) (v_n r^n)^{\frac{1}{p}} \right)^{-1} < \infty$, since $w \in \Omega_{p\infty}$.

(2) For any $x_1, x_2 \in B(0,r)$, by Hölder's inequality, we have

$$|(M_r f)(x_1) - (M_r f)(x_2)| = \frac{1}{v_n r^n} \left| \int\limits_{B(x_1,r)} f(y)dy - \int\limits_{B(x_2,r)} f(y)dy \right|$$

$$= (v_n r^n)^{-1} \left| \int\limits_{B(0,r)} f(z + x_1)dz - \int\limits_{B(0,r)} f(z + x_2)dz \right|$$

$$\le (v_n r^n)^{-1} \int\limits_{B(0,r)} |f(z + x_1) - f(z + x_2)| dz$$

$$= (v_n r^n)^{-1} \int\limits_{B(x_2,r)} |f(s + x_1 - x_2) - f(s)| ds$$

$$\le (v_n r^n)^{-\frac{1}{p}} \|f(\cdot + x_1 - x_2) - f(\cdot)\|_{L_p(B(x_2,r))}.$$

Therefore, similar to the first part of the proof, we obtain

$$|(M_r f)(x_1) - (M_r f)(x_2)| \le C_1 \|f(\cdot + x_1 - x_2) - f(\cdot)\|_{M_p^{w(\cdot)}}.$$

Hence,

$$\sup_{x_1,x_2 \in \mathbb{R}^n,\ |x_1 - x_2| \le \delta} |(M_r f)(x_1) - (M_r f)(x_2)|$$

$$\le C_1 \sup_{x_1,x_2 \in \mathbf{R}^n,\ |x_1 - x_2| \le \delta} \|f(\cdot + x_1 - x_2) - f(\cdot)\|_{M_p^{w(\cdot)}}$$

$$= C_1 \sup_{u\in B(0,\delta)} \|f(\cdot+u)-f(\cdot)\|_{M_p^{w(\cdot)}}.$$

Lemma 3 is proved. □

Lemma 4. *Let $1 \le p < \infty$, $w \in \Omega_{p\infty}$. Then, there exists $C_2 > 0$, depending only on n, p, w, such that for any $r, R > 0$ and for any $f, g \in M_p^{w(\cdot)}$*

$$\|M_r f - M_r g\|_{M_p^{w(\cdot)}} \le C_2(1+R^{\frac{n}{p}})\|M_r f - M_r g\|_{C\left(\overline{B(0,R)}\right)}$$

$$+ \sup_{u\in B(0,r)} \|f(\cdot+u)-f(\cdot)\|_{M_p^{w(\cdot)}} + \sup_{u\in B(0,r)} \|g(\cdot+u)-g(\cdot)\|_{M_p^{w(\cdot)}}$$

$$+ \left\|f\chi_{{}^c B(0,R)}\right\|_{M_p^{w(\cdot)}} + \left\|g\chi_{{}^c B(0,R)}\right\|_{M_p^{w(\cdot)}}.$$

Proof. Indeed,

$$\|M_r f - M_r g\|_{M_p^{w(\cdot)}}$$

$$\le \left\|(M_r f - M_r g)\chi_{B(0,R)}\right\|_{M_p^{w(\cdot)}} + \left\|(M_r f - M_r g)\chi_{{}^c B(0,R)}\right\|_{M_p^{w(\cdot)}} := I_1 + I_2.$$

First, we will estimate I_1. By using $B(x,\rho)\cap B(0,R) \subset B(0,R)$, $B(x,\rho)\cap B(0,R) \subset B(x,\rho)$, for any $\rho > 0$, $R > 0$, we have

$$I_1 = \sup_{x\in\mathbb{R}^n,\ \rho>0} \Big(w(\rho)\|M_r f - M_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\Big)$$

$$\le \sup_{x\in\mathbb{R}^n,\ 0<\rho<1} \Big(w(\rho)\|M_r f - M_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\Big)$$

$$+ \sup_{x\in\mathbb{R}^n,\ 1\le\rho<\infty} \Big(w(\rho)\|M_r f - M_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\Big)$$

$$\le \|M_r f - M_r g\|_{C(\overline{B(0,R)})} \cdot \left(\sup_{0<\rho<1} w(\rho)(v_n\rho^n)^{\frac{1}{p}} + \sup_{1\le\rho<\infty} w(\rho)(v_n R^n)^{\frac{1}{p}}\right)$$

$$\le \|M_r f - M_r g\|_{C(\overline{B(0,R)})} \cdot v_n^{\frac{1}{p}}\left(\sup_{0<\rho<1} w(\rho)\rho^{\frac{n}{p}} + \sup_{1\le\rho<\infty} w(\rho)R^{\frac{n}{p}}\right).$$

Therefore,

$$I_1 \le \|M_r f - M_r g\|_{C(\overline{B(0,R)})} \cdot v_n^{\frac{1}{p}}\left(\sup_{0<\rho<1} w(\rho)\rho^{\frac{n}{p}} + \sup_{1\le\rho<\infty} w(\rho)\right)\times$$

$$\times\left(\frac{\sup_{0<\rho<1} w(\rho)\rho^{\frac{n}{p}}}{\sup_{0<\rho<1} w(\rho)\rho^{\frac{n}{p}} + \sup_{1<\rho<\infty} w(\rho)} + \frac{\sup_{1\le\rho<\infty} w(\rho)}{\sup_{0<\rho<1} w(\rho)\rho^{\frac{n}{p}} + \sup_{1\le\rho<\infty} w(\rho)}\cdot R^{\frac{n}{p}}\right)$$

$$\le C_2\Big(1+R^{\frac{n}{p}}\Big)\|M_r f - M_r g\|_{C(\overline{B(0,R)})},$$

where

$$C_2 = v_n^{\frac{1}{p}}\left(\sup_{0<\rho<1} w(\rho)\rho^{\frac{n}{p}} + \sup_{1\le\rho<\infty} w(\rho)\right) < \infty,$$

since, by $w \in \Omega_{p\infty}$.

For estimate I_2, using Lemma 1, we have

$$I_2 = \left\| (M_r f - M_r g)\chi_{{}^c B(0,R)} \right\|_{M_p^{w(\cdot)}}$$

$$\leq \left\| (M_r f - f)\chi_{{}^c B(0,R)} \right\|_{M_p^{w(\cdot)}} + \left\| (f-g)\chi_{{}^c B(0,R)} \right\|_{M_p^{w(\cdot)}} + \left\| (M_r g - g)\chi_{{}^c B(0,R)} \right\|_{M_p^{w(\cdot)}}$$

$$\leq \| M_r f - f\|_{M_p^{w(\cdot)}} + \left\| (f-g)\chi_{{}^c B(0,R)} \right\|_{M_p^{w(\cdot)}} + \| M_r g - g\|_{M_p^{w(\cdot)}}$$

$$\leq \sup_{u\in B(0,r)} \|f(\cdot+u) - f(\cdot)\|_{M_p^{w(\cdot)}} + \sup_{u\in B(0,r)} \|g(\cdot+u) - g(\cdot)\|_{M_p^{w(\cdot)}}$$

$$+ \left\| f\chi_{{}^c B(0,R)} \right\|_{M_p^{w(\cdot)}} + \left\| g\chi_{{}^c B(0,R)} \right\|_{M_p^{w(\cdot)}}.$$

From estimates of I_1 and I_2, we obtain the inequality of Lemma 4.

Lemma 4 is proved. □

Lemma 5. *Let $1 \leq p < \infty$, $w \in \Omega_{p\infty}$. Then, for any $r, R > 0$ and for any $f, g \in M_p^{w(\cdot)}$*

$$\|f - g\|_{M_p^{w(\cdot)}} \leq C_2\left(1 + R^{\frac{n}{p}}\right)\|M_r f - M_r g\|_{C(\overline{B(0,R)})}$$

$$+2 \sup_{u\in B(0,r)} \|f(\cdot+u) - f(\cdot)\|_{M_p^{w(\cdot)}} + 2 \sup_{u\in B(0,r)} \|g(\cdot+u) - g(\cdot)\|_{M_p^{w(\cdot)}} \tag{9}$$

$$+ \left\| f\chi_{{}^c B(0,R)} \right\|_{M_p^{w(\cdot)}} + \left\| g\chi_{{}^c B(0,R)} \right\|_{M_p^{w(\cdot)}},$$

where $C_2 > 0$ is the same as in Lemma 4.

Proof. It is sufficient to note that

$$\|f - g\|_{M_p^{w(\cdot)}} \leq \|M_r f - f\|_{M_p^{w(\cdot)}} + \|M_r f - M_r g\|_{M_p^{w(\cdot)}} + \|M_r g - g\|_{M_p^{w(\cdot)}}$$

and use Lemmas 1 and 4. □

Proof of Theorem 1. Let $S \subset M_p^{w(\cdot)}$ and let conditions (1)–(3) hold.

Step 1. First, we show that the set $S_r = \{M_r f : f \in S\}$ is a strongly pre-compact set in $C(\overline{B(0,R)})$.

Let $0 < r < r_0$, where r_0 is defined in Lemma 3 and $R > 0$ is fixed. Due to inequality (6) and condition (1), it follows that

$$\sup_{f\in S} \|M_r f\|_{C(\overline{B(0,R)})} \leq \sup_{f\in S} \|M_r f\|_{C(\mathbb{R}^n)} \leq C_1 \sup_{f\in S} \|f\|_{M_p^{w(\cdot)}} < \infty.$$

In addition, due to inequality (7) and condition (2), it follows that

$$\sup_{u\in B(0,\delta)} \|M_r f(\cdot+u) - M_r f(\cdot)\|_{C(\overline{B(0,R)})} \leq \sup_{u\in B(0,\delta)} \|M_r f(\cdot+u) - M_r f(\cdot)\|_{C(\mathbb{R}^n)}$$

$$\leq C_1 \sup_{u\in B(0,\delta)} \|f(\cdot+u) - f(\cdot)\|_{M_p^{w(\cdot)}}.$$

Therefore, by using condition (2), we have

$$\lim_{u\to 0} \sup_{f\in S} \|M_r f(\cdot+u) - M_r f(\cdot)\|_{C(\overline{B(0,R)})} = 0.$$

As such, we obtained that the set S_r is uniformly bounded and equicontinuous in $C(\overline{B(0,R)})$.

Therefore, by the Ascoli–Arzela theorem, the set S_r is pre-compact in $C(\overline{B(0,R)})$, then the set S_r is totally bounded in $C(\overline{B(0,R)})$. Hence, for any $\varepsilon > 0$, there exists $f_1, ..., f_m \in S$ (depending on ε, r and R) such that $\{M_r f_1, M_r f_2, ..., M_r f_m\}$ is a finite ε-net in S_r with respect to norm of $C(\overline{B(0,R)})$. Therefore, for any $f \in S$, there is $1 \leq j \leq m$ such that

$$\|M_r f - M_r f_j\|_{C(\overline{B(0,R)})} < \varepsilon.$$

Hence,

$$\min_{j=1,...,m} \|M_r f - M_r f_j\|_{C(\overline{B(0,R)})} < \varepsilon.$$

Step 2. Let us show that the set S is a relative compact set in $M_p^{w(\cdot)}$. Let $\{\varphi_1, ..., \varphi_m\}$ be an arbitrary finite subset of S. By inequality (9) for any $f \in S$ and any $j = 1, ..., m$ we have

$$\|f - \varphi_j\|_{M_p^{w(\cdot)}} \leq C_2(1 + R^{\frac{n}{p}})\|M_r f - M_r \varphi_j\|_{C(\overline{B(0,R)})}$$

$$+2 \sup_{u \in B(0,r)} \|f(\cdot + u) - f(\cdot)\|_{M_p^{w(\cdot)}} + 2 \sup_{u \in B(0,r)} \|\varphi_j(\cdot + u) - \varphi_j(\cdot)\|_{M_p^{w(\cdot)}}$$

$$+\left\|f \chi_{^c B(0,R)}\right\|_{M_p^{w(\cdot)}} + \left\|\varphi_j \chi_{^c B(0,R)}\right\|_{M_p^{w(\cdot)}}$$

$$\leq C_2(1 + R^{\frac{n}{p}})\|M_r f - M_r \varphi_j\|_{C(\overline{B(0,R)})}$$

$$+4 \sup_{g \in S} \sup_{u \in B(0,r)} \|g(\cdot + u) - g(\cdot)\|_{M_p^{w(\cdot)}} + 2 \sup_{g \in S} \left\|g \chi_{^c B(0,R)}\right\|_{M_p^{w(\cdot)}},$$

where C_2 is the same as in Lemma 4, $C_2 = v_n^{\frac{1}{p}} \left(\sup_{0<\rho<1} w(\rho)\rho^{\frac{n}{p}} + R^{\frac{n}{p}} \sup_{1 \leq \rho < \infty} w(\rho) \right)$.

Hence, for any $f \in S$:

$$\min_{j=1,...,m} \|f - \varphi_j\|_{M_p^{w(\cdot)}} \leq C_2(1 + R^{\frac{n}{p}}) \min_{j=1,...,m} \|M_r f - M_r \varphi_j\|_{C(\overline{B(0,R)})}$$

$$+4 \sup_{g \in S} \sup_{u \in B(0,r)} \|g(\cdot + u) - g(\cdot)\|_{M_p^{w(\cdot)}} + 2 \sup_{g \in S} \left\|g \chi_{^c B(0,R)}\right\|_{M_p^{w(\cdot)}}. \quad (10)$$

Let $\varepsilon > 0$. First, using condition (3) we find $R(\varepsilon) > 0$ such that

$$\sup_{g \in S} \left\|g \chi_{^c B(0,R(\varepsilon))}\right\|_{M_p^{w(\cdot)}} < \frac{\varepsilon}{6}.$$

Next, using condition (2), we find $r(\varepsilon)$ such that

$$\sup_{u \in B(0,r(\varepsilon))} \sup_{g \in S} \|g(\cdot + u) - g(\cdot)\|_{M_p^{w(\cdot)}} < \frac{\varepsilon}{12}.$$

Finally, by the pre-compactness of the set $S_{r(\varepsilon)}$ in $C(\overline{B(0, R(\varepsilon))})$, there exist $m(\varepsilon) \in \mathbb{N}$ and $f_{1,\varepsilon}, ..., f_{m(\varepsilon),\varepsilon} \in S$, such that for any $f \in S$

$$\min_{j=1,...,m(\varepsilon)} \left\|M_{r(\varepsilon)} f - M_{r(\varepsilon)} f_{j,\varepsilon}\right\|_{C(\overline{B(0,R(\varepsilon))})} < \frac{\varepsilon}{3C_2(1 + R^{\frac{n}{p}})}.$$

Therefore, setting $\varphi_j = f_{j,\varepsilon}, j = 1, ..., m(\varepsilon)$, by inequality (10), for any $f \in S$ we obtain

$$\min_{j=1,...,m(\varepsilon)} \|f - f_{j,\varepsilon}\|_{M_p^{w(\cdot)}} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Then, we have that $\varphi_j = f_{j,\varepsilon}, j = 1, ..., m(\varepsilon)$ is a finite ε-net in S in the norm of $M_p^{w(\cdot)}$.

Therefore, the set S is a pre-compact set in $M_p^{w(\cdot)}$. Theorem 1 is proved. □

3. Compactness of the Commutator for the Riesz Potential on Generalized Morrey Spaces

The main goal of this section is to find sufficient conditions for the compactness of the commutator $[b, I_\alpha]$ from $M_p^{w_1(\cdot)}$ to $M_q^{w_2(\cdot)}$.

The Riesz potential I_α of order $\alpha(0 < \alpha < n)$ is defined by

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

The boundedness of I_α on Morrey spaces was investigated in [13,14].

The sufficient conditions for the boundedness of I_α from $M_p^{w_1(\cdot)}$ to $M_q^{w_2(\cdot)}$ were obtained by T. Mizuhara [8], E. Nakai [9], and V.S. Guliyev [10].

The following theorems give sufficient conditions for the boundedness of the Riesz potential and its commutator in generalized Morrey spaces.

Theorem 2 ([10])**.** *Let $1 < p < q < \infty$ and $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$. Moreover, let functions $w_1 \in \Omega_{p,\infty}$, $w_2 \in \Omega_{q,\infty}$ satisfy the condition*

$$\left\| w_1^{-1}(r) r^{-\frac{n}{q}-1} \right\|_{L_1(t,\infty)} \lesssim w_2^{-1}(t) t^{-\frac{n}{p}} \tag{11}$$

uniformly in $t \in (0, \infty)$. Then, the operator I_α is bounded from $M_p^{w_1(\cdot)}$ to $M_q^{w_2(\cdot)}$.

Theorem 3 ([16])**.** *Let $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$ and $w_1(\cdot), w_2(\cdot)$ satisfy the following condition*

$$\int\limits_r^\infty \ln\left(e + \frac{l}{r}\right) \frac{ess \inf\limits_{t<s<\infty} w_1(s) dt}{t} \lesssim w_2(r). \tag{12}$$

Then, the operator $[b, I_\alpha]$ is bounded from $M_p^{w_1(\cdot)}$ to $M_q^{w_2(\cdot)}$.

Theorem 4. *Let $1 < p < q < \infty$, $0 < \alpha < n(1 - \frac{1}{q})$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in VMO(\mathbb{R}^n)$ and functions $w_1 \in \Omega_{p,\infty}$, $w_2 \in \Omega_{q,\infty}$ satisfy conditions (11) and (12). Then, the commutator $[b, I_\alpha]$ is a compact operator from $M_p^{w_1(\cdot)}$ to $M_q^{w_2(\cdot)}$.*

To prove Theorem 4, we need the following auxiliary statements.

Lemma 6. *Let $n \in \mathbb{N}, 1 < p < q < \infty, 0 < \alpha < n\left(1 - \frac{1}{q}\right), \beta > 0$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then, there is $C_5 > 0$, depending only on n, p, q, α, such that for some $f \in L_p(B(0, \beta))$ satisfying the condition $supp f \subset \overline{B(0, \beta)}$, and for some $\gamma \geq 2\beta$, $t \in \mathbb{R}^n$, $r > 0$*

$$\left\| (I_\alpha f) \chi_{{}^c B(0,\gamma)} \right\|_{L_q(B(t,r))} \leq C_5 \gamma^{\alpha - n} (min\{\gamma, r\})^{\frac{n}{q}} \|f\|_{L_p(B(0,\beta))}. \tag{13}$$

Proof. Let $f \in L_p(B(t, r))$. By definition of the operator I_α, we have

$$I := \left\| (I_\alpha f) \chi^c_{B(0,\gamma)} \right\|_{L_q(B(t,r))}$$

$$= \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left| \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}}$$

$$\leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left| \int\limits_{B(0,\beta)} \frac{f(y)}{|x-y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}}.$$

Since $\beta \leq \frac{\gamma}{2}$ for $x \in {}^c B(0,\gamma)$, $y \in B(0,\beta)$, we have

$$|x-y| \geq |x| - |y| \geq |x| - \beta = \frac{|x|}{2} + \frac{|x|}{2} - \beta \geq \frac{|x|}{2}. \tag{14}$$

By $(n-\alpha)q - n > 0$, we have

$$I \leq 2^{n-\alpha} \left(\int\limits_{^c B(0,\gamma)} \frac{dx}{|x|^{(n-\alpha)q}} \right)^{\frac{1}{q}} \int\limits_{B(0,\beta)} |f(y)| dy$$

$$\leq 2^{n-\alpha} \left(\int\limits_{\gamma}^{\infty} \rho^{-(n-\alpha)q+n-1} d\rho \right)^{\frac{1}{q}} (v_n \beta^n)^{1-\frac{1}{p}} \|f\|_{L_p(B(0,\beta))}$$

$$\equiv C_6 \gamma^{\alpha - n(1-\frac{1}{q})} \|f\|_{L_p(B(0,\beta))}. \tag{15}$$

Since $\beta \leq \frac{\gamma}{2}$ for $x \in {}^c B(0,\gamma), y \in B(0,\beta)$, by (14) $|x-y| \geq \frac{|x|}{2}$. Therefore,

$$I \leq 2^{n-\alpha} \gamma^{\alpha-n} \left(\int\limits_{B(t,r)} dx \right)^{\frac{1}{q}} \int\limits_{B(0,\beta)} |f(y)| dy$$

$$\leq 2^{n-\alpha} \gamma^{\alpha-n} (v_n r^n)^{\frac{1}{q}} (v_n \beta^n)^{1-\frac{1}{p}} \|f\|_{L_p(B(0,\beta))}$$

$$= C_4 \gamma^{\alpha-n} r^{\frac{n}{q}} \|f\|_{L_p(B(0,\beta))}. \tag{16}$$

Inequalities (15) and (16) imply inequality (13), where $C_5 = \max\{C_6, C_4\}$ □

Lemma 7. *Let $n \in \mathbb{N}, 1 < p < q < \infty, 0 < \alpha < n\left(1 - \frac{1}{q}\right), \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \beta > 0$. Then, there is $C_7 > 0$ depending only on n, p, q, α such that for some $f \in L_p(B(0,\beta))$, $b \in L_\infty(\mathbb{R}^n)$ satisfying the condition $supp\, b \subset \overline{B(0,\beta)}$, and for some $\gamma \geq 2\beta$, $t \in \mathbb{R}^n$, $r > 0$*

$$\left\| ([b, I_\alpha] f) \chi_{^c B(0,\gamma)} \right\|_{L_q(B(t,r))} \leq C_7 \gamma^{\alpha-n} (min\{\gamma, r\})^{\frac{n}{q}} \|b\|_{L_\infty(\mathbb{R}^n)} \|f\|_{L_p(B(0,\beta))}. \tag{17}$$

Proof. Let $\gamma > \beta$, $supp\, b \subset B(0,\beta)$, for $x \in {}^c B(0,\gamma), b(x) = 0$. Then

$$\left\| [b, I_\alpha] f \chi^c_{B(0,\gamma)} \right\|_{L_q(B(t,r))}$$

$$= \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left| \int\limits_{\mathbb{R}^n} \frac{(b(x) - b(y)) f(y)}{|x-y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}}$$

$$\leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left| \int\limits_{\mathbb{R}^n} \frac{b(y)f(y)}{|x-y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}}$$

$$\leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left| \int\limits_{B(0,\beta)} \frac{|b(y)|\cdot|f(y)|}{|x-y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}}$$

$$\leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left| \int\limits_{B(0,\beta)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}} \|b\|_{L_\infty(\mathbb{R}^n)}.$$

Finally, by proof of Lemma 6, we obtain estimate (17). □

Proof of Theorem 4. Let us prove that for $[b, I_\alpha]f$, conditions (1)–(3) of Theorem 1 are satisfied.

Let F be an arbitrary bounded set in $M_p^{w_1(\cdot)}$. Due to the density, it is sufficient to prove the statement of the theorem under the condition $b \in C_0^\infty(\mathbb{R}^n)$; i.e., under this condition, the set $G = \{[b, I_\alpha]f : f \in F\}$ is pre-compact in $M_q^{w_2(\cdot)}$.

Let

$$\|f\|_{M_p^{w_1(\cdot)}} \leq D, \quad \text{for} \quad f \in F.$$

By Theorem 3, we have

$$\|[b, I_\alpha]f\|_{M_q^{w_2(\cdot)}} \leq C_8 \cdot \sup_{f\in F} \|f\|_{M_p^{w_1(\cdot)}} \leq C_8 \cdot D < \infty.$$

This implies condition (1) of Theorem 1.

Now let us prove that condition (3) of Theorem 1 holds for $[b, I_\alpha]$. On the other hand, suppose that $suppb \subset \{x : |x| \leq \beta\}$. For any $0 < \varepsilon < 1$, we take $\gamma > \beta + 1$ such that $(\gamma - \beta)^{-(n-\alpha)+n/q} < \varepsilon$. Below, we show that for every $t \in \mathbb{R}^n$ and $r > 0$,

$$\|[b, I_\alpha]f\chi^c_{B(0,\gamma)}\|_{M_q^{w_2(\cdot)}} < C_9 \cdot D \cdot \varepsilon,$$

hence

$$\lim_{\gamma\to\infty} \left\| ([b, I_\alpha]f)\chi^c_{B(0,\gamma)} \right\|_{M_q^{w_2(\cdot)}} = 0.$$

By Lemma 7, we have

$$\left\| ([b, I_\alpha]f)\chi_{^cB(0,\gamma)} \right\|_{M_q^{w_2(\cdot)}} = \sup_{x\in(\mathbb{R}^n)} \left\| w_2(r) \|([b, I_\alpha]f)\chi_{^cB(0,\gamma)}\|_{L_p(B(x,r))} \right\|_{L_\infty(0,\infty)}$$

$$\leq C_5 \gamma^{\alpha-n} \sup_{x\in(\mathbb{R}^n)} \left\| w_2(r)(min\{\gamma, r\})^{\frac{n}{q}} \right\|_{L_\infty(0,\infty)} \|b\|_{L_\infty(\mathbb{R}^n)} \|f\|_{L_p(B(0,\beta))}.$$

For $r < t < \gamma$, we have $(min\{\gamma, r\})^{\frac{n}{q}} = r^{\frac{n}{q}}$. Using condition $w_2 \in \Omega_{q,\infty}$, we obtain

$$\|w_2(r)r^{\frac{n}{q}}\|_{L_\infty(0,t)} < \infty.$$

For $\gamma < t < r$, we have $(min\{\gamma, r\})^{\frac{n}{q}} = \gamma^{\frac{n}{q}}$. Using condition $w_2 \in \Omega_{q,\infty}$, we obtain

$$\|w_2(r)\gamma^{\frac{n}{q}}\|_{L_\infty(t,\infty)} = \gamma^{\frac{n}{q}} \|w_2(r)\|_{L_\infty(t,\infty)} < \infty.$$

$$\lim_{\gamma\to\infty}\left\|([b,I_\alpha]f)\chi_{{}^cB(0,\gamma)}\right\|_{M_q^{w_2(\cdot)}}=0.$$

Consequently, we have the required condition (3) of Theorem 1.

Now, let us prove that condition (2) of Theorem 1 holds for the set $[b, I_\alpha]$, where $f \in F$. That is, we will show that for all $\varepsilon > 0$ and for all $f \in F$, the inequality

$$\|[(b,I_\alpha f)(\cdot+z)]-[b,I_\alpha]f(\cdot)\|_{M_q^{w_2(\cdot)}}\le C_{10}\cdot\varepsilon,$$

is satisfied for sufficiently small $|z|$.

Let ε be an arbitrary number such that $0<\varepsilon<\frac{1}{2}$. For $|z|\in\mathbb{R}^n$, we have

$$[b,I_\alpha]f(x+z)-[b,I_\alpha]f(x)=\int\limits_{|x-y|>\frac{|z|}{\varepsilon}}\frac{[b(x+z)-b(x)]f(y)}{|x-y|^{n-\alpha}}dy$$

$$+\int\limits_{|x-y|>\frac{|z|}{\varepsilon}}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x+z-y|^{n-\alpha}}\right)\cdot[b(y)-b(x+z)]f(y)dy$$

$$+\int\limits_{|x-y|\le\frac{|z|}{\varepsilon}}\frac{[b(y)-b(x)]f(y)}{|x-y|^{n-\alpha}}dy-\int\limits_{|x-y|\le\frac{|z|}{\varepsilon}}\frac{[b(y)-b(x+z)]f(y)}{|x+z-y|^{n-\alpha}}dy$$

$$=J_1+J_2+J_3-J_4.$$

Due to $b\in C_0^\infty(\mathbb{R}^n)$, we have

$$|b(x)-b(x+z)|\le|\nabla f(x)|\cdot|z|\le C_{11}|z|.$$

Then,

$$|J_1|\le C_{11}|z|I_\alpha(|f|)(x).$$

By Theorem 2,

$$\|J_1\|_{M_q^{w_2(\cdot)}}\le C_{11}|z|\|I_\alpha(f)\|_{M_q^{w_2(\cdot)}}\le C_{11}|z|\|f\|_{M_p^{w_1(\cdot)}}\le C_{11}D|z|.$$

For J_2, we have that

$$(b(x+z)-b(y))\le 2\|b\|_\infty\le C_{10}.$$

Therefore,

$$|J_2|\le C_{12}|z|\int\limits_{|x-y|>\frac{|z|}{\varepsilon}}\frac{f(y)}{|x-y|^{n-\alpha}}dy\le C_{12}\varepsilon I_\alpha(|f|)(x).$$

Again, based on Theorem 2, we obtain

$$\|J_2\|_{M_q^{w_2(\cdot)}}\le C_{12}\varepsilon\|I_\alpha(f)\|_{M_p^{w_1(\cdot)}}\le C_{12}\varepsilon\|f\|_{M_p^{w_1(\cdot)}}\le C_{12}\cdot D\cdot\varepsilon.$$

Now, consider J_3. Since $b\in C_0^\infty$, we have $|b(x)-b(y)|\le C_{13}|x-y|$.

Then, for $|J_3|$, we have

$$|J_3|\le C_{13}\int\limits_{|x-y|\le\frac{|z|}{\varepsilon}}\frac{f(y)}{|x-y|^{n-\alpha-1}}dy$$

$$\le C_{13}\varepsilon^{-1}|z|\int\limits_{|x-y|\le\frac{|z|}{\varepsilon}}\frac{f(y)}{|x-y|^{n-\alpha}}dy$$

$$\leq C_{13} \cdot \frac{|z|}{\varepsilon} I_\alpha(|f|)(x).$$

Therefore, by Theorem 2

$$\|J_3\|_{M_q^{w_2(\cdot)}} \leq C_{13} \cdot \varepsilon^{-1}|z| \|I_\alpha(f)\|_{M_q^{w_2(\cdot)}} \leq C_{13} \cdot \varepsilon^{-1}|z| \|f\|_{M_p^{w_1(\cdot)}} \leq \varepsilon^{-1}|z|.$$

Similarly, using the estimate

$$|b(x+z) - b(y)| \leq C_{14}|x+z-y|,$$

we obtain

$$|J_4| \leq C_{14} \int\limits_{|x-y| \leq \varepsilon^{-1}|z|} |x+z-y|^{-n+\alpha+1} |b(y)| dy \leq C_{14}(\varepsilon^{-1}|z| + |z|) I_\alpha |f|(x+z).$$

Therefore,

$$\|J_4\|_{M_q^{w_2(\cdot)}} \leq C_{14} \cdot (\varepsilon^{-1}|z| + |z|) \|f\|_{M_p^{w_1(\cdot)}} \leq C_{14} \cdot D \cdot (\varepsilon^{-1}|z| + |z|).$$

Here, the constants do not depend on z and ε.

Taking $|z|$ small enough, we finally obtain

$$\|[b, I_\alpha(f)(\cdot + z)] - [b, I_\alpha] f(\cdot)\|_{M_q^{w_2(\cdot)}}$$

$$\leq \|J_1\|_{M_q^{w_2(\cdot)}} + \|J_2\|_{M_q^{w_2(\cdot)}} + \|J_3\|_{M_q^{w_2(\cdot)}} + \|J_4\|_{M_q^{w_2(\cdot)}} \leq C_{15} \cdot D \cdot \varepsilon,$$

that is, the set $[b, I_\alpha](f),\ f \in F$ also satisfies condition (2) of Theorem 1. Then, according to Theorem 1, the set $[b, I_\alpha](f),\ f \in F$ is compact in $M_q^{w_2(\cdot)}$. Theorem 4 is proved.
□

Remark 1. *When proving Theorem 4, we used the method from [19], taking into account the specifics of the generalized Morrey space.*

4. Conclusions

In this paper we have obtained the sufficient conditions for the compactness of sets in generalized Morrey spaces . Moreover, we have obtained the sufficient conditions for the compactness of the commutator $[b, I_\alpha]$ for the Riesz potential operator on generalized Morrey spaces $M_p^{w(\cdot)}(R^n)$. More precisely, we prove that if $b \in VMO(R^n)$, then $[b, I_\alpha]$ is a compact operator from $M_p^{w_1(\cdot)}$ to $M_q^{w_2(\cdot)}$.

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