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Cones generated by a generalized fractional maximal function

The paper considers the space of generalized fractional-maximal function, constructed on the basis of a rearrangement-invariant space. Two types of cones generated by a nonincreasing rearrangement of a generalized fractional-maximal function and equipped with positive homogeneous functionals are constructed. The question of embedding the space of generalized fractional-maximal function in a rearrangement-invariant space is investigated. This question reduces to the embedding of the considered cone in the corresponding rearrangement-invariant spaces. In addition, conditions for covering a cone generated by generalized fractional-maximal function by the cone generated by generalized Riesz potentials are given. Cones from non-increasing rearrangements of generalized potentials were previously considered in the works of M. Goldman, E. Bakhtigareeva, G. Karshygina and others.

Keywords: rearrangement-invariant spaces, non-increasing rearrangements of functions, cones generated by generalized fractional-maximal function, covering of cones.

Introduction

In this work two types of cones of non-negative monotonically non-increasing functions on the positive semiaxis generated by generalized fractional maximal functions and equipped with corresponding positively homogeneous functionals are introduced. We give the conditions on the function Φ, under which there are pointwise mutual covering of these cones.

In the work of Hakim D.I., Nakai E., Sawano Y. [1], Kucukaslan A. [2], Mustafayev R., Bilgicli N. [3], Gogatishvili A. [4] a generalized fractional-maximal functions of another type were defined, a particular case of which is the classical fractional-maximal function.

It is known that the maximal function is a very important operator in the theory of functions. With their help, many important issues of the theory of function and harmonic analysis are solved. The generalized fractional-maximal functions are also closely related to the generalized Riesz potentials, considered in the works of Goldman M.L. [5–7] (see also [8–10]).

The study of various properties of operators using a generalized fractional-maximal function is sometimes easier than the study of such operators using a generalized potential.

In this paper, we aim to determine the cones of non-negative measurable functions generated by a generalized fractional-maximal function and to investigate the properties of such cones.

1 Definitions, notation, and auxiliary statements

Let (S, Σ, μ) be space with a measure. Here is Σ is σ-algebra of subsets of the set S, on which is determined a non-negative σ-finite, σ-additive measure μ. By $L_0 = L_0(S, \Sigma, \mu)$ denotes the set of μ-measurable real-valued functions $f : S \to R$, and by L_0^+ a subset of the set L_0 consisting of non-negative functions:

$$L_0^+ = \{f \in L_0 : f \geq 0\}.$$

By $L_0^+(0, \infty; \downarrow)$ we denote the set of all non-increasing functions belonging to L_0^+.

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Definition 1. [11] A mapping $\rho : L_0^+ \to [0,\infty]$ is called a *functional norm* (short: FN), if the next conditions are met for all $f, g, f_n \in L_0^+,\ \ n \in N$:

$(P1)$ $\rho(f) = 0 \Rightarrow f = 0$, $\mu-$ almost everywhere (briefly: $\mu-$ a.e.);
$\rho(\alpha f) = \alpha\rho(f),\ \alpha \geq 0;\ \rho(f+g) \leq \rho(f) + \rho(g)$ (properties of the norm);

$(P2)$ $f \leq g$, ($\mu-$ a.e.) $\Rightarrow \rho(f) \leq \rho(g)$ (monotony of the norm);

$(P3)$ $f_n \uparrow f \Rightarrow \rho(f_n) \to \rho(f) (n \to \infty)$ (the Fatou property);

$(P4)$ $0 < \mu(\sigma) < \infty \Rightarrow \int\limits_\sigma f d\mu \leq c_\sigma \rho(f),\ f \in L_0^+$ (Local integrability);

$(P5)$ $0 < \mu(\sigma) < \infty \Rightarrow \rho(\chi_\sigma) < \infty$ (finiteness of the FN for characteristic functions (χ_σ) of sets of finite measure).

Here $f_n \uparrow f$ means that $f_n \leq f_{n+1}, \lim\limits_{n\to\infty} f_n = f$ (μ-a.e.).

Definition 2. Let ρ be a functional norm. The set of functions $X = X(\rho)$ from L_0, for which $\rho(|f|) < \infty$ is called a *Banach function space* (briefly: BFS), generated by the FN ρ. For $f \in X$ we assume

$$\|f\|_X = \rho(|f|).$$

Let $L_0 = L_0(R^n)$ be the set of all Lebesgue measurable functions $f : R^n \to C$; $\dot{L}_0 = \dot{L}_0(R^n)$ be the set of functions $f \in L_0$, for which the non-increasing rearrangement of the f^* is not identical to infinity. Non-increasing rearrangement f^* is defined by the equality:

$$f^*(t) = inf\{y \in [0;\infty) :\ \ \lambda_f(y) \leq t\},\ \ t \in R_+ = (0;\infty),$$

where

$$\lambda_f(y) = \mu_n \{x \in R^n :\ \ |f(x)| > y\},\quad y \in [0,\infty)$$

is the Lebesgue distribution function. It is known that f^* is a non-negative, non-increasing and right-continuous function on $\mathbb{R}_+$; f^* is equimeasurable with $|f|$, i.e.

$$\mu_1 \{t \in \mathbb{R}_+ :\ \ f^*(t) > y\} = \mu_n \{x \in \mathbb{R}^n :\ \ |f(x)| > y\},$$

here μ is the Lebesgue measure (on $\mathbb{R}^n$ or on $\mathbb{R}_+$, respectively, see [1]).

Let $f^\# : \mathbb{R}^n \to \mathbb{R}^n$ denote a symmetric rearrangement of f, i.e. a radially symmetric non-negative non-increasing right continuous function (as a function of $r = |x|$, $x \in \mathbb{R}^n$) that is equimeasurable with f. That is

$$f^\#(r) = f^*(v_n r^n);\ f^*(t) = f^\#\left((\frac{t}{v_n})^{\frac{1}{n}} \right),\ \ r, t \in \mathbb{R}_+,$$

here v_n is the volume of the n-dimensional unit ball.

The function $f^{**} : (0,\infty) \to [0,\infty]$ is defined as

$$f^{**}(t) = \frac{1}{t}\int\limits_0^t f^*(\tau)d\tau;\ \ t \in R_+.$$

It is clear that f^{**} is a non-increasing function on R_+.

Really, let $t_1 \leq t_2$, then

$$\begin{aligned} f^{**}(t_2) &= \frac{1}{t_2}\int\limits_0^{t_2} f^*(\tau)d\tau = \frac{1}{t_2}\int\limits_0^{t_1} f^*(\tau)d\tau + \frac{1}{t_2}\int\limits_{t_1}^{t_2} f^*(\tau)d\tau \leq \\ &\leq \frac{1}{t_2}\int\limits_0^{t_1} f^*(\tau)d\tau + f^*(t_1)\cdot\frac{t_2 - t_1}{t_2}. \end{aligned}$$

Hence, we have

$$f^{**}(t_2) \le \frac{1}{t_2}\int\limits_0^{t_1} f^*(\tau)d\tau + \frac{t_2-t_1}{t_2t_1}\int\limits_0^{t_1} f^*(\tau)d\tau \le \left(\frac{1}{t_2}+\frac{t_2-t_1}{t_2t_1}\right)\int\limits_0^{t_1} f^*(\tau)d\tau =$$
$$= \frac{1}{t_1}\int\limits_0^{t_1} f^*(\tau)d\tau = f^{**}(t_1).$$

Definition 3. A functional norm ρ is said to be *rearrangement-invariant* if

$$f^* \le g^* \Rightarrow \rho(f) \le \rho(g).$$

Banach function space $X = X(\rho)$, generated by a rearrangement invariant functional norm ρ will be called a *rearrangement invariant space* (in short: RIS).

Example 1. Let $S = R^n$, $\mu \equiv \mu_n$ be the Lebesgue measure in R^n, $1 \le p \le \infty$; $u \in L_0(R^n)$, $0 < u < \infty$, (μ-a.e.); $u \in L_p^{loc}(R^n)$, $\frac{1}{u} \in L_{p'}^{loc}(R^n)$, $\frac{1}{p}+\frac{1}{p'}=1$.

The space $X = L_{p,u}(R^n)$ with a norm $f_X = f_{L_{p,u}}$ i.e.:

$$\|f\|_X = \left(\int\limits_{R^n} |fu|^p d\mu\right)^{\frac{1}{p}},\ 1 \le p < \infty;\ \|f\|_X = \|fu\|_{L_\infty},\ p = \infty$$

is a BFS. Associated space:

$$X' = L_{p',\frac{1}{u}}(R^n).$$

Everywhere in this work, we denote rearrangement invariant space (in short: RIS) by $E = E(R^n)$, and by $E' = E'(R^n)$ the associated rearrangement-invariant space and $\tilde{E} = \tilde{E}(R_+)$, $\tilde{E}' = \tilde{E}'(R_+)$ their Luxembourg representation, i.e. such RIS that

$$\|f\|_E = \|f^*\|_{\tilde{E}},\ \ \|g\|_{E'} = \|g^*\|_{\tilde{E}'}. \tag{1}$$

Let Ω_0 be a set of all nonnegative, finite on R_+, decreasing and right continuous functions:

$$\Omega_0 = \{g : R_+ \to [0;\infty);\ \ g\downarrow,\ \ g(t+0) = g(t),\ \ t \in R_+\}.$$

Definition 4. A function $f : R_+ \to R_+$ is called *quasi-decreasing* and is denoted by $f \downarrow$ (*quasi-increasing and is denoted by* $f \uparrow$) if there exists $C > 1$, such that

$$f(t_2) < Cf(t_1) \quad \text{if } t_1 < t_2$$
$$(f(t_1) < Cf(t_2) \ \text{ if } t_1 < t_2).$$

Throughout this work we will denote by C, C_1, C_2 positive constants, generally speaking, different in different places.

By the notation $f(x) \cong g(x)$ we mean that there are constants $C_1 > 0$, $C_2 > 0$ such that

$$C_1 f(t) \le g(t) \le C_2 f(t),\ \ t \in \mathbb{R}_+.$$

Definition 5. Let $n \in \mathbb{N}$ and $R \in (0;\infty]$. We say that a function $\Phi : (0;R) \to R_+$ belongs to the class $A_n(R)$ if:

(1) Φ is non-increasing and continuous on $(0;R)$;

(2) the function $\Phi(r)r^n$ is quasi-increasing on $(0,R)$.

For example, $\Phi(t) = t^{-\alpha} \in A_n(\infty)$, $0 < \alpha < n$.

Definition 6. [12] Let $n \in \mathbb{N}$ and $R \in (0;\infty]$. A function $\Phi : (0;R) \to R_+$ belongs to the class $B_n(R)$ if the following conditions hold:

(1) Φ is non-increasing and continuous on $(0;R)$;

(2) there exists $C > 0$ such that

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \le C\Phi(r)r^n, \quad r \in (0,R). \tag{2}$$

For example,

$$\Phi(\rho) = \rho^{\alpha-n} \in B_n(\infty)\ (0<\alpha<n); \quad \Phi(\rho) = ln\frac{eR}{\rho} \in B_n(R),\ R \in R_+.$$

For $\Phi \in B_n(R)$ the following estimate also holds

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \ge n^{-1}\Phi(r)r^n,\ r \in (0,R).$$

Therefore

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \cong \Phi(r)r^n,\ r \in (0,R), \tag{3}$$

$$\Phi \in B_n(R) \Rightarrow \{0 \le \Phi \downarrow;\ \Phi(r)r^n.\uparrow,\ r \in (0,R)\}. \tag{4}$$

Definition 7. Let $\Phi \in A_n(\infty)$. The *generalized fractional-maximal function* $M_\Phi f$ is defined for the function $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is a ball with the center at the point x and radius r. That is, consider the operator M_Φ: $L^1_{loc}(R^n) \to \dot{L}_0(R^n)$.

In the case $\Phi(r) = r^{\alpha-n}$, $\alpha \in (0;n)$ we obtain the classical fractional maximal function $M_\alpha f$:

$$(M_\alpha f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} |f(y)|dy.$$

We denote by $M_E^\Phi = M_E^\Phi(R^n)$ the set of the functions u, for which there is a function $f \in E(R^n)$ such that

$$u(x) = (M_\Phi f)(x),$$

$$\|u\|_{M_E^\Phi} = inf\{\|f\|_E : f \in E(R^n),\ \ M_\Phi f = u\} \tag{5}$$

such a space M_E^Φ will be called space of generalized fractional-maximal function.

Note that in the works of Goldman M.L., Bakhtigareeva E.G [4–5], the generalized Riesz potential was considered using the convolution operator:

$$A : E_1(R^n) \to \dot{L}_0(R^n),$$

$$Af(x) = (G * f)(x) = 2\pi^{-n/2} \int_{R^n} G(x-y)f(y)dy,$$

where the kernel $G(x)$ satisfies the conditions:

$$G(x) \cong \Phi(|x|), \ \ x \in R^n, \tag{6}$$

$$\Phi \in B_n(\infty); \ \ \exists c \in R_+.$$

The kernel of the classical Riesz potential has the form

$$G(x) = |x|^{\alpha-n}, \ \ \alpha \in (0; n).$$

Note that, unlike the operator A the operator M_Φ is not linear.

Definition 8. Define $\Im_T = \{K(T)\}$ for $T \in (0, \infty]$ as a set of cones considering from measurable non-negative functions on $(0, T)$, equipped with positive homogeneous functionals $\rho_{KM(T)} : K(T) \to [0, \infty)$ with properties:
(1) $h \in K(T), \ \ \alpha \geq 0 \Rightarrow \alpha h \in K(T), \quad \rho_{K(T)}(\alpha h) = \alpha \rho_{K(T)}(h);$
(2) $\rho_{K(T)}(h) = 0 \Rightarrow h = 0$ almost everywhere on $(0, T)$.

Definition 9. [5] Let $K(T), M(T) \in \Im_T$. The cone $K(T)$ *covers the cone* $M(T)$ (notation: $M(T) \prec K(T)$) if there exist $C_0 = C_0(T) \in R_+$, and $C_1 = C_1(T) \in [0, \infty)$ with $C_1(\infty) = 0$ such that for each $h_1 \in M(T)$ there is $h_2 \in K(T)$ satisfying

$$\rho_{K(T)}(h_2) \leq C_0 \rho_{M(T)}(h_1), \ \ h_1(t) \leq h_2(t) + C_1 \rho_{M(T)}(h_1), \ \ t \in (0, T).$$

The equivalence of the cones means mutual covering:

$$M(T) \approx K(T) \Leftrightarrow M(T) \prec K(T) \prec M(T).$$

Let E is rearrangement-invariant space (briefly: RIS). We consider the following two cones of decreasing rearrangements of generalized fractional maximal function equipped with homogeneous functionals, respectively:

$$K_1 \equiv KM_E^\Phi := \{h \in L^+(\mathbb{R}_+) : h(t) = u^*(t), \ t \in \mathbb{R}_+, \ u \in M_E^\Phi\},$$
$$\rho_{K_1}(h) = \inf\{\|u\|_{M_E^\Phi} : u \in M_E^\Phi; \ u^*(t) = h(t), \ t \in \mathbb{R}_+\}; \tag{7}$$

$$K_2 \equiv K\widetilde{M}_E^\Phi := \{h : h(t) = u^{**}(t), \ t \in \mathbb{R}_+, \ u \in M_E^\Phi\},$$
$$\rho_{K_2}(h) = \inf\{\|u\|_{M_E^\Phi} : u \in M_E^\Phi; \ u^{**}(t) = h(t), \ t \in \mathbb{R}_+\}. \tag{8}$$

This means that the cones K_1 and K_2 consist of non-increasing rearrangements of generalized fractional maximal functions.

Note that in the works of Goldman M.L. [5], Bokayev N.A., Goldman M.L., Karshygina G.Zh. [9–10] cones generated by generalized potentials are considered. They study the space of potentials $H_E^G \equiv H_E^G(\mathbb{R}^n)$ in n-dimensional Euclidean space:

$$H_E^G(\mathbb{R}^n) = \{u = G * f : f \in E(\mathbb{R}^n)\},$$

where $E(\mathbb{R}^n)$ is an rearrangement invariant space (RIS).

$$\|u\|_{H_E^G} = \inf\{\|f\|_E : f \in E(\mathbb{R}^n); G * f = u\},$$

$$M(T) \equiv KM_E^G(T) = \{h(t) = u^*(t), t \in (0; T), u \in H_E^G\},$$
$$\rho_{M(T)}(h) = \inf\{\|u\|_{H_E^G} : u \in H_E^G; u^*(t) = h(t), t \in (0; T)\};$$

$$\widetilde{M}(T) \equiv K\widetilde{M}_E^G(T) = \{h(t) = u^{**}(t), t \in (0;T) : u \in H_E^G\},$$

$$\rho_{\widetilde{M}}(h) = \inf\{\|u\|_{H_E^G} : u \in H_E^G; u^{**}(t) = h(t)), t \in (0;T)\}.$$

In the following Theorem 1 [13] gives the estimate for a non-increasing rearrangement of a generalized fractional maximal function $(M_\Phi f)$ by non-increasing rearrangement of the function f.

Theorem 1. Let $\Phi \in A_n(\infty)$. Then there exist a positive constant C, depending from $n \in \mathbb{N}$ such that

$$(M_\Phi f)^*(t) \le C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \quad t \in (0,\infty),$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$.

In the following theorem we give the compares of the cone generated by a generalized fractional-maximal function and the cone generated by the generalized Riesz potential.

Theorem 2. Let $\Phi \in B_n(\infty)$ and kernel $G(x)$ satisfies the condition (6). Then cone generated by the generalized potential covers the cone generated by the generalized maximal function, i.e. $KM_E^\Phi \prec KM_E^G$.

Proof. Let $h_1 \in KM_E^\Phi$, then according to the definition of KM_E^Φ there is a function $u_1 \in M_E^\Phi$, such that $h_1(t) = u_1^*(t)$. So there is a function $f \in E(R^n)$ such that

$$u_1(x) = (M_\Phi f_1)(x),$$

$$\|f_1\|_E \le 2\|u_1\|_{M_E^\Phi}.$$

Therefore

$$\|u_1\|_{K_1} \le 4\rho_{K_1}(h_1),$$

$$\|f\|_E \le C\rho_{KM}(h_1).$$

Therefore, by Theorem 1 and taking into account the monotonicity of the function $\Phi \downarrow$ and denoting

$$\psi(t) = \int_t^\infty \frac{\Phi(\xi^{1/n})}{\xi} d\xi,$$

we have:

$$\begin{aligned}
h_1(t) &= (M_\Phi f)^*(t) \le C \sup_{t\le s<\infty} \Phi(s^{1/n}) \int_0^s f^*(\tau)d\tau = \\
&= \sup_{t\le s<\infty} C\left(\frac{1}{\ln 2}\int_s^{2s} \frac{d\xi}{\xi}\right)\Phi(s^{1/n}) \cdot \int_0^s f^*(\tau)d\tau \le \\
&\le C \cdot \sup_{t\le s<\infty} \frac{1}{\ln 2}\int_s^{2s} \frac{\Phi(\xi^{1/n})}{\xi} d\xi \cdot \int_0^s f^*(\tau)d\tau \le \\
&\le C \cdot \sup_{t\le s<\infty} \int_s^{2s} \frac{\Phi(\xi^{1/n})}{\xi} \cdot \int_0^s f^*(\tau)d\tau d\xi \le
\end{aligned}$$

$$\leq \quad C \cdot \int_t^\infty \frac{\Phi(\xi^{1/n})}{\xi} \cdot \int_0^\infty f^*(\tau)d\tau d\xi = \int_t^\infty \frac{\Phi(\xi^{1/n})}{\xi} d\xi \cdot \int_0^t f^*(\tau)d\tau +$$

$$+ \quad \int_t^\infty f^*(\tau)d\tau \cdot \int_t^\infty \frac{\Phi(\xi^{1/n})}{\xi} d\xi \leq C \cdot \left(\psi(t) \cdot \int_0^t f^*(\tau)d\tau + \right.$$

$$\left. + \quad \int_t^\infty \psi(s) f^*(\tau)d\tau \right) \leq C \cdot \left(G * f^{\#}\right)^*(t) = h_2(t).$$

We put that

$$h_2(t) = C \cdot \left(G * f^{\#}\right)^*(t), t \in \mathbb{R}_+.$$

Consequently $h_1(t) \leq h_2(t)$. So

$$K_1 = KM_E^\Phi \prec KM_E^G,$$

$$\rho_{M(T)}(h_2) \leq C\|f^{\#}\|_E = C\|f\|_E \leq 2C\rho_{KM}(h_1).$$

Theorem 2 is proved.

Lemma 1. The following covering takes place

$$K_1 \prec K_2.$$

Proof. Let $h_1 \in K_1$. Then there is a function $u_1 \in M_E^\Phi$ such that

$$h_1(t) = u_1^*(t), \quad \|u_1\|_{M_E^\Phi} \leq 2\rho_{K_1}(h_1).$$

For $u_1 \in M_E^\Phi$ we find $f_1 \in E(\mathbb{R}^n)$ satisfying

$$u_1(x) = (M_\Phi f_1)(x) = \sup_{r>0} \Phi(r) \int_{B(x,r)} f_1(\xi)d\xi,$$

$$\|f_1\|_{E(\mathbb{R}^n)} \leq 2\|u_1\|_{M_E^\Phi}.$$

Hence $h_1(t) = (M_\Phi f_1)^{**}(t)$ and (see (1), (2), (3)),

$$\|f_1\|_{E(\mathbb{R}^n)} = \|f_1\|_{\widetilde{E}(R^+)} \leq 4\rho_{K_1}(h_1).$$

By inequality

$$u_1^*(t) \leq u_1^{**}(t), \ t \in \mathbb{R}_+.$$

We set

$$h_2(t) = u_1^{**}(t) \in K_2.$$

Then we have $h_1(t) \leq h_2(t)$. Moreover (see (8), (5), (4), (7))

$$\rho_{K_2}(h_2) = \|u_1\|_{M_E^\Phi} \leq \|f_1\|_{E(\mathbb{R}^n)} \leq 4\rho_{K_1}(h_1).$$

We proved $K_1 \prec K_2$. Lemma 1 is proved.

The following theorem shows that the embedding of the space of generalized fractional linear spaces in the RIS $X(\mathbb{R}^n)$ is reduced to the embedding of the cone $K_1 = K_1 M_E^\Phi$ in the space of the RIS $\widetilde{X}(\mathbb{R}_+)$.

Theorem 3. Let $\Phi \in B_n(\infty)$. The embedding

$$M_E^\Phi(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n) \tag{9}$$

is equivalence to the next embedding

$$K_1 M_E^\Phi(\mathbb{R}_+) \mapsto \widetilde{X}(\mathbb{R}_+). \tag{10}$$

Proof. From embedding (9) it follows that there is a constant $C_1 \in \mathbb{R}_+$ such that for any $u \in M_E^\Phi$

$$\|u^*\|_{\widetilde{X}(\mathbb{R}_+)} = \|u\|_{X(\mathbb{R}^n)} \leq C_1 \|u\|_{M_E^\Phi(\mathbb{R}^n)}. \tag{11}$$

For $h \in K_1 = K_1 M_E^\Phi$ we find the function $u \in M_E^\Phi$ such that $u^* = h$ and

$$\|u\|_{M_E^\Phi(\mathbb{R}^n)} \leq 2\rho_{K_1}(h). \tag{12}$$

From (11) and (12) it follows that $h = u^* \in \widetilde{X}(\mathbb{R}_+)$

$$\|h\|_{\widetilde{X}(\mathbb{R}_+)} \leq 2C_1 \rho_{K_1}(h),$$

i.e. holds (10).

Conversely let the embedding (10) hold. For $u \in M_E^\Phi(\mathbb{R}^n)$ and $h = u^* \in K_1 M_E^\Phi(\mathbb{R}_+)$ we get

$$\|h\|_{\widetilde{X}(\mathbb{R}_+)} \leq C_0 \rho_{K_1}(h),$$

but $\rho_{K_1}(h) \leq \|u\|_{M_E^\Phi(\mathbb{R}^n)}$, so the last estimate is

$$\|u\|_{X(\mathbb{R}^n)} = \|u^*\|_{\widetilde{X}(\mathbb{R}_+)} \leq C_0 \|u\|_{M_E^\Phi(\mathbb{R}^n)}, \quad \forall u \in M_E^\Phi(\mathbb{R}^n).$$

That is $M_E^\Phi(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n)$. Theorem 3 is proved.

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Жалпыланған бөлшекті-максималды функциямен туындаған конустар

Жұмыста ауыстырмалы-инварианттық кеңістік негізінде жалпыланған бөлшекті-максималды функция кеңістігі қарастырылған. Жалпыланған бөлшекті-максималды функцияның өспейтін алмастыруымен құрылған және оң біртекті функциялармен жабдықталған конустардың екі түрі құрастырылған. Жалпыланған бөлшекті-максималды функция кеңістігін ауыстырмалы-инварианттық кеңістікке енгізу мәселесі зерттелді. Бұл сұрақ қарастырылатын конусты сәйкес ауыстырмалы-инварианттық кеңістіктерге енгізуге әкеледі. Сонымен қатар, жалпыланған бөлшекті-максималды функция арқылы туындаған конусты жалпыланған Рисс потенциалы арқылы туындаған конуспен жабу шарттары берілген. Жалпыланған потенциалдардың өспейтін алмастыруларының конустары бұрын М. Гольдман, Э. Бахтигареева, Г. Каршыгина және т.б. еңбектерінде қарастырылған.

Кілт сөздер: ауыстырмалы-инварианттық кеңістіктер, функцияның өспейтін алмастырулары, жалпыланған бөлшекті-максималды функциялар арқылы туындаған конустар, конустардың жабулары.

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Конусы, порожденные обобщенной дробно-максимальной функцией

В работе рассмотрено пространство обобщенной дробно-максимальной функции на основе перестановочно-инвариантного пространства. Построены два вида конусов, порожденных невозрастающей перестановкой обобщенной дробно-максимальной функцией и снабженных положительными однородными функционалами. Исследован вопрос о вложении пространства обобщенных дробно-максимальных функций в перестановочно-инвариантное пространство. Этот вопрос сводится к вложению рассматриваемого конуса в соответствующие перестановочно-инвариантные пространства. Кроме того, приведены условия для покрытия конуса, порожденного обобщенной дробно-максимальной функцией, конусом, порожденным обобщенным потенциалом Рисса. Конусы из невозрастающих перестановок обобщенных потенциалов были изучены ранее в работах М. Гольдмана, Э. Бахтигареевой, Г. Каршыгиной и других.

Ключевые слова: перестановочно-инвариантные пространства, невозрастающие перестановки функций, конусы, порожденные обобщенными дробно-максимальными функциями, покрытие конусов.