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Correctness conditions for high-order differential equations with unbounded coefficients

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Abstract

We give some sufficient conditions for the existence and uniqueness of the solution of a higher-order linear differential equation with unbounded coefficients in the Hilbert space. We obtain some estimates for the weighted norms of the solution and its derivatives. Using these estimates, we show the conditions for the compactness of some integral operators associated with the resolvent.

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1 Introduction and formulation of the main results

We consider the following equation:

$$\rho(x)\big(\rho(x)y^{(k)}\big)' + \sum_{j=0}^{k-1} r_j(x)y^{(k-j)} + r_k(x)y = f(x), \tag{1}$$

where $k = 1, 2, \dots$, $x \in R = (-\infty, +\infty)$, $f(x) \in L_2 := L_2(R)$. In what follows, we assume that $\rho(x) > 0$ is $(k+1)$ times, and $r_j = r_j(x)$ $(j = \overline{1, k-1})$ is $(k-j)$ times continuously differentiable, and $r_k = r_k(x)$ is a continuous functions. Equation (1) is given in an infinite domain, and its coefficients can be unbounded functions. Hence, it is a singular differential equation.

Let L_0 be a differential operator from $C_0^{(k+1)}(R)$ to L_2, which is defined by the following formula:

$$L_0 y = \rho(x)\big(\rho(x)y^{(k)}\big)' + \sum_{j=0}^{k-1} r_j(x)y^{(k-j)} + r_k(x)y.$$

Since the coefficients ρ and r_j $(j = \overline{0, k})$ are smooth functions, the operator L_0 is a closable operator (see [1, Sect. 6 of Chap. 2]). We denote by L the closure of L_0.

A function $y(x)$ is called a solution of differential equation (1) if there exists a sequence $\{y_m(x)\}_{m=1}^{\infty} \subseteq C_0^{(k+1)}(R)$ such that $y_m \to y$ and $Ly_m \to f$ in the norm of L_2 as $m \to \infty$. It is clear that $y \in L_2$.

The following more general equation

$$\rho_0(x)y^{(k+1)} + \tilde{r}_0(x)y^{(k)} + \sum_{j=1}^{k-1} r_j y^{(k-j)} + r_k(x)y = f(x), \quad \rho_0(x) > 0,$$

can be reduced to equation (1). Indeed, set $\rho(x) = \sqrt{\rho_0(x)}$ and $r_0 = \tilde{r}_0 - \rho\rho'$, then

$$\begin{aligned} &\rho_0(x)y^{(k+1)} + \tilde{r}_0(x)y^{(k)} + \sum_{j=1}^{k-1} r_j(x) y^{(k-j)} + r_k(x)y \\ &\quad = \rho(x)\bigl(\rho(x)y^{(k)}\bigr)' + \bigl(\tilde{r}_0(x) - \rho(x)\rho'(x)\bigr)y^{(k)} + \sum_{j=1}^{k-1} r_j(x)y^{(k-j)} + r_k(x)y \\ &\quad = \rho(x)\bigl(\rho(x)y^{(k)}\bigr)' + \sum_{j=0}^{k-1} r_j(x)y^{(k-j)} + r_k(x)y, \end{aligned}$$

this is the left-hand side of equation (1).

We study equation (1) in the case that the intermediate coefficients $r_j(x)$ $(j = \overline{1, k-1})$ can be unbounded and their growth do not depend on the extreme coefficients $\rho_0(x)$ and $r_k(x)$. When $r_j(x)$ $(j = \overline{1, k-1})$ are bounded or they are unbounded and are controlled by the extreme coefficients $\rho_0(x)$ and $r_k(x)$, this equation has been studied systematically. For more details, see [2–4].

A number of problems of stochastic analysis and stochastic differential equations lead to singular elliptic equations and ordinary differential equations and their systems with unbounded intermediate coefficients. Specific representatives of such equations are the stationary equations of Ornstein–Uhlenbeck (see [5]) and Fokker–Planck–Kolmogorov (see [6]). In the case $k = 1$, equation (1) is the simplest model of Brownian motion of particles with a covariance matrix determined by the function $\rho(x)$, and $r_0(x)$ is called the drift coefficient.

For applications of equation (1) to various practical processes, it is important to investigate the correctness of equation (1) with coefficients $\rho(x)$ and $r_j(x)$ $(j = \overline{0, k-1})$ from wider classes. In the case that the intermediate coefficients do not depend on the potential and the diffusion coefficient and can grow as a linear function, the correctness of the second-order singular elliptic equations was studied in [7–10]. The correctness conditions for the second-order and third-order one-dimensional differential equations with rapidly growing intermediate coefficients were obtained in [11–16]. However, in [11–16] the condition of weak oscillation is imposed on the intermediate and senior coefficients. In this paper, sufficient conditions for the existence and uniqueness of a solution $y(x)$ of (1) are obtained. Moreover, for the solution, we proved the following inequality:

$$\bigl\| \sqrt{r_0} y^{(k)} \bigr\|_2 + \sum_{j=1}^{k} \bigl\| r_j y^{(k-j)} \bigr\|_2 \le C \|f\|_2.$$

Using this estimate, we obtained compactness conditions for operators $\theta(x)L^{-1}$ and $\theta(x)\frac{d^\alpha}{dx^\alpha}L^{-1}$ ($\alpha = \overline{1,k-1}$).

The difference between this result and the results in [7–16] is that equation (1) is high order, and the coefficients $r_j(x)$ ($j = \overline{0,k-1}$) can grow rapidly, and all coefficients can be fluctuating (see Example 4.1). In addition, the leading coefficient can tend to zero at infinity. In other words, the cases of some degenerate equations are covered. Note that the criteria for the existence of positive periodic solutions for differential equations with indefinite singularity and pseudo almost periodic solutions of an iterative functional differential equations, respectively, were found in [17] and [18].

We introduce the following notation:

$$T_{u,v,s}(x) = \left[\int_0^x u^2(t)\,dt\right]^{1/2}\left[\int_x^{+\infty} t^{2(s-1)}v^{-2}(t)\,dt\right]^{1/2}, \quad x > 0,$$
$$M_{u,v,s}(\tau) = \left[\int_\tau^0 u^2(t)\,dt\right]^{1/2}\left[\int_{-\infty}^{\tau} t^{2(s-1)}v^{-2}(t)\,dt\right]^{1/2}, \quad \tau < 0,$$
$$\gamma_{u,v,s} = \max\Bigl(\sup_{x>0} T_{u,v,s}(x), \sup_{\tau<0} M_{u,v,s}(\tau)\Bigr),$$

where $u = u(x)$ and $v = v(x) \neq 0$ are real continuous functions, s is a positive integer number. The following statements are the main results of this paper.

Theorem 1.1 *Let $\rho(x) > 0$ be $(k+1)$ times, $r_j(x)$ $(j = \overline{1,k-1})$ be $(k-j)$ times continuously differentiable, $r_k(x)$ be a continuous function, and the following conditions be satisfied*:

(a) $r_0 \geq 1$, $\gamma_{1,\sqrt{r_0},k} < \infty$;

(b) *there exists a point $x_0 \in R$ such that* $\sup_{x<x_0}[\rho(x)\exp\int_{x_0}^x \frac{r_0(t)}{\rho^2(t)}\,dt] < \infty$;

(c) $\max_{j=\overline{1,k}} \gamma_{r_j,\sqrt{r_0},j} < \infty$.

Then, for any $f \in L_2$, equation (1) *has a unique solution y and*

$$\bigl\|\sqrt{r_0}y^{(k)}\bigr\|_2 + \sum_{j=1}^{k-1}\bigl\|r_jy^{(k-j)}\bigr\|_2 + \|r_ky\|_2 \leq C\|f\|_2 \tag{2}$$

holds, where $\|\cdot\|_2$ is the norm of L_2.

Changing variable in Theorem 1.1, we obtain the following result.

Theorem 1.2 *Let $\rho(x) > 0$ be $(k+1)$ times, $r_j(x)$ $(j = \overline{1,k-1})$ be $(k-j)$ times continuously differentiable, and $r_k(x)$ be a continuous function, and the following conditions be satisfied*:

(a) *respectively*, $r_0(x) \leq -1$, $\gamma_{1,\sqrt{|r_0|},k} < \infty$;

(b) *there exists a point $x_1 \in R$ such that* $\sup_{x>x_1}[\rho(x)\exp\int_{x_1}^x \frac{r_0(t)}{\rho^2(t)}\,dt] < \infty$;

(c) $\max_{j=\overline{1,k}} \gamma_{r_j,\sqrt{|r_0|},j} < \infty$.

Then, for any $f \in L_2$, equation (1) *has a unique solution y and*

$$\bigl\|\sqrt{|r_0|}y^{(k)}\bigr\|_2 + \sum_{j=1}^{k-1}\bigl\|r_jy^{(k-j)}\bigr\|_2 + \|r_ky\|_2 \leq C\|f\|_2 \tag{3}$$

holds.

Theorem 1.3 *Suppose that the coefficients ρ and r_j $(j=\overline{0,k})$ satisfy the conditions of Theorem 1.1 or Theorem 1.2 and the function $\theta(x)$ is continuous, and let for some $j\in[1,k]$ the equality*

$$\max\Big(\lim_{x\to+\infty}T_{\theta,\sqrt{|r_0|},j}(x),\ \lim_{\tau\to-\infty}M_{\theta,\sqrt{|r_0|},j}(\tau)\Big)=0 \tag{4}$$

hold. Then $\theta(x)\frac{d^{k-j}}{dx^{k-j}}L^{-1}$ $(\theta(x)L^{-1}$ for $j=k)$ is the compact operator in L_2.

2 Some auxiliary statements

Let $C_+^{(s)}(0,+\infty)=\{m(x)\in C^{(s)}(0,+\infty):\exists\tau>0 m(x)=0\ \forall x>\tau\}$ $(s\in N)$. The next lemma is a particular case of Theorem 2.1 in [19].

Lemma 2.1 *Suppose that the functions $g(x)$, $v(x)\neq 0$ $(x>0)$ are continuous, and for a natural number s,*

$$\tilde T_{g,v,s}=\sup_{x>0}T_{g,v,s}(x)<\infty$$

holds. Then, for any $y\in C_+^{(s)}(0,+\infty)$,

$$\bigg(\int_0^{+\infty}\big|g(x)y(x)\big|^2\,dx\bigg)^{1/2}\le C\bigg(\int_0^{+\infty}\big|v(x)y^{(s)}(x)\big|^2\,dx\bigg)^{1/2} \tag{5}$$

holds. Moreover, if C is the smallest constant for which inequality (5) is valid, then

$$T_{0,g,v,s}\le C\le 2\tilde T_{g,v,s}, \tag{6}$$

where

$$T_{0,g,v,s}=\sup_{x>0}\bigg[\int_0^x g^2(t)\,dt\bigg]^{1/2}\bigg[\int_x^{+\infty}(t-x)^{2(s-1)}v^{-2}(t)\,dt\bigg]^{1/2}.$$

Remark 2.1 If $s=1$ and C is the smallest constant for which inequality (5) is valid, then, instead of (6), the inequalities $\tilde T_{g,v,s}\le C\le 2\tilde T_{g,v,s}$ hold (see [20]).

Lemma 2.2 *Suppose that the functions $u(x)$, $h(x)\neq 0$ $(x<0)$ are continuous and $\tilde M_{u,h,s}=\sup_{x<0}M_{u,h,s}(x)<\infty$ for a natural number s. Then, for any $y\in C_-^{(s)}(-\infty,0)=\{\eta\in C^{(s)}(-\infty,0):\exists\tau<0 \eta(x)=0\ \forall x<\tau\}$,*

$$\bigg(\int_{-\infty}^0\big|u(x)y(x)\big|^2\,dx\bigg)^{1/2}\le C_1\bigg(\int_{-\infty}^0\big|h(x)y^{(s)}(x)\big|^2\,dx\bigg)^{1/2} \tag{7}$$

holds. Moreover, if C_1 is the smallest constant for which (7) is valid, then

$$M_{0,u,h,s}\le C_1\le 2\tilde M_{u,h,s},$$

where

$$M_{0,u,h,s}=\sup_{\tau<0}\bigg[\int_\tau^0 u^2(t)\,dt\bigg]^{1/2}\bigg[\int_{-\infty}^\tau(\tau-t)^{2(s-1)}h^{-2}(t)\,dt\bigg]^{1/2}.$$

Proof Changing variable in Lemma 2.1, we obtain the desired result. □

Remark 2.2 If $s = 1$ and C_1 is the smallest constant for which inequality (7) is valid, then the inequalities $\tilde{M}_{u,h,s} \le C_1 \le 2\tilde{M}_{u,h,s}$ hold.

The following statement is proved by application of Lemmas 2.1 and 2.2.

Lemma 2.3 *Let continuous functions $u(x)$, $v(x) \neq 0$ $(x \in R)$ satisfy the conditions $\tilde{T}_{u,v,s} < \infty$, $\tilde{M}_{u,v,s} < \infty$ for some natural number s. Then, for any $y \in C_0^{(s)}(R)$,*

$$\left(\int_{-\infty}^{+\infty} \left|u(x)y(x)\right|^2 dx\right)^{1/2} \le C_2 \left(\int_{-\infty}^{+\infty} \left|v(x)y^{(s)}(x)\right|^2 dx\right)^{1/2}. \tag{8}$$

Moreover, if C_2 is the smallest constant for which inequality (8) *is valid, then*

$$\min(T_{0,u,v,s}, M_{0,u,v,s}) \le C_2 \le 2\gamma_{u,v,s}.$$

Remark 2.3 If $s = 1$ and C_2 is the smallest constant for which inequality (8) is valid, then

$$\min(\tilde{T}_{u,v,s}, \tilde{M}_{u,v,s}) \le C_2 \le 2\gamma_{u,v,s}.$$

3 On a two-term differential operator

Let l_0 be a differential operator from the set $C_0^{(k+1)}(R)$ to L_2, which is defined by

$$l_0 y = \rho(x)\big(\rho(x)y^{(k)}\big)' + r_0(x)y^{(k)}.$$

We denote its closure by l.

Lemma 3.1 *Let $\rho(x) > 0$ and the function $r_0(x)$ satisfy condition* (a) *of Theorem* 1.1. *Then the operator l is invertible, and for $y \in D(l)$, the inequality*

$$\left\|\sqrt{r_0}y^{(k)}\right\|_2 + \|y\|_2 \le C\|ly\|_2 \tag{9}$$

holds.

Proof Let $y \in C_0^{(k+1)}(R)$. Integrating by parts, we get that

$$\big(l_0 y, y^{(k)}\big) = \left\|\sqrt{r_0}y^{(k)}\right\|_2^2.$$

By Hölder's inequality,

$$\left|\big(l_0 y, y^{(k)}\big)\right| \le \left\|\sqrt{r_0}y^{(k)}\right\|_2 \left\|\frac{1}{\sqrt{r_0}} l_0 y\right\|_2.$$

Hence,

$$\left\|\sqrt{r_0}y^{(k)}\right\|_2 \le \left\|\frac{1}{\sqrt{r_0}} l_0 y^{(k)}\right\|_2. \tag{10}$$

According to Lemma 2.3, we have that

$$\|y\|_2 \le 2\gamma_{1,\sqrt{r_0},k} \left\| \sqrt{r_0} y^{(k)} \right\|_2.$$

Therefore,

$$\left\| \sqrt{r_0} y^{(k)} \right\|_2 + \|y\|_2 \le C \|l_0 y\|_2, \tag{11}$$

where $C = 2\gamma_{1,\sqrt{r_0},k} + 1$.

Now let $y \in D(l)$. Then there is a sequence $\{y_m\} \subseteq C_0^{(k+1)}(R)$ such that $\|y_m - y\|_2 \to 0$, $\|l_0 y_m - ly\|_2 \to 0$ as $m \to \infty$. So, it follows that

$$\big| \|y_m\|_2 - \|y\|_2 \big| \to 0, \qquad \big| \|l_0 y_m\|_2 - \|ly\|_2 \big| \to 0 \quad (m \to \infty). \tag{12}$$

According to (11), we have

$$\left\| \sqrt{r_0} y_m^{(k)} - \sqrt{r_0} y_s^{(k)} \right\|_2 + \|y_m - y_s\|_2 \le C \|l_0 y_m - l_0 y_s\|_2, \quad s, m \in N. \tag{13}$$

We denote by $W_{2,r_0}^k(R)$ the completion of $(C_0^{(k)}(R), \|\cdot\|_{W_{2,r_0}^k})$, where $\|y\|_{W_{2,r_0}^k} = \|\sqrt{r_0} y^{(k)}\|_2 + \|y\|_2$. From (12) and (13) it follows that $\{y_m\}$ is a Cauchy sequence in the space $W_{2,r_0}^k(R)$. Consequently, there is $y \in W_{2,r_0}^k(R)$ such that $\{y_m\}$ converges to y in the norm of $W_{2,r_0}^k(R)$. Using (11) and (12), we obtain that for y inequality (9) holds. According to (9), l is invertible. Similarly, by (10),

$$\left\| \sqrt{r_0} y^{(k)} \right\|_2 \le \left\| \frac{1}{\sqrt{r_0}} ly \right\|_2, \quad y \in D(l). \tag{14}$$

□

Inequality (9) gives $D(l) \subseteq W_{2,r_0}^k(R)$. For $y \in D(l)$, set $z = y^{(k)}$ and $\tilde{L}z = \rho(x)(\rho(x)z)' + r_0(x)z$.

Lemma 3.2 *Suppose that $\rho(x)$ and $r_0(x)$ satisfy the conditions of Lemma 3.1. Then $\tilde{L}$ is a closed operator in L_2.*

Proof Let $\{z_n\}_{n=1}^\infty \subseteq D(\tilde{L})$ such that $\|z_n - z\|_2 \to 0$, $\|\tilde{L}z_n - w\|_2 \to 0$ as $n \to \infty$. According to our choice, there is a sequence $\{y_n\}_{n=1}^\infty \subseteq D(l)$ such that $y_n^{(k)} = z_n$ and

$$\left\| y_n^{(k)} - z \right\|_2 \to 0, \qquad \|ly_n - w\|_2 \to 0 \quad (n \to \infty).$$

By Lemma 3.1, y_n $(n \in N)$ satisfies (9). Hence, $\{y_n\}_{n=1}^\infty \subseteq W_{2,r_0}^k(R)$ is a Cauchy sequence. Therefore, there exists $y \in L_2$ such that

$$\left\| y_n^{(k)} - z \right\|_2 + \|y_n - y\|_2 \to 0 \quad (n \to \infty). \tag{15}$$

So

$$\|y_n - y\|_2 \to 0, \qquad \left\| y_n^{(k)} - z \right\|_2 \to 0, \qquad \|ly_n - w\|_2 \to 0 \quad (n \to \infty).$$

Since the operator l and the generalized differentiation operator are closed, we have $y \in D(l)$, $z = y^{(k)} \in D(\tilde{L})$ and

$$w = \tilde{L}z. \tag{16}$$

Thus, $\tilde{L}$ is a closed operator. □

Lemma 3.3 *If functions $\rho(x)$ and $r_0(x)$ satisfy the conditions of Lemma 3.1, then*

$$R(l) = R(\tilde{L}).$$

The proof follows from the following equalities:

$$\begin{aligned} R(\tilde{L}) &= \{w \in L_2 : \exists z \in D(\tilde{L}), w = \tilde{L}z\} \\ &= \{w \in L_2 : \exists y \in D(l), y^{(k)} = z \in D(\tilde{L}), w = ly\} \\ &= \{w \in L_2 : \exists y \in D(l), w = ly\} = R(l). \end{aligned}$$

Lemma 3.4 *Suppose that the functions $\rho(x)$ and $r_0(x)$ satisfy conditions* (a) *and* (b) *of Theorem 1.1. Then l is invertible and its inverse l^{-1} is bounded.*

Proof By Lemma 3.1, l has an inverse l^{-1}. Since l is a closed operator, using (9), we deduce that $R(l)$ is a closed set. By Lemma 3.3, it suffices to prove $R(\tilde{L}) = L_2$. If $R(\tilde{L}) \neq L_2$, then, according to [1, p. 284], there is a nonzero element $v(x) \in L_2$ such that

$$(\tilde{L}z, v) = (z, \tilde{L}^* v) = 0$$

(where $\tilde{L}^*$ is the adjoint of $\tilde{L}$) for any $z \in D(\tilde{L})$. Since $C_0^{(1)}(R) \subseteq D(\tilde{L})$, the set $D(\tilde{L})$ is dense in L_2. Therefore,

$$\tilde{L}^* v = -\rho(x)\big(\rho(x)v\big)' + r_0(x)v = 0.$$

This implies that $v(x)$ is continuously differentiable and

$$v(x) = \frac{C}{\rho(x)} \exp \int_{x_0}^{x} \frac{r_0}{\rho^2}\, dt.$$

Since $v \neq 0$, we have $C \neq 0$. Taking into account condition b) of Theorem 1.1, we have that $|v(x)| \geq \frac{|C|}{K} > 0$ for all $x < x_0$, where

$$K = \sup_{x<x_0} \rho(x) \exp \int_{x_0}^{x} \frac{r_0(t)}{\rho^2(t)}\, dt.$$

Hence $v \notin L_2$. This is a contradiction. □

Remark 3.1 If in Lemma 3.4 the condition $r_0 \geq 1$ is replaced with the condition $r_0 \geq \delta > 0$, then the lemma remains valid.

4 Proofs of the main results

Proof of Theorem 1.1 Set $x = mt$, $\hat{y}(t) = y(mt)$, $\hat{\rho}(t) = \rho(mt)$, $\hat{r}_j(t) = r_j(mt)$ $(j = \overline{0,k})$, $\hat{f}(t) = f(mt)m^{-(k+1)}$ $(m > 0)$. Then equation (1) changes to

$$\begin{aligned} P_m\hat{y} &= \hat{\rho}(t)\big(\hat{\rho}(t)\hat{y}^{(k)}(t)\big)' \\ &\quad + \sum_{j=0}^{k-1} m^{-(j+1)}\hat{r}_j(t)\hat{y}^{(k-j)}(t) + m^{-(k+1)}\hat{r}_k(t)\hat{y}(t) = \hat{f}. \end{aligned} \tag{17}$$

Let $\hat{l}$ be a closure of $\hat{l}_0$, where $\hat{l}_0 : D(\hat{l}_0) \to L_2$ is defined by

$$\hat{l}_0\hat{y} = \hat{\rho}(t)\big(\hat{\rho}(t)\hat{y}^{(k)}(t)\big)' + m^{-1}\hat{r}_0(t)\hat{y}^{(k)}, \qquad D(\hat{l}_0) = C_0^{(k+1)}(R).$$

By the conditions of the theorem, we can choose a number m so that

$$m \geq \max\Big(2, 8\max_{j=\overline{1,k}} \gamma_{\hat{r}_j,\sqrt{\hat{r}_0},j}\Big). \tag{18}$$

Then, according to condition c) of the theorem, Lemma 2.3, and estimate (14), we obtain that, for any $\hat{y} \in D(\hat{l})$,

$$\begin{aligned} \sum_{j=1}^{k-1}\left\| \frac{1}{m^{j+1}}\hat{r}_j\hat{y}^{(k-j)}\right\|_2 + \left\| \frac{1}{m^{k+1}}\hat{r}_k\hat{y}\right\|_2 &\leq \frac{2}{m}\left(\sum_{\theta=0}^{k-1}\frac{1}{m^\theta}\right)\max_{j=\overline{1,k}}\gamma_{\hat{r}_j,\sqrt{\hat{r}_0},j}\left\|\frac{1}{m}\sqrt{\hat{r}_0}\hat{y}^{(k)}\right\|_2 \\ &\leq \frac{1}{2}\left\|\frac{1}{m}\sqrt{\hat{r}_0}\hat{y}^{(k)}\right\|_2 \leq \frac{1}{2}\|\hat{l}\hat{y}\|_2. \end{aligned} \tag{19}$$

By (19) and Lemma 3.1, we get that

$$\begin{aligned} \|S\hat{y}\|_2 &= \left\|\sum_{j=1}^{k-1} m^{-(k-j)}r_j(t)\hat{y}^{(k-j)}(t) + m^{-(k+1)}\hat{r}_k(t)\hat{y}(t)\right\|_2 \\ &\leq \frac{1}{2}\|\hat{l}\hat{y}\|_2, \quad \hat{y} \in D(\hat{l}). \end{aligned} \tag{20}$$

According to Lemma 3.4 and Remark 3.1, the operator $\hat{l}$ is invertible, and its inverse $\hat{l}^{-1}$ is defined on the whole L_2. Then, by inequality (20) and the well-known statement on small perturbations [21, Chap. 4, Theorem 1.16], the following operator

$$P_m\hat{y} = \hat{l}\hat{y} + \sum_{j=1}^{k-1} m^{-(k-j)}r_j(t)\hat{y}^{(k-j)}(t) + m^{-(k+1)}\hat{r}_k(t)\hat{y}(t)$$

is also closed and invertible, and the inverse operator P_m^{-1} is defined on the whole space L_2. So, it follows that, for each $\hat{f} \in L_2$, $\hat{y} = P_m^{-1}\hat{f} \in D(P_m)$ and $\hat{y}$ is a solution of equation (17). By (19), we deduce that

$$\left\|\frac{1}{m}\sqrt{\hat{r}_0}\hat{y}^{(k)}\right\|_2 + \sum_{j=1}^{k-1}\left\|\frac{1}{m^{j+1}}\hat{r}_j\hat{y}^{(k-j)}\right\|_2 + \left\|\frac{1}{m^{k+1}}\hat{r}_k\hat{y}\right\|_2 \leq C\|\hat{f}\|_2. \tag{21}$$

Using the substitution $t = m^{-1}x$, we obtain that the function $y(x) = \hat{y}(\frac{1}{m}x)$ is a solution to equation (1). Inequality (21) implies (3). □

Proof of Theorem 1.3 Let the conditions of Theorem 1.1 be satisfied. Without loss of generality, we assume that $\theta(x)$ is a real function. Let

$$Q_j = \left\{\theta(x)\frac{d^{k-j}y}{dx^{k-j}} : y \in D(L), \|Ly\|_2 \le 1\right\} \quad (j = \overline{1,k}).$$

By Theorem 1.1 and (3), for any $y \in C_0^{(k+1)}(R)$: $\|Ly\|_2 \le 1$, we obtain

$$\|z\|_2 = \left\|\theta y^{(k-j)}\right\|_2 \le C_1 \cdot \left\|\sqrt{r_0} y^{(k)}\right\|_2 \le CC_1\|Ly\|_2 \le C_2.$$

These inequalities are valid for any $y \in D(L)$ such that $\|Ly\|_2 \le 1$, since L is a closed operator. Therefore, Q_j is bounded in L_2. Let us show that Q_j is compact in L_2. By the Frechet–Kolmogorov theorem, it suffices to show that, for each $\varepsilon > 0$, there is a number N_ε such that, for any $y \in C_0^{(k+1)}(R)$, $\|Ly\|_2 \le 1$, and $N \ge N_\varepsilon$, the following inequality

$$\|z\|_{L_2(R\setminus[-N,N])} = \left\|\theta y^{(k-j)}\right\|_{L_2(R\setminus[-N,N])} < \varepsilon \tag{22}$$

holds. We have that

$$\left\|\theta y^{(k-j)}\right\|_{L_2(R\setminus[-N,N])} = \left\|\theta y^{(k-j)}\right\|_{L_2(-\infty,-N)} + \left\|\theta y^{(k-j)}\right\|_{L_2(N,+\infty)}. \tag{23}$$

According to Lemma 2.1,

$$\begin{aligned}
&\int_N^{+\infty} \left|\theta(t)y^{(k-j)}(t)\right|^2 dt \\
&\quad = \int_0^{+\infty} \left|\theta(t+N)y^{(k-j)}(t+N)\right|^2 dt \\
&\quad \le \sup_{x>0}\left[\int_0^x \theta^2(t+N)\,dt \cdot \int_x^{+\infty} (t+N)^{2(j-1)}r_0^{-1}(t+N)\,dt\right] \times \\
&\qquad \times \int_0^{+\infty} \left|\sqrt{r_0(t+N)}y^{(k)}(t+N)\right|^2 dt \\
&\quad = \sup_{x\ge N}\left(\int_0^x \theta^2(t)\,dt \cdot \int_x^{+\infty} t^{2(j-1)}r_0^{-1}(t)\,dt\right) \cdot \int_N^{+\infty} \left|\sqrt{r_0(t)}y^{(k)}(t)\right|^2 dt \\
&\quad \le \sup_{x\ge N}\left(\int_0^x \theta^2(t)\,dt \cdot \int_x^{+\infty} t^{2(j-1)}r_0^{-1}(t)\,dt\right) \cdot \int_0^{+\infty} \left|\sqrt{r_0(t)}y^{(k)}(t)\right|^2 dt.
\end{aligned} \tag{24}$$

Similarly, using Lemma 2.2, we obtain

$$\begin{aligned}
&\int_{-\infty}^{-N} \left|\theta(t)y^{(k-j)}(t)\right|^2 dt \\
&\quad \le \sup_{\tau\le -N}\left(\int_\tau^0 \theta^2(t)\,dt \cdot \int_{-\infty}^{\tau} t^{2(j-1)}r_0^{-1}(t)\,dt\right) \cdot \int_{-\infty}^0 \left|\sqrt{r_0(t)}y^{(k)}(t)\right|^2 dt.
\end{aligned} \tag{25}$$

Set

$$A_{s,r_0,j}(N) = \max\Big(\sup_{t\geq N} T_{\theta,\sqrt{r_0},j}(x), \sup_{\tau\leq -N} M_{\theta,\sqrt{r_0},j}(\tau)\Big).$$

By virtue of (23), (24), (25), and (3), we have that

$$\|\theta y^{(k-j)}\|_{L_2(R\setminus[-N,N])} \leq A_{\theta,r_0,j}(N).$$

Taking into account this inequality and condition (4), we see that the number N_ε for given $\varepsilon > 0$ can be chosen so that, for all $y \in C_0^{(k+1)}(R)$, $\|Ly\|_2 \leq 1$, and $N : N \geq N_\varepsilon$, inequality (22) holds. □

Example 4.1 We consider the equation

$$\tilde{L}_0 y = \rho_0(x)\big(\rho_0(x)y^{(3)}\big)' + r(x)y^{(3)} - g(x)y' - h(x)y = f(x), \tag{26}$$

where

$$\rho_0(x) = \begin{cases} (1+x^4)(2-\sin^8 10x^6), & x < 0, \\ (1+x^{11})^{-1}(1+x^3\cos^2 7x^{10}), & x \geq 0, \end{cases}$$

$$r(x) = \big[9 + x^2\big(4 - \sin^{10} x^8\big)\big]^4, \qquad g(x) = \sqrt{x}\cos^4\big(3\exp x^2\big), \qquad h(x) = 6\sin\big(\exp|x|\big).$$

We denote by $\tilde{L}$ the closure of the operator $\tilde{L}_0$ ($D(\tilde{L}_0) = C_0^{(4)}(R)$) corresponding to (26). It is easy to check that $\gamma_{1,\sqrt{r},3} < \infty$, $\gamma_{g,\sqrt{r},2} < \infty$ and

$$\sup_{x<0}\left[\rho_0(x)\exp\int_0^x \frac{r(t)}{\rho_0^2(t)}\,dt\right] < \infty$$

hold. Hence, by Theorem 1.1, equation (26) has a unique solution for any $f(x) \in L_2$. By direct calculation we obtain that

$$\max\Big(\lim_{x\to+\infty} T_{(|x|+1)^\alpha,\sqrt{r},3}(x), \lim_{\tau\to-\infty} M_{(|\tau|+1)^\alpha,\sqrt{r},3}(\tau)\Big) = 0 \quad \text{if } \alpha < 1, \quad \text{and}$$
$$\max\Big(\lim_{x\to+\infty} T_{(|x|+1)^\beta,\sqrt{r},2}(x), \lim_{\tau\to-\infty} M_{(|\tau|+1)^\beta,\sqrt{r},2}(\tau)\Big) = 0 \quad \text{if } \beta < 2.$$

Therefore, by Theorem 1.3, the operators $(|x|+1)^\alpha\tilde{L}^{-1}$ and $(|x|+1)^\beta\frac{d}{dx}\tilde{L}^{-1}$ are compact in L_2 for $\alpha < 1$ and $\beta < 2$, where $\tilde{L}^{-1}$ is the inverse of $\tilde{L}$. Note that the coefficients of (26) are fluctuating and $\rho(x)$ tends to zero as $x \to +\infty$ and is unbounded as $x \to -\infty$.

Example 4.2 We consider the following higher-order equation:

$$\begin{aligned}\tilde{l}_0 y = {}& \big(1+x^2\big)^s\big(\big(1+x^2\big)^s y^{(k)}\big)' + \big[2 - \big(11k + 3x^2\big)^{2k}\big]y^{(k)} \\ & + \sum_{j=1}^{k-1}\big[(-1)^j + 2jx^2\big]^{k-\frac{j}{2}} y^{(k-j)} - \frac{5}{3+4x^2}y = f(x),\end{aligned} \tag{27}$$

where $x \in R$, k and s are natural numbers, and $f \in L_2$. Let $\tilde{l}$ be the closure of the operator $\tilde{l}_0$ with $D(\tilde{l}_0) = C_0^{(k+1)}(R)$. Direct calculations show that all conditions of Theorem 1.2 are satisfied. Hence, equation (27) is uniquely solvable and its solution y satisfies the following estimate:

$$\left\| \sqrt{\left(11k + 3x^2\right)^{2k} - 2}\, y^{(k)} \right\|_2 + \sum_{j=1}^{k-1} \left\| \left[(-1)^j + 2jx^2\right]^{k-\frac{j}{2}} y^{(k-j)} \right\|_2 + \|y\|_2 \le C\|f\|_2.$$

Moreover, for continuous functions $\theta_j(x)$ ($j = \overline{1,k}$) such that $|\theta_j(x)| \le (1 + x^2)^{\omega_j}$ with $\omega_j < k - \frac{j}{2}$, the equality

$$\max\Bigl(\lim_{x\to+\infty} T_{\theta_j,[2-(11k+3x^2)^k],j}(x), \lim_{\tau\to-\infty} M_{\theta_j,[2-(11k+3x^2)^k],j}(\tau)\Bigr) = 0$$

holds. Then, according to Theorem 1.3, the operators $(1 + x^2)^\alpha \tilde{l}^{-1}$ ($\alpha < \frac{k}{2}$) and $\theta_j(x)\frac{d^{k-j}}{dx^{k-j}}\tilde{l}^{-1}$ ($j = \overline{1,k-1}$) are compact in L_2, where $\tilde{l}^{-1}$ is the inverse of $\tilde{l}$.

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References
1. Yosida, K.: Functional Analysis. Springer, Berlin (1995)
2. Naimark, M.: Linear Differential Operators. Nauka, Moscow (1969)
3. Reed, M., Simon, B.: Methods of Modern Mathematical Physics: Harmonic Analysis, Self-Adjointness, vol. 2. Mir, Moscow (1978) (in Russian)
4. Fedoruk, M.V.: Asymptotic Analysis: Linear Ordinary Differential Equations. Springer, Berlin (1993)
5. Ornstein, G.E., Uhlenbeck, L.S.: On the theory of Brownian motion. Phys. Rev. **36**, 823–841 (1930)
6. Bogachev, V.I., Krylov, N.V., Rökner, M., Shaposhnikov, S.V.: Fokker–Planck–Kolmogorov Equations. Mathematical Surveys and Monographs, vol. 207. Am. Math. Soc., Providence (2015)
7. Fornaro, S., Lorenzi, L.: Generation results for elliptic operators with unbounded diffusion coefficients in L_p- and C_b-spaces. Discrete Contin. Dyn. Syst. **18**(4), 747–772 (2007)
8. Hieber, M., Sawada, O.: The Navier–Stokes equations in R^n with linearly growing initial data. Arch. Ration. Mech. Anal. **175**, 269–285 (2005)
9. Metafune, G., Pallara, D., Vespri, V.: L_p-estimates for a class of elliptic operators with unbounded coefficients in R^n. Houst. J. Math. **31**, 605–620 (2005)
10. Hieber, M., Lorenzi, L., Prüss, J., Rhandi, A., Schnaubelt, R.: Global properties of generalized Ornstein–Uhlenbeck operators on $L_p(R^N, R^N)$ with more than linearly growing coefficients. J. Math. Anal. Appl. **350**(1), 100–121 (2009)
11. Ospanov, K.N., Akhmetkaliyeva, R.D.: Some inequalities for second order differential operators with unbounded drift. Eurasian Math. J. **6**(2), 63–71 (2015)
12. Ospanov, K.N.: L_1-maximal regularity for quasilinear second order differential equation with damped term. Electron. J. Qual. Theory Differ. Equ. **2015**, 39 (2015)
13. Ospanov, K.N.: Maximal L_p-regularity for a second-order differential equation with unbounded intermediate coefficient. Electron. J. Qual. Theory Differ. Equ. **2019**, 65 (2019)

14. Ospanov, K., Yesbayev, A.: Solvability and maximal regularity results for a differential equation with diffusion coefficient. Turk. J. Math. **44**(4), 1304–1316 (2020)
15. Ospanov, K.N., Yeskabylova, Z.B., Bekjan, T.N.: The solvability results for the third-order singular non-linear differential equation. Eurasian Math. J. **10**(4), 84–90 (2019)
16. Ospanov, K.N., Eskabylova, Z.B., Beisenova, D.R.: Maximal regularity estimates for a higher order differential equation with the fluctuating coefficients. Eurasian Math. J. **10**(2), 65–74 (2019)
17. Lu, S., Yu, X.: Periodic solutions for second order differential equations with indefinite singularities. Adv. Nonlinear Anal. **9**(1), 994–1007 (2020)
18. Zhao, H.Y.: Pseudo almost periodic solutions for a class of differential equation with delays depending on state. Adv. Nonlinear Anal. **9**(1), 1251–1258 (2020)
19. Apyshev, O., Otelbaev, M.: On the spectrum of a class of differential operators and some embedding theorems. Izv. Akad. Nauk SSSR, Ser. Mat. **43**(4), 739–764 (1979) (in Russian)
20. Muckenhoupt, B.: Hardy's inequality with weights. Stud. Math. **44**, 31–38 (1972)
21. Kato, T.: Perturbation Theory for Linear Operators. Classics in Mathematics. Springer, Berlin (1995)