Turkish Journal of Mathematics

Volume 43 | Number 1 Article 25

1-1-2019

Kernel operators and their boundedness from weighted Sobolev space to weighted Lebesgue space

AIGERIM KALYBAY

RYSKUL OINAROV

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 301 – 315
© TÜBİTAK
doi:10.3906/mat-1807-187

Kernel operators and their boundedness from weighted Sobolev space to weighted Lebesgue space

Aigerim KALYBAY[1,*], **Ryskul OINAROV**[2]
[1]KIMEP University, Almaty, Kazakhstan
[2]L.N. Gumilyov Eurasian National University, Astana, Kazakhstan

Received: 25.07.2018 • **Accepted/Published Online:** 07.12.2018 • **Final Version:** 18.01.2019

Abstract: In this paper, for a wide class of integral operators, we consider the problem of their boundedness from a weighted Sobolev space to a weighted Lebesgue space. The crucial step in the proof of the main result is to use the equivalence of the basic inequality and certain Hardy-type inequality, so we first state and prove this equivalence.

Key words: Integral operator, kernel, weighted Lebesgue space, weighted Sobolev space, boundedness, compactness

1. Introduction

Let $I=(a,b)$ and $-\infty\leq a<b\leq\infty$. Let $1<p,r,q<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Suppose that υ, ρ, u and ω are functions nonnegative on I such that υ^p, ρ^p, ω^p, u^r, $\rho^{-p'}$, and $\omega^{-q'}$ are locally summable on I.

Denote by $W^1_{p,r}(u,\upsilon)\equiv W^1_{p,r}(u,\upsilon,I)$ the space of all functions locally absolutely continuous on I having the finite norm

$$\|f\|_{W^1_{p,r}}=\|uf'\|_r+\|\upsilon f\|_p,$$

where $\|\cdot\|_p$ is the usual norm of the Lebesgue space $L_p(I)$. In the case $p=r$ and $u\equiv\rho$, we suppose that $W^1_{p,p}(\rho,\upsilon)\equiv W^1_p(\rho,\upsilon)$ and $\|f\|_{W^1_{p,p}}\equiv\|f\|_{W^1_p}$.

Let $L_{p,\upsilon}\equiv L_p(\upsilon,I)$ be the set of all functions measurable on I such that $\|f\|_{p,\upsilon}\equiv\|\upsilon f\|_p<\infty$.

Let $\overset{\circ}{AC}(I)$ be the set of all locally absolutely continuous functions with compact supports on I.

Denote by $\overset{\circ}{W}{}^1_p(\rho,\upsilon)\equiv\overset{\circ}{W}{}^1_p(\rho,\upsilon,I)$ the closure of the set $\overset{\circ}{AC}(I)\cap W^1_p(\rho,\upsilon)$ with respect to the norm of the space $W^1_p(\rho,\upsilon)$.

Consider the operators

$$\mathcal{K}^+f(x)=\int_a^x K(x,s)f(s)ds,\quad x\in I,\tag{1}$$

$$\mathcal{K}^-g(s)=\int_s^b K(x,s)g(x)dx,\quad s\in I,\tag{2}$$

with a kernel $K(\cdot,\cdot)\geq 0$ measurable on $\Omega=\{(x,s):a<s\leq x<b\}$.

*Correspondence: o_ryskul@mail.ru
2010 *AMS Mathematics Subject Classification:* Primary 47B38, Secondary 26D10, 46E30, 46E35.

Let $T \equiv \mathcal{K}^+$ or $T \equiv \mathcal{K}^-$. In this paper, under some assumptions on the kernel $K(\cdot,\cdot)$, we establish a criterion of the boundedness of the operator T from $\mathring{W}_p^1(\rho,\upsilon)$ to $L_{q,}(\omega,I)$, i.e. the validity of the inequality

$$\|\omega Tf\|_q \le C(\|\rho f'\|_p + \|\upsilon f\|_p), \quad f \in \mathring{W}_p^1(\rho,\upsilon). \tag{3}$$

In the case where $\rho \equiv 0$, the validity of the inequality (3) means the boundedness of the integral operator T from $L_{p,\upsilon}$ to $L_{q,\omega}$. For the last few years, this problem has been the subject of many papers and monographs (see, e.g., the papers [4, 10] and monographs [2, 3, 9]). In the works [5] and [7], the inequality (3) for the operator (1) is studied for a more restricted class of kernels $K(\cdot,\cdot)$ than in this paper. In [6], the inequality (3) is characterized for the same class of kernels $K(\cdot,\cdot)$ as here. However, the technique applied in [6] assumes the validity of some strict condition on the weights. The purpose in this paper is to reduce this strict condition. We achieve this purpose by using a new technique based on the equivalence of certain inequalities of independent interest.

The paper is organized as follows: in Section 2, we present the notations, definitions, and known statements required to prove the main results; in Section 3, we obtain necessary and sufficient conditions for the validity of the inequality (3) for the operators (1) and (2).

In the work, the notation $A \approx B$ means $c_1 B \le A \le c_2 B$, where constants $c_2 > c_1 > 0$ possibly depend only on some nonessential parameters.

2. Required definitions, statements, and facts

As in [7] (see also [5, 6]), we introduce the function

$$\delta(x,y) = \sup\left\{ d > 0 : \int\limits_{x-d}^{x} \rho^{-p'}(t)dt \le \int\limits_{x}^{x+y} \rho^{-p'}(t)dt, (x-d,x] \subset I \right\},$$

with the domain $D(\delta) = \{(x,y) : x \in I, y > 0, [x, x+y) \in I\}$. If we fix $x \in I$, then at least for a sufficient small $y > 0$, we have

$$\int\limits_{x-\delta(x,y)}^{x} \rho^{-p'}(t)dt = \int\limits_{x}^{x+y} \rho^{-p'}(t)dt. \tag{4}$$

Let $x \in I$ and D_x be a set of $y > 0$ for which $x + y \in I$ and (4) is fulfilled. For all $x \in I$ we define

$$d^+(x) = \sup\{d : \|\rho^{-1}\|_{p',(x-\delta(x,d),x+d)} \|\upsilon\|_{p,(x-\delta(x,d),x+d)} \le 1, \ d \in D_x\},$$

and assume that $d^-(x) = \delta(x, d^+(x))$, $\mu^-(x) = x - d^-(x)$ and $\mu^+(x) = x + d^+(x)$.

Let for some $c \in I$,

$$\|\rho^{-1}\|_{p',(a,c)} + \|\upsilon\|_{p,(a,c)} = \infty, \quad \|\rho^{-1}\|_{p',(c,b)} + \|\upsilon\|_{p,(c,b)} = \infty. \tag{5}$$

For simplicity, we assume that (5) holds that is equivalent to the condition $\mathring{W}_p^1(\rho,\upsilon) = W_p^1(\rho,\upsilon)$ (see [7]). How to overcome the difficulties that arise when the condition (5) does not hold is given in [7].

On the basis of lemmas 1.1–1.3 of [7], the functions $\mu^-(x) = x - d^-(x)$ and $\mu^+(x) = x + d^+(x)$ are continuous and strictly increasing on I. Moreover,

$$\lim_{x\to a} \mu^{\pm}(x) = a, \quad \lim_{x\to b} \mu^{\pm}(x) = b. \tag{6}$$

This gives that $a < \mu^{\pm}(x) < b$ for any $x \in I$. Furthermore, we need the following statement of [5, 6].

Lemma 2.1 *Let the condition (5) hold. Then the functions $\mu^-(x)$ and $\mu^+(x)$ are locally absolutely continuous on I.*

We denote the inverse functions of the functions μ^- and μ^+ by φ^+ and φ^-, respectively. Then the functions φ^+ and φ^- are continuous and strictly increasing on I and $\varphi^+(x) > \varphi^-(x)$, $x \in I$, $\lim\limits_{x\to a^+} \varphi^{\pm}(x) = a$ and $\lim\limits_{x\to b^-} \varphi^{\pm}(x) = b$.

On the basis of theorem 2 of [5], we have the following theorem.

Theorem A. *Let $1 < p, q < \infty$. The inequality (3) for all functions $f \in \mathring{W}_p^1(\rho, \upsilon)$ is equivalent to the inequality*

$$\left(\int_a^b \left(\int_{\varphi^-(x)}^{\varphi^+(x)} (T^*g)(t)dt\right)^{p'} \rho^{-p'}(x)dx\right)^{\frac{1}{p'}} \le C_1 \left(\int_a^b \omega^{-q'}(t) g^{q'}(t)dt\right)^{\frac{1}{q'}} \tag{7}$$

for all nonnegative functions $g \in L_{q'}(\omega^{-1}, I)$, where T^* is the dual operator to the operator T with respect to the bilinear form $\int_a^b f(t)g(t)dt$. Moreover, $C \approx C_1$, where $C > 0$ and $C_1 > 0$ are the best constants in (3) and (7), respectively.

Remark 2.2 *Theorem A was firstly proved in [5], but there, the functions φ^+ and φ^- are inverse to the functions $\mu^-(\mu^-)$ and $\mu^+(\mu^+)$, respectively. In [1], it is shown that Theorem A is also correct when the functions φ^+ and φ^- are inverse of the functions μ^- and μ^+, respectively.*

For every integer $n \ge 0$, we define the classes $\mathcal{O}_n^{\pm}(\Omega)$ (see [4]) of the kernels of the operators (1) and (2). We agree to write $K(\cdot,\cdot) \equiv K_n(\cdot,\cdot)$ if $K(\cdot,\cdot) \in \mathcal{O}_n^{\pm}(\Omega)$.

Let $K^+(\cdot,\cdot)$ and $K^-(\cdot,\cdot)$ be nonnegative measurable functions defined on Ω such that $K^+(\cdot,\cdot)$ is nondecreasing in the first argument and $K^-(\cdot,\cdot)$ is nonincreasing in the second argument.

The functions $K^{\pm}(\cdot,\cdot) \equiv K_0^{\pm}(\cdot,\cdot)$ belong to the classes $\mathcal{O}_0^{\pm}(\Omega)$ if and only if $K_0^+(x,s) = \upsilon(s) \ge 0$ and $K_0^-(x,s) = u(x) \ge 0$ for all $(x,s) \in \Omega$.

Suppose that the classes $\mathcal{O}_i^{\pm}(\Omega)$, $i = 0, 1, ..., n-1$, $n \ge 1$, are defined. The functions $K(\cdot,\cdot) \equiv K_n^{\pm}(\cdot,\cdot)$ belong to the classes $\mathcal{O}_n^{\pm}(\Omega)$ if and only if there exist functions $K_i^{\pm}(\cdot,\cdot) \in \mathcal{O}_i^{\pm}(\Omega)$, $i = 0, 1, ..., n-1$, such that

$$K_n^+(x,s) \approx \sum_{i=0}^{n} K_{n,i}^+(x,t) K_i^+(t,s), \tag{8}$$

$$K_n^-(x,s) \approx \sum_{i=0}^{n} K_i^-(x,t) K_{i,n}^-(t,s) \tag{9}$$

for $a < s \le t \le x < b$. Moreover, $K_{n,n}^{\pm}(\cdot,\cdot) \equiv 1$, where the functions $K_{n,i}^{+}(\cdot,\cdot)$ and $K_{i,n}^{-}(\cdot,\cdot)$, $i = 0,1,...,n$, are, generally speaking, arbitrary nonnegative measurable functions defined on Ω satisfying the conditions (8) and (9), respectively.

On the basis of theorems 5 and 6 of [4], we have the following theorems.

Theorem B^+**.** *Let $1 < p \le q < \infty$ and the kernel of the operator (1) belong to the class $\mathcal{O}_n^+(\Omega) \bigcup \mathcal{O}_n^-(\Omega)$, $n \ge 0$. Then the operator (1) is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if the condition $B_i^+ = \sup_{z\in I} B_i^+(z) < \infty$ holds at least for one of $i = 1,2$, where*

$$B_1^+(z) = \left(\int_z^b \omega^q(x) \left(\int_a^z K^{p'}(x,s)\rho^{-p'}(s)ds \right)^{\frac{q}{p'}} dx \right)^{\frac{1}{q}},$$

$$B_2^+(z) = \left(\int_a^z \rho^{-p'}(s) \left(\int_z^b K^{q}(x,s)\omega^{q}(x)dx \right)^{\frac{p'}{q}} ds \right)^{\frac{1}{p'}}.$$

Moreover, for the norm $\|\mathcal{K}^+\|$ of the operator $\mathcal{K}^+$ from $L_p(\rho,I)$ to $L_q(\omega,I)$ the relation $\|\mathcal{K}^+\| \approx B_1^+ \approx B_2^+$ is valid.

Theorem B^-**.** *Let $1 < p \le q < \infty$ and the kernel of the operator (2) belong to the class $\mathcal{O}_n^+(\Omega) \bigcup \mathcal{O}_n^-(\Omega)$, $n \ge 0$. Then the operator (2) is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if the condition $B_i^- = \sup_{z\in I} B_i^-(z) < \infty$ holds at least for one of $i = 1,2$, where*

$$B_1^-(z) = \left(\int_a^z \omega^q(s) \left(\int_z^b K^{p'}(x,s)\rho^{-p'}(x)dx \right)^{\frac{q}{p'}} ds \right)^{\frac{1}{q}},$$

$$B_2^-(z) = \left(\int_z^b \rho^{-p'}(x) \left(\int_a^z K^{q}(x,s)\omega^{q}(s)ds \right)^{\frac{p'}{q}} dx \right)^{\frac{1}{p'}}.$$

Moreover, for the norm $\|\mathcal{K}^-\|$ of the operator $\mathcal{K}^-$ from $L_p(\rho,I)$ to $L_q(\omega,I)$ the relation $\|\mathcal{K}^-\| \approx B_1^- \approx B_2^-$ is valid.

Consider the integral operators

$$(\mathcal{K}_+ f)(x) = \int_{\alpha(x)}^{\beta(x)} K(x,s) f(s) ds, \quad x \in I, \tag{10}$$

$$(\mathcal{K}_{-}g)(s)=\int\limits_{\alpha(s)}^{\beta(s)}K(x,s)g(x)dx,\quad s\in I, \tag{11}$$

where $\alpha(x)$ and $\beta(x)$ are locally absolute continuous functions strictly increasing on I such that $\alpha(x)<\beta(x)$, $\forall x\in I$, and $\lim\limits_{x\to a+}\alpha(x)=\lim\limits_{x\to a+}\beta(x)=a$, $\lim\limits_{x\to b-}\alpha(x)=\lim\limits_{x\to b-}\beta(x)=b$.

Let $\Omega^{+}=\{(x,s): a<x<b,\ \alpha(x)\le s\le\beta(x)\}$ and $\Omega^{-}=\{(x,s): a<s<b,\ \alpha(s)\le x\le\beta(s)\}$.

Let $K^{\pm}(\cdot,\cdot)\ge 0$ be measurable functions defined on $\Omega^{\pm}$ such that $K^{+}(\cdot,\cdot)$ is nondecreasing in the first argument and $K^{-}(\cdot,\cdot)$ is nonincreasing in the second argument. As above, we define [8] the classes $\mathcal{O}_n^{\pm}(\Omega^{\pm})$, $n\ge 0$, of kernels of the operators (10) and (11). The classes $\mathcal{O}_0^{+}(\Omega^{+})$ and $\mathcal{O}_0^{-}(\Omega^{-})$ consist of the functions $K_0^{+}(x,s)=\upsilon(s)$ and $K_0^{-}(x,s)=u(x)$ for all $(x,s)\in\Omega^{+}$ and $(x,s)\in\Omega^{-}$, respectively.

Let the classes $\mathcal{O}_i^{\pm}(\Omega^{\pm})$, $i=0,1,...,n-1$, $n\ge 1$ be defined. The functions $K_n^{\pm}(\cdot,\cdot)$ belong to the classes $\mathcal{O}_n^{\pm}(\Omega^{\pm})$ if and only if for $K_n^{+}(\cdot,\cdot)$ and $K_n^{-}(\cdot,\cdot)$ there exist the functions $K_i^{+}(\cdot,\cdot)\in\mathcal{O}_i^{+}(\Omega^{+})$ and $K_i^{-}(\cdot,\cdot)\in\mathcal{O}_i^{-}(\Omega^{-})$, $i=0,1,...,n-1$, respectively, and the relations (8) and (9) hold for $a<t\le x<b$, $\alpha(x)\le s\le\beta(t)$ and $a<s\le t<b$, $\alpha(t)\le x\le\beta(s)$, respectively, where $K_{n,n}^{\pm}(\cdot,\cdot)\equiv 1$. As above, the functions $K_{n,i}^{+}(\cdot,\cdot)$ and $K_{i,n}^{-}(\cdot,\cdot)$, $i=0,1,...,n-1$ are, generally speaking, arbitrary nonnegative measurable functions defined on Ω satisfying the conditions (8) and (9), respectively.

Remark 2.3 *If $K_n^{\pm}(x,s)\in\mathcal{O}_n^{\pm}(\Omega)$, then $K^{+}(\beta(x),s)\in\mathcal{O}_n^{+}(\Omega^{+})$ and $K^{-}(x,\ \alpha(s))\in\mathcal{O}_n^{-}(\Omega^{-})$.*

Assume that

$$D_1^{+}(z)=\sup_{y\in\Delta^{+}(z)}\left(\int\limits_{y}^{z}\omega^{q}(x)\left(\int\limits_{\alpha(z)}^{\beta(y)}K^{p'}(x,s)\rho^{-p'}(s)ds\right)^{\frac{q}{p'}}dx\right)^{\frac{1}{q}},$$

$$D_2^{+}(z)=\sup_{y\in\Delta^{+}(z)}\left(\int\limits_{\alpha(z)}^{\beta(y)}\rho^{-p'}(s)\left(\int\limits_{y}^{z}K^{q}(x,s)\omega^{q}(x)dx\right)^{\frac{p'}{q}}ds\right)^{\frac{1}{p'}},$$

$$D_1^{-}(z)=\sup_{y\in\Delta^{-}(z)}\left(\int\limits_{z}^{y}\omega^{q}(s)\left(\int\limits_{\alpha(y)}^{\beta(z)}K^{p'}(x,s)\rho^{-p'}(x)dx\right)^{\frac{q}{p'}}ds\right)^{\frac{1}{q}},$$

$$D_2^{-}(z)=\sup_{y\in\Delta^{-}(z)}\left(\int\limits_{\alpha(y)}^{\beta(z)}\rho^{-p'}(x)\left(\int\limits_{z}^{y}K^{q}(x,s)\omega^{q}(s)ds\right)^{\frac{p'}{q}}dx\right)^{\frac{1}{p'}},$$

where $\Delta^{+}(z)=[\beta^{-1}(\alpha(z)),z]$ and $\Delta^{-}(z)=[z,\alpha^{-1}(\beta(z))]$.

From the results of [8], we have the following theorems.

Theorem C^+. *Let $1 < p \leq q < \infty$ and the kernel of the operator (10) belong to $\mathcal{O}_n^+(\Omega^+)\bigcup\mathcal{O}_n^-(\Omega^-)$, $n \geq 0$. Then the operator (10) is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if the condition $D_i^+ = \sup_{z\in I} D_i^+(z) < \infty$ holds at least for one of $i = 1, 2$. Moreover, for the norm $\|\mathcal{K}_+\|$ of the operator $\mathcal{K}_+$ from $L_p(\rho, I)$ to $L_q(\omega, I)$ the relation $\|\mathcal{K}_+\| \approx D_1^+ \approx D_2^+$ is valid.*

Theorem C^-. *Let $1 < p \leq q < \infty$ and the kernel of the operator (11) belong to $\mathcal{O}_n^+(\Omega^+)\bigcup\mathcal{O}_n^-(\Omega^-)$, $n \geq 0$. Then the operator (11) is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if the condition $D_i^- = \sup_{z\in I} D_i^-(z) < \infty$ holds at least for one of $i = 1, 2$. Moreover, for the norm $\|\mathcal{K}_-\|$ of the operator $\mathcal{K}_-$ from $L_p(\rho, I)$ to $L_q(\omega, I)$ the relation $\|\mathcal{K}_-\| \approx D_1^- \approx D_2^-$ is valid.*

3. Criteria of validity of inequality (3) for operators (1) and (2)

Here and in the sequel, we suppose that the conditions in (5) are fulfilled.

Theorem 3.1 *Let $1 < p \leq q < \infty$ and the kernel of the operator (1) belong to the class $\mathcal{O}_n^-(\Omega)$, $n \geq 0$. Then for the operator (1) the inequality (3) holds if and only if $\max\{F_i^+, G_j^+\} < \infty$ at least for one of the pairs (i, j), $i, j = 1, 2$. Moreover, for the best constant $C > 0$ in (3), the relation $C \approx \max\{F_i^+, G_j^+\}$, $i, j = 1, 2$, is valid. Here, $F_i^+ = \sup_{z\in I} F_i^+(z)$, $G_j^+ = \sup_{z\in I} G_j^+(z)$,*

$$F_1^+(z) = \left(\int_a^{\mu^-(z)} \rho^{-p'}(x) \left(\int_z^b \left(\int_{\varphi^-(x)}^{\varphi^+(x)} K(t,s)ds\right)^q \omega^q(t)dt\right)^{\frac{p'}{q}} dx\right)^{\frac{1}{p'}},$$

$$F_2^+(z) = \left(\int_z^b \omega^q(t) \left(\int_a^{\mu^-(z)} \left(\int_{\varphi^-(x)}^{\varphi^+(x)} K(t,s)ds\right)^{p'} \rho^{-p'}(x)dx\right)^{\frac{q}{p'}} dt\right)^{\frac{1}{q}},$$

$$G_1^+(z) = \sup_{y\in\Delta_\mu^+(z)} \left(\int_{\mu^-(z)}^{\mu^+(y)} \rho^{-p'}(x) \left(\int_y^z \left(\int_{\varphi^-(x)}^{t} K(t,s)ds\right)^q \omega^q(t)dt\right)^{\frac{p'}{q}} dx\right)^{\frac{1}{p'}},$$

$$G_2^+(z) = \sup_{y\in\Delta_\mu^+(z)} \left(\int_y^z \omega^q(t) \left(\int_{\mu^-(z)}^{\mu^+(y)} \left(\int_{\varphi^-(x)}^{t} K(t,s)ds\right)^{p'} \rho^{-p'}(x)dx\right)^{\frac{q}{p'}} dt\right)^{\frac{1}{q}},$$

where $\Delta_\mu^+(z) = [\varphi^-(\mu^-(z)), z]$.

Theorem 3.2 *Let $1 < p \leq q < \infty$ and the kernel of the operator (2) belong to the class $\mathcal{O}_n^+(\Omega)$, $n \geq 0$. Then the inequality (3) for the operator (2) holds if and only if $\max\{F_i^-, G_j^-\} < \infty$ at least for one of the pairs (i,j), $i,j = 1,2$. Moreover, for the best constant $C > 0$ in (3) the relation $C = \max\{F_i^-, G_j^-\}$, $i,j = 1,2$, is valid. Here $F_i^- = \sup\limits_{z\in I} F_i^-(z)$, $G_j^- = \sup\limits_{z\in I} G_j^-(z)$,*

$$F_1^-(z) = \left(\int\limits_{\mu^+(z)}^{b} \rho^{-p'}(x) \left(\int\limits_a^z \left(\int\limits_{\varphi^-(x)}^{\varphi^+(x)} K(t,s)dt \right)^q \omega^q(s)ds \right)^{\frac{p'}{q}} dx \right)^{\frac{1}{p'}},$$

$$F_2^-(z) = \left(\int\limits_a^z \omega^q(s) \left(\int\limits_{\mu^+(z)}^{b} \left(\int\limits_{\varphi^-(x)}^{\varphi^+(x)} K(t,s)dt \right)^{p'} \rho^{-p'}(x)dx \right)^{\frac{q}{p'}} ds \right)^{\frac{1}{q}},$$

$$G_1^-(z) = \sup_{y\in\Delta_\mu^-(z)} \left(\int\limits_{\mu^-(y)}^{\mu^-(z)} \rho^{-p'}(x) \left(\int\limits_z^y \left(\int\limits_s^{\varphi^+(x)} K(t,s)dt \right)^q \omega^q(s)ds \right)^{\frac{p'}{q}} dx \right)^{\frac{1}{p'}},$$

$$G_2^-(z) = \sup_{y\in\Delta_\mu^-(z)} \left(\int\limits_z^y \omega^q(s) \left(\int\limits_{\mu^-(y)}^{\mu^-(z)} \rho^{-p'}(x) \left(\int\limits_s^{\varphi^+(x)} K(t,s)dt \right)^{p'} dx \right)^{\frac{q}{p'}} ds \right)^{\frac{1}{q}},$$

where $\Delta_\varphi^-(z) = [z, \varphi^+(\mu^+(z)))]$.

Theorems 3.1 and 3.2 are proved in [6] under the condition:

$$E_J^{\pm} = \sup_{\omega\subset J} \left(\int\limits_\omega \rho^{-p'}(t)dt \right)^{-1} \int\limits_{\mu^{\pm}(\omega)} \rho^{-p'}(s)ds < 2.$$

This condition is a strong restriction on the weight functions. Here, using a method different from that of the work [6], we reprove theorems 3.1 and 3.2 without this restriction.

We first prove a statement that is equivalent to Theorem A.

Lemma 3.3 *Let $1 < p, q < \infty$. The inequality (3) for all functions $f \in \overset{\circ}{W}{}_p^1(\rho, \upsilon)$ is equivalent to the inequality:*

$$\left(\int\limits_a^b \left(\omega(x) T \left(\int\limits_{\mu^-(\cdot)}^{\mu^+(\cdot)} f(t)dt \right)(x) \right)^q dx \right)^{\frac{1}{q}} \leq C_1 \left(\int\limits_a^b \rho^p(t) f^p(t)dt \right)^{\frac{1}{p}} \tag{12}$$

for all nonnegative functions $f \in L_p(\rho, I)$. Moreover, $C \approx C_1$, where $C > 0$ and $C_1 > 0$ are the best constants in (3) and (12), respectively.

Proof [Proof of Lemma 3.3] We find the dual operator to the operator $\int\limits_{\varphi^-(x)}^{\varphi^+(x)}(T^*g)(t)dt$ with respect to bilinear form $\int\limits_a^b f(t)u(t)dt$, where $f\in L_p(\rho,I)$ and $u\in L_{p'}(\rho^{-1},\ I)$. Let $f\in L_p(\rho,I)$ and $g\in L_{q'}(\omega^{-1},I)$. Then by Theorem A, we have that

$$\int\limits_{\varphi^-(x)}^{\varphi^+(x)}(T^*g)(t)dt\in L_{p'}(\rho^{-1},I)$$

and

$$\int\limits_a^b f(x)\int\limits_{\varphi^-(x)}^{\varphi^+(x)}(T^*g)(t)dtdx=\int\limits_a^b(T^*g)(t)\int\limits_{\mu^-(t)}^{\mu^+(t)}f(x)dxdt=\int\limits_a^b g(t)T\left(\int\limits_{\mu^-(\cdot)}^{\mu^+(\cdot)}f(x)dx\right)(t)dt,$$

i.e. the operator $T\left(\int\limits_{\mu^-(\cdot)}^{\mu^+(\cdot)}f(x)dx\right)(t)$ is the dual operator to $\int\limits_{\varphi^-(x)}^{\varphi^+(x)}(T^*g)(t)dt$. Since by Theorem A, the operator $\int\limits_{\varphi^-(x)}^{\varphi^+(x)}(T^*g)(t)dt$ acts from $L_{q'}(\omega^{-1},I)$ to $L_{p'}(\rho^{-1},I)$, then the dual operator $T\left(\int\limits_{\mu^-(\cdot)}^{\mu^+(\cdot)}f(x)dx\right)(t)$ acts from $L_p(v,I)$ to $L_q(\omega,I)$, i.e. (12) and (7) are equivalent, and the best constants in (12) and (7) coincide. Then by Theorem A, the best constants in (12) and (3) are equivalent. The proof of Lemma 3.3 is complete. □

Proof [Proof of Theorem 3.1] Let $T=\mathcal{K}^+$. Then by Lemma 3.3 the inequality (3) holds if and only if the inequality (12) holds for $T=\mathcal{K}^+$, i.e. when the operator $\mathcal{K}^+\left(\int\limits_{\mu^-(\cdot)}^{\mu^+(\cdot)}f(x)dx\right)(t)$ is bounded from $L_p(v,I)$ to $L_q(\omega,I)$. Since

$$\mathcal{K}^+\left(\int\limits_{\mu^-(\cdot)}^{\mu^+(\cdot)}f(x)dx\right)(t)=\int\limits_a^t K(t,s)\int\limits_{\mu^-(s)}^{\mu^+(s)}f(x)dxds, \tag{13}$$

the change of order of integration gives

$$\int\limits_a^t K(t,s)\int\limits_{\mu^-(s)}^{\mu^+(s)}f(x)dxds=\int\limits_a^{\mu^-(t)}f(x)\int\limits_{\varphi^-(x)}^{\varphi^+(x)}K(t,s)dsdx+\int\limits_{\mu^-(t)}^{\mu^+(t)}f(x)\int\limits_{\varphi^-(x)}^{t}K(t,s)dsdx. \tag{14}$$

From (12)–(14), it follows that the operator

$$\widetilde{\mathcal{K}}^+f(t)\equiv\mathcal{K}^+\left(\int\limits_{\mu^-(\cdot)}^{\mu^+(\cdot)}f(x)dx\right)(t)$$

is bounded from $L_p(v,I)$ to $L_q(\omega,I)$ if and only if

$$\widetilde{\mathcal{K}}_1^+ f(t) \equiv \int\limits_a^{\mu^-(t)} f(x) \int\limits_{\varphi^-(x)}^{\varphi^+(x)} K(t,s) ds dx$$

and

$$\widetilde{\mathcal{K}}_2^+ f(t) \equiv \int\limits_{\mu^-(t)}^{\mu^+(t)} f(x) \int\limits_{\varphi^-(x)}^{t} K(t,s) ds dx$$

are bounded from $L_p(v,I)$ to $L_q(\omega,I)$. Moreover, between the norms of the operators $\|\widetilde{\mathcal{K}}^+\|$, $\|\widetilde{\mathcal{K}}_1^+\|$ and $\|\widetilde{\mathcal{K}}_2^+\|$ from $L_p(v,I)$ to $L_q(\omega,I)$ the relation

$$\|\widetilde{\mathcal{K}}^+\| \approx \|\widetilde{\mathcal{K}}_1^+\| + \|\widetilde{\mathcal{K}}_2^+\| \tag{15}$$

is valid. Let us estimate the values $\|\widetilde{\mathcal{K}}_1^+\|$ and $\|\widetilde{\mathcal{K}}_2^+\|$:

$$\widetilde{\mathcal{K}}_1^+ f(s) = \int\limits_a^{\mu^-(s)} \widetilde{K}_1^+(s,x) f(x) dx, \quad \widetilde{K}_1^+(s,x) = \int\limits_{\varphi^-(x)}^{\varphi^+(x)} K(s,t) dt,$$

$$\widetilde{\mathcal{K}}_2^+ f(s) = \int\limits_{\mu^-(s)}^{\mu^+(s)} \widetilde{K}_2^+(s,x) f(x) dx, \quad \widetilde{K}_2^+(s,x) = \int\limits_{\varphi^-(x)}^{s} K(s,t) dt.$$

In $\widetilde{\mathcal{K}}_1^+ f(s)$, the variables x and t change within the bounds $a \le x \le \mu^-(s)$ and $\varphi^-(x) \le t \le \varphi^+(x)$. Therefore, $a \le \varphi^+(x) \le s$ and $\varphi^-(x) \le t \le \varphi^+(x) \le s$. Then from $K(s,t) \in \mathcal{O}_n^-(\Omega)$, $n \ge 0$, we have that

$$\widetilde{K}_1^+(s,x) \approx \sum_{i=0}^{n} K_i^-(s,\varphi^+(x)) \int\limits_{\varphi^-(x)}^{\varphi^+(x)} K_{i,n}^-(\varphi^+(x),t) dt, \tag{16}$$

where $K_i^-(\cdot,\cdot) \in \mathcal{O}_i^-(\Omega)$, $i = 0,1,...,n$, and $K_n^-(\cdot,\cdot) \equiv K(\cdot,\cdot)$. Hence,

$$\widetilde{\mathcal{K}}_1^+ f(s) \approx \sum_{i=0}^{n} \int\limits_a^{\mu^-(s)} K_i^-(s,\varphi^+(x)) \Phi_i^-(x) f(x) dx, \tag{17}$$

where $\Phi_i^-(x) = \int\limits_{\varphi^-(x)}^{\varphi^+(x)} K_{i,n}^-(\varphi^+(x),t) dt$, $i = 0,1,...,n$.

Denote by $\|\mathcal{K}_{i,\varphi^+}^+\|$ the norm of the operator

$$\mathcal{K}_{i,\varphi^+}^+ f(s) = \int\limits_a^{\mu^-(s)} K_i^-(s,\varphi^+(x)) \Phi_i^-(x) f(x) dx, \quad i = 0,1,...,n,$$

from $L_p(\rho,I)$ to $L_q(\omega,I)$.

In these integrals, we change the variables $\varphi^+(x) = y$ or $x = \mu^-(y)$ and get

$$\mathcal{K}^+_{i,\varphi^+} f(s) = \int_a^s K_i^-(s,y)\Phi_i^-(\mu^-(y))f(\mu^-(y))(\mu^-(y))' dy, \quad i = 0,1,...,n.$$

Therefore,

$$\|\mathcal{K}^+_{i,\varphi^+}\| = \|\mathcal{K}^+_i\|, \quad i = 0,1,...,n, \tag{18}$$

where $\|\mathcal{K}^+_i\|$ is a norm of the operator

$$\mathcal{K}^+_i f(s) = \int_a^s K_i^-(s,y)\Phi_i^-(\mu^-(y))f(y)dy, \quad i = 0,1,...,n,$$

from $L_p(\widetilde{\rho}, I)$ to $L_q(\omega, I)$ and $\widetilde{\rho}(y) = \rho(\mu^-(y))[(\mu^-(y))']^{-\frac{1}{p'}}$.

Then from (17) and (18), it follows that

$$\|\widetilde{\mathcal{K}}^+_1\| \approx \sum_{i=0}^n \|\mathcal{K}^+_i\|. \tag{19}$$

Since $K_i^-(\cdot,\cdot) \in \mathcal{O}_i^-(\Omega)$, $0 \le i \le n$, then on the basis of Theorem B^+ we have

$$\|\mathcal{K}^+_i\| \approx \sup_{z\in I}\left(\int_z^b \omega^q(x)\left(\int_a^z (K_i^-(x,y))^{p'}(\Phi_i^-(\mu^-(y)))^{p'}\widetilde{\rho}^{-p'}(y)dy\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}}$$

(we change the variables in the inside integral $t = \mu^-(y)$)

$$= \sup_{z\in I}\left(\int_z^b \omega^q(x)\left(\int_a^{\mu^-(z)} (K_i^-(x,\varphi^+(t)))^{p'}(\Phi_i^-(t))^{p'}\rho^{-p'}(t)dt\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}}$$

$$= \sup_{z\in I}\left(\int_z^b \omega^q(x)\left(\int_a^{\mu^-(z)} \left(K_i^-(x,\varphi^+(t)) \times \int_{\varphi^-(t)}^{\varphi^+(t)} K^-_{i,n}(\varphi^+(t),s)ds\right)^{p'} \rho^{-p'}(t)dt\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}}, \tag{20}$$

and similarly,

$$\|\mathcal{K}^+_i\| \approx \sup_{z\in I}\left(\int_a^{\mu^-(z)} \rho^{-p'}(t)\left(\int_{\varphi^-(t)}^{\varphi^+(t)} K^-_{i,n}(\varphi^+(t),s)ds\right)^{p'} \times \left(\int_z^b \left(K_i^-(x,\varphi^+(t))\right)^q \omega^q(x)dx\right)^{\frac{p'}{q}} dt\right)^{\frac{1}{p'}}. \tag{21}$$

From (20), (19), (18), (16) and (9), we have

$$\|\widetilde{\mathcal{K}}_1^+\| \approx \sup_{z\in I}\left(\int_z^b \omega^q(x)\left(\int_a^{\mu^-(z)}\left(\int_{\varphi^-(t)}^{\varphi^+(t)}\sum_{i=0}^n K_i^-(x,\varphi^+(t))K_{i,n}^-(\varphi^+(t),s)ds\right)^{p'} \times \rho^{-p'}(t)dt\right)^{\frac{q}{p'}}dx\right)^{\frac{1}{q}} \approx F_2^+.$$

Similarly, from (21), (19), (18), (16) and (9), it follows that $\|\widetilde{\mathcal{K}}_1^+\| \approx F_1^+$, i.e.,

$$\|\widetilde{\mathcal{K}}_1^+\| \approx F_1^+ \approx F_2^+. \tag{22}$$

Now, we estimate the value $\|\widetilde{\mathcal{K}}_2^+\|$. Consider the kernel of the operator $\widetilde{\mathcal{K}}_2^+$:

$$\widetilde{K}_2^+(s,x) \equiv K_{n+1}^-(s,x) = \int_{\varphi^-(x)}^s K(s,t)dt.$$

In the operator $\widetilde{\mathcal{K}}_2^+$, the variables x, t and s change within the bounds $a < s < b$, $a < \varphi^-(x) \le t \le s < b$ and $\mu^-(s) \le x \le \mu^+(x)$. Let $a < x \le y < b$ and $\varphi^-(y) \le s \le \varphi^+(x)$. Then

$$K_{n+1}^-(s,x) = \int_{\varphi^-(x)}^{\varphi^-(y)} K(s,t)dt + \int_{\varphi^-(y)}^s K(s,t)dt = \int_{\varphi^-(x)}^{\varphi^-(y)} K(s,t)dt + K_{n+1}^-(s,y). \tag{23}$$

From the conditions $K(\cdot,\cdot) \in \mathcal{O}_n^-(\Omega)$, $n \ge 0$, and $\varphi^-(x) \le t \le \varphi^-(y) \le s \le \varphi^+(x)$, we have

$$\int_{\varphi^-(x)}^{\varphi^-(y)} K(s,t)dt \approx \sum_{i=0}^n K_i^-(s,\varphi^-(y))\int_{\varphi^-(x)}^{\varphi^-(y)} K_{i,n}^-(\varphi^-(y),t)dt = \sum_{i=0}^n K_i^-(s,\varphi^-(y))K_{i,n+1}^-(y,x), \tag{24}$$

where $K_{i,n+1}^-(y,x) = \int\limits_{\varphi^-(x)}^{\varphi^-(y)} K_{i,n}^-(\varphi^-(y),t)dt$, $i = 0,1,...,n$, $K_i^-(\cdot,\cdot) \in \mathcal{O}_i^-(\Omega)$, $i = 0,1,...,n$, and $K_n^-(s,\varphi^-(y)) = K(s,\varphi^-(y))$. From (23) and (24), we get $\widetilde{K}_2^+(s,x) \equiv K_{n+1}^-(s,x) \approx K_{n+1}^-(s,y) + \sum\limits_{i=0}^n \widehat{K}_i^-(s,y)K_{i,n+1}^-(y,x)$ for $a < x \le y < b$ and $\varphi^-(y) \le s \le \varphi^+(x)$, where $\widehat{K}_i^-(s,y) \equiv K_i^-(s,\varphi^-(y))$, $i = 0,1,...,n$, and $K_{n+1,n+1}^-(\cdot,\cdot) \equiv 1$. Due to Remark 2.3, the functions $\widehat{K}_i^-(\cdot,\cdot)$ belong to the class $\mathcal{O}_i^-(\Omega^-)$, $i = 0,1,...,n$. Then by the definition of the class $\mathcal{O}_{n+1}^-(\Omega^-)$, the kernel $\widetilde{K}_2^+(s,x)$ of the operator $\widetilde{\mathcal{K}}_2^+$ belongs to the class $\mathcal{O}_{n+1}^-(\Omega^-)$ and by Lemma 2.1, the functions μ^- and μ^+ are locally absolute continuous on I. Therefore, by Theorem C^+ for $\alpha = \mu^-$ and $\beta = \mu^+$, we have

$$\|\widetilde{\mathcal{K}}_2^+\| \approx G_1^+ \approx G_2^+. \tag{25}$$

From (25), (22), (14) and (15), it follows that the inequality (12) for $T = \mathcal{K}^+$ holds if and only if $\max\{F_i^+, G_j^+\} < \infty$ at least for one of the pairs (i,j), $i,j = 1,2$, and $C_1 \approx \max\{F_i^+, G_j^+\}$, $i,j = 1,2$, where C_1 is the best constant in (12). Hence, by Lemma 3.3, it follows the validity of Theorem 3.1. The proof of Theorem 3.1 is complete. □

Proof [Proof of Theorem 3.2] Let $T = \mathcal{K}^-$. Then by Lemma 3.3 the inequality (3) holds if and only if the inequality (12) holds, i.e. the operator

$$\mathcal{K}^- \left(\int\limits_{\mu^-(\cdot)}^{\mu^+(\cdot)} f(x)dx \right)(s)$$

is bounded from $L_p(v, I)$ to $L_q(\omega, I)$. Since

$$\widetilde{\mathcal{K}}^- f(s) \equiv \mathcal{K}^- \left(\int\limits_{\mu^-(\cdot)}^{\mu^+(\cdot)} f(x)dx \right)(s) = \int\limits_s^b K(t,s) \int\limits_{\mu^-(t)}^{\mu^+(t)} f(x)dxdt, \tag{26}$$

the change of order of integration gives

$$\int\limits_s^b K(t,s) \int\limits_{\mu^-(t)}^{\mu^+(t)} f(x)dxdt = \int\limits_{\mu^+(s)}^b f(x) \int\limits_{\varphi^-(x)}^{\varphi^+(x)} K(t,s)dtdx + \int\limits_{\mu^-(s)}^{\mu^+(s)} f(x) \int\limits_s^{\varphi^+(x)} K(t,s)dtdx. \tag{27}$$

From (12), (26), and (27), as in the proof of theorem 3.1, it follows that the operator $\widetilde{\mathcal{K}}^- f(s)$ is bounded from $L_p(v, I)$ to $L_q(\omega, I)$ if and only if the operators $\widetilde{\mathcal{K}}_1^- f(s) \equiv \int\limits_{\mu^+(s)}^b f(x) \int\limits_{\varphi^-(x)}^{\varphi^+(x)} K(t,s)dtdx$ and $\widetilde{\mathcal{K}}_2^- f(s) \equiv \int\limits_{\mu^-(s)}^{\mu^+(s)} f(x) \int\limits_s^{\varphi^+(x)} K(t,s)dtdx$ are bounded from $L_p(v, I)$ to $L_q(\omega, I)$, and between the norms of the operators $\|\widetilde{\mathcal{K}}^-\|$, $\|\widetilde{\mathcal{K}}_1^-\|$, and $\|\widetilde{\mathcal{K}}_2^-\|$ from $L_p(v, I)$ to $L_q(\omega, I)$ the relation

$$\|\widetilde{\mathcal{K}}^-\| \approx \|\widetilde{\mathcal{K}}_1^-\| + \|\widetilde{\mathcal{K}}_2^-\| \tag{28}$$

is valid. We estimate the values $\|\widetilde{\mathcal{K}}_1^-\|$ and $\|\widetilde{\mathcal{K}}_2^-\|$ and get

$$\widetilde{\mathcal{K}}_1^- f(s) = \int\limits_{\mu^+(s)}^b \widetilde{K}_1^-(x,s) f(x)dx, \quad \widetilde{K}_1^-(x,s) = \int\limits_{\varphi^-(x)}^{\varphi^+(x)} K(t,s)dt,$$

$$\widetilde{\mathcal{K}}_2^- f(s) = \int\limits_{\mu^-(s)}^{\mu^+(s)} \widetilde{K}_2^-(x,s) f(x)dx, \quad \widetilde{K}_2^-(x,s) = \int\limits_s^{\varphi^+(x)} K(t,s)dt.$$

In the operator $\widetilde{\mathcal{K}}_1^-$, the variables x, s, and t change within the bounds $a < x < b$ and $a < s \leq \varphi^-(x) \leq t \leq \varphi^+(x) < b$. Hence, by taking into account that $K(\cdot,\cdot) \in \mathcal{O}_n^+(\Omega)$, we have

$$\widetilde{K}_1^-(x,s) \approx \sum_{i=0}^n \int\limits_{\varphi^-(x)}^{\varphi^+(x)} K_{n,i}^+(t, \varphi^-(x))dt K_i^+(\varphi^-(x), s) = \sum_{i=0}^n \Phi_i^+(x) K_i^+(\varphi^-(x), s), \tag{29}$$

where $\Phi_i^+(x) = \int\limits_{\varphi^-(x)}^{\varphi^+(x)} K_{n,i}^+(t,\varphi^-(x))dt$, $i = 0,1,...,n$.

Then

$$\widetilde{\mathcal{K}}_1^- f(s) \approx \sum_{i=0}^n \mathcal{K}_{i,\varphi^-}^- f(s), \quad \mathcal{K}_{i,\varphi^-}^- f(s) = \int\limits_{\mu^+(s)}^b \Phi_i^+(x) K_i^+(\varphi^-(x),s) f(x)dx,$$

$i = 0,1,...,n$. This, together with the change of variables $\varphi^-(x) = y$, gives

$$\|\widetilde{\mathcal{K}}_1^-\| \approx \sum_{i=0}^n \|\mathcal{K}_i^-\|, \tag{30}$$

where $\|\mathcal{K}_i^-\|$ is a norm of the operator

$$\mathcal{K}_i^- f(s) = \int\limits_s^b \Phi_i^+(\mu^+(y)) K_i^+(y,s) f(y)dy, \quad s \in I,$$

from $L_p(\widehat{\rho}, I)$ to $L_q(\omega, I)$ and $\widehat{\rho}(y) = \rho(\mu^+(y))[(\mu^+(y))']^{-\frac{1}{p'}}$. Since $K_i^+(\cdot,\cdot) \in \mathcal{O}_i^+(\Omega)$, $i = 0,1,...,n$, by Theorem B^- we have

$$\|\mathcal{K}_i^-\| \approx \sup_{z\in I}\left(\int\limits_a^z \omega^q(s)\left(\int\limits_z^b (K_i^+(y,s)\Phi_i^+(\mu^+(y)))^{p'}\widehat{\rho}^{-p'}(y)dy\right)^{\frac{q}{p'}} ds\right)^{\frac{1}{q}}$$

$$= \sup_{z\in I}\left(\int\limits_a^z \omega^q(s)\left(\int\limits_{\mu^+(z)}^b \left(K_i^+(\varphi^-(x),s)\Phi_i^+(x)\right)^{p'} \rho^{-p'}(x)dx\right)^{\frac{q}{p'}} ds\right)^{\frac{1}{q}}$$

$$= \sup_{z\in I}\left(\int\limits_a^z \omega^q(s)\left(\int\limits_{\mu^+(z)}^b \left[K_i^+(\varphi^-(x),s) \times \int\limits_{\varphi^-(x)}^{\varphi^+(x)} K_{n,i}^+(t,\varphi^-(x))dt\right]^{p'} \rho^{-p'}(x)dx\right)^{\frac{q}{p'}} ds\right)^{\frac{1}{q}}, \tag{31}$$

$i = 0,1,...,n$, and similarly,

$$\|\mathcal{K}_i^-\| \approx \sup_{z\in I}\left(\int\limits_{\mu^+(z)}^b \left[\int\limits_{\varphi^-(x)}^{\varphi^+(x)} K_{n,i}^+(t,\varphi^-(x))dt\right]^{p'} \rho^{-p'}(x) \times \left(\int\limits_a^z \left[K_i^+(\varphi^-(x),s)\right]^q \omega^q(s)ds\right)^{\frac{p'}{q}} dx\right)^{\frac{1}{p'}}, \tag{32}$$

$i = 0,1,...,n$. From (32), (31), and (30), by taking into account (29), we have

$$\|\widetilde{\mathcal{K}}_1^-\| \approx F_1^- \approx F_2^-. \tag{33}$$

Now, we consider the operator $\widetilde{\mathcal{K}}_2^-$ and its kernel. Let $a < \tau \leq x < b$ and $\varphi^-(x) \leq s \leq \varphi^+(\tau)$. Then

$$\widetilde{K}_2^-(x,s) \equiv K_{n+1}^+(x,s) = \int\limits_s^{\varphi^+(x)} K(t,s)dt = \int\limits_{\varphi^+(\tau)}^{\varphi^+(x)} K(t,s)dt + \int\limits_s^{\varphi^+(\tau)} K(t,s)dt$$

$$= \int\limits_{\varphi^+(\tau)}^{\varphi^+(x)} K(t,s)dt + K_{n+1}^+(\tau,s). \tag{34}$$

In the expression $\int\limits_{\varphi^+(\tau)}^{\varphi^+(x)} K(t,s)dt$, the variables t and s change within the bounds $\varphi^+(x) \geq t \geq \varphi^+(\tau) \geq s \geq \varphi^-(x)$. Therefore, by taking into account $K(\cdot,\cdot) \in \mathcal{O}_n^+(\Omega)$, $n \geq 0$, we have

$$\int\limits_{\varphi^+(\tau)}^{\varphi^+(x)} K(t,s)ds \approx \sum_{i=0}^{n} \int\limits_{\varphi^+(\tau)}^{\varphi^+(x)} K_{n,i}^+(t,\varphi^+(\tau))dt K_i^+(\varphi^+(\tau),s)$$

$$= \sum_{i=0}^{n} K_{n+1,i}^+(x,\tau) K_i^+(\varphi^+(\tau),s), \tag{35}$$

where $K_{n+1,i}^+(x,\tau) = \int\limits_{\varphi^+(\tau)}^{\varphi^+(x)} K_{n,i}(t,\varphi^+(\tau))dt$, $K_i^+(\cdot,\cdot) \in \mathcal{O}_i^+(\Omega)$, $i = 0,1,...,n$, and $K_n^+(\cdot,\cdot) \equiv K(\cdot,\cdot)$. From (34) and (35), we obtain

$$\begin{gathered} \widetilde{K}_2^+(x,s) \equiv K_{n+1}^+(x,s) \approx \sum_{i=0}^{n+1} K_{n+1,i}^+(x,\tau) K_i^+(\varphi^+(\tau),s), \\ K_{n+1,n+1}^+(\cdot,\cdot) \equiv 1 \end{gathered} \tag{36}$$

for $a < \tau \leq x < b$ and $\varphi^-(x) \leq s \leq \varphi^+(\tau)$. Since due to Remark 2.3, we have that $K_i^+(\varphi^+(\tau),s) \in \mathcal{O}_i^+(\Omega^+)$, $i = 0,1,...,n$, from (36) and the definition of the class $\mathcal{O}_{n+1}^+(\Omega^+)$ it follows $\widetilde{K}_2^+(\cdot,\cdot) \equiv K_{n+1}^+(\cdot,\cdot) \in \mathcal{O}_{n+1}^+(\Omega^+)$. Then by Theorem C^-, we have

$$\|\widetilde{\mathcal{K}}_2^-\| \approx G_1^- \approx G_2^-. \tag{37}$$

From (37), (33) and (28) we get the validity of theorem 3.2.

The proof of theorem 3.2 is complete. □

Acknowledgments

The paper was written under financial support by the Scientific Committee of the Ministry of Education and Science of the Republic of Kazakhstan, Grant No. AP05130975/GF5, in priority area "Intellectual potential of the country".

References

[1] Eveson SP, Stepanov VD, Ushakova EP. A duality principle in weighted Sobolev spaces on the real line. Math Nachr 2015; 288: 877-897 doi: 10.1002/mana.201400019.

[2] Kufner A, Maligranda L, Persson LE. The Hardy Inequality. About its History and Some Related Results. Pilsen, Czech Republic: Vydavatelsky Servis, 2007.

[3] Kufner A, Persson LE. Weighted Inequalities of Hardy type. London, UK: World Scientific, 2003.

[4] Oinarov R. Boundedness and compactness of Volterra type integral operators. Siberian Math J 2007; 48: 884-896.

[5] Oinarov R. Boundedness of integral operators from weighted Sobolev space to weighted Lebesgue space. Complex Var Elliptic 2011; 56: 1021-1038.

[6] Oinarov R. Boundedness of integral operators in weighted Sobolev spaces. Izv Math 2014; 78: 836-853.

[7] Oinarov R. On weighted norm inequalities with three weights. J London Math Soc 1993; 48: 103-116.

[8] Oinarov R. Boundedness and compactness in weighted Lebesgue spaces of integral operators with variable integration limits. Siberian Math J 2011; 52: 1042-1055.

[9] Prokhorov DV, Stepanov VD, Ushakova EP. Hardy-Steklov integral operators. Part I. Proc Steklov Inst Math 2018; 300: 1-111.

[10] Stepanov VD, Ushakova EP. Kernel operators with variable intervals of integration in Lebesgue spaces and applications. Math Inequal Appl 2010; 13: 449-510.