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Maximal Regularity Estimates and the Solvability of Nonlinear Differential Equations

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Abstract: We study a type of third-order linear differential equations with variable and unbounded coefficients, which are defined in an infinite interval. We also consider a non-linear generalization with coefficients that depends on an unknown function. We establish sufficient conditions for the correctness of this linear equation and the maximal regularity estimate for their solution. Using these results, we prove the solvability of a nonlinear differential equation and estimate the norms of its terms.

Keywords: third-order differential equation; unbounded coefficient; correctness; maximal regularity estimate; qualitative property of solution

MSC: 34A30; 34A34; 34C11

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1. Introduction

We consider the following differential equations:

$$Ly = -(r(x)y')'' + p(x)y' + q(x)y = f(x) \quad (1)$$

and

$$-(r(x)y')'' + p(x, y)y' + q(x, y)y = F(x), \quad (2)$$

where $x \in \mathbb{R} = (-\infty, +\infty)$, $r(x)$, p and q are given smooth functions, and $f(x),\ F(x) \in L^2(\mathbb{R})$.

Equations (1) and (2) are singular differential equations, i.e., these equations are given in an infinite interval and in general the coefficients r, p, and q are unbounded functions. The assumption that the coefficients in (1) and (2) are not bounded is due to the essence of the matter. If in (1) $q = 0$, then at least one of the coefficients r and p must increase near infinity. Otherwise, (1) may not have a solution in $L^2(\mathbb{R})$.

Third-order differential equations appear in many practical problems, for example, finding the deflection of a three-layer beam, describing small charge fluctuations taking into account the braking force of radiation, automatic control by spacecraft controls, a steam turbine, etc. (see [1,2] and the references cited therein). In [1–5], the authors in particular have obtained asymptotic estimates for solutions of third-order linear and nonlinear differential equations defined in the interval $[0, +\infty)$. In these studies, it was assumed that either the coefficients of a linear equation are bounded functions or the nonlinear terms of the equation are controlled by its linear part. In this paper, we consider the case that the functions r, p, and q can grow strongly near infinity.

We denote by $C_0^{(3)}(\mathbb{R})$ the set of all three times continuously differentiable and compactly supported functions.

Definition 1. *A function* $y \in L^2(\mathbb{R})$ *is called a solution to Equation (1) if there is a sequence* $\{y_n\}_{n=1}^{\infty}$ *from* $C_0^{(3)}(\mathbb{R})$ *such that* $\|y_n - y\|_2 \to 0$ *,* $\|Ly_n - f\|_2 \to 0$ *(* $n \to \infty$*), where* $\|\cdot\|_2$ *is the norm of* $L^2(\mathbb{R})$*.*

We will study the following questions: the solvability of Equation (1), the uniqueness of the solution, and the following estimate for the solution

$$\left\|(r(x)y')''\right\|_2 + \left\|p(x)y'\right\|_2 + \|[1 + |q(x)|]y\|_2 \leq C\|f\|_2, \quad (3)$$

where C is a positive constant independent of y. If (3) holds, the solution to Equation (1) is called L^2-maximally regular (see [6,7]), the linear operator L is called separable in $L^2(\mathbb{R})$ (see [8,9]), and inequality (3) is called a coercive estimate.

Inequality (3) is very important in the theory of differential equations. For example, this inequality carries exact information about the smoothness of solution (1) and gives a complete description of the differential operator L corresponding to Equation (1). If inequality (3) holds, then we can use the methods of function spaces to study properties of the solution to Equation (1). For example, using inequality (3), one can find approximate properties of a solution to Equation (1) (see [10–12]). In this paper, we find sufficient conditions for inequality (3) and use them to study solvability of the singular nonlinear differential Equation (2) (see also [13]).

On the real axis R when $p = 0$ and $r = 1$, the correctness and regularity properties for Equation (1) were considered in [14]. The maximal regularity of solutions of (1) was proved in the case that q is the positive continuous function such that

$$\sup_{|x-z|\leq 1} \frac{|q(x) - q(z)|}{|x - z|^{\alpha} q^{\theta}(z)} \leq M < \infty, \quad (4)$$

where $0 < \theta < 1 + \frac{\alpha}{3}$ and $\alpha \in (0, 1]$. This result was essentially used to prove the solvability of nonlinear Equation (2) with $p = 0$ and $r = 1$. In [15], the maximal regularity estimate was obtained for (1) in the case that $p = 0$, q is a positive function satisfying (4), and r is a sign-definite smooth function. In the case that $r = 1$, $p \neq 0$ and grows rapidly, and q can change sign and does not satisfy condition (4), the maximal regularity estimate for the solution y of (1) was established (see [16]). In [17], authors using the results of [16] showed the solvability of quasilinear Equation (2) in the case $r = 1$.

In [14,15], the authors considered equations without an intermediate term, but their methods are not applicable to our case. In this paper, the coefficients of Equations (1) and (2) are assumed to be smooth, but there are not any restrictions on their derivatives. Our conditions are formulated in terms of the coefficients themselves.

Equations (1) and (2) with an unbounded coefficient r have many applications. For example, they apply to the interaction of elastic bodies, the phenomena of the input flow in hydrodynamics, and the propagation of electrical impulses in a living organism (the Hodgkin–Huxley and Nagumo models), etc. [1,2].

For a study of the maximal regularity of differential equations of other types, see [10–12].

The paper is organized as follows. In Section 2, we prove the auxiliary statement (Lemma 3) on the continuously invertibility of the linear operator generated by the two highest terms of Equation (1). Section 3 discusses the well-posedness property of linear Equation (1) depending on the relationships between its coefficients. In Section 4, we give the conditions for the maximal regularity estimate (3) for the solution. In Section 5, essentially using inequality (3), we find conditions for the existence of a solution y to nonlinear Equation (2) and prove the following relation:

$$\left\|(r(x)y')''\right\|_2 + \left\|p(x,y)y'\right\|_2 + \|[1 + q(x,y)]y\|_2 < \infty.$$

In Sections 2–4 we will assume that $r(x)$ is twice continuously differentiable, $p(x)$ is continuously differentiable, and $q(x)$ is a continuous function; C , C_1, C_2, etc., everywhere denote positive constants, which may be different in different places.

2. On One Two-Term Linear Operator

Let $l_0 y = -(r(x)y')'' + p(x)y'$ be the linear operator with $D(l_0) = C_0^{(3)}(\mathbb{R})$. Since the functions $r(x)$ and $p(x)$ are smooth, l_0 is closable in norm in $L^2(\mathbb{R})$. We denote its closure by l . Let g and $h \neq 0$ be continuous functions. We introduce the following notation:

$$\alpha_{g,h}(x) = \left(\int_0^x g^2(t)dt \right)^{1/2} \left(\int_x^{+\infty} h^{-2}(t)dt \right)^{1/2}, \quad x > 0,$$

$$\beta_{g,h}(\tau) = \left(\int_\tau^0 g^2(t)dt \right)^{1/2} \left(\int_{-\infty}^{\tau} h^{-2}(t)dt \right)^{1/2}, \quad \tau < 0,$$

$$\gamma_{g,h} = \max\left(\sup_{\{x>0\}} \alpha_{g,h}(x), \sup_{\{\tau<0\}} \beta_{g,h}(\tau) \right).$$

The following result proved in [13].

Lemma 1. *Let the functions g and h satisfy the condition $\gamma_{g,h} < \infty$. Then for $y \in C_0^{(1)}(\mathbb{R})$ the following inequality holds:*

$$\left(\int_{-\infty}^{+\infty} |g(x)y(x)|^2 dx \right)^{1/2} \leq C \left(\int_{-\infty}^{+\infty} |h(x)y'(x)|^2 dx \right)^{1/2}. \tag{5}$$

Moreover, if C is the smallest positive constant in (5), then $\gamma_{g,h} \leq C \leq 2\gamma_{g,h}$.

The next result is known. For easy reference, we give its proof.

Lemma 2. *Let X be a Banach space, Y be a normed linear space, and $A : X \to Y$ be closed linear operator such that*

$$\|y\|_X \leq C\|Ay\|_Y, \tag{6}$$

for each $y \in D(A)$, where C_1 is a positive constant independent of y. Then the set $R(A)$ is closed.

Proof of Lemma 2. Let $\{w_n\}_{n=1}^{\infty} \subseteq R(A)$ and $\|w_n - w\|_Y \to 0$ $(n \to \infty)$. Since $w_n \in R(A)$, there exits $z_n \in D(A)$ such that $w_n = Az_n$ $(n = 1, 2, \ldots)$. By (6),

$$\|z_n - z_m\|_X \leq C_1\|Az_n - Az_m\|_Y \to 0(n, m \to \infty),$$

$\{z_n\}_{n=1}^{\infty}$ is the Cauchy sequence, it converges to some element $z \in X$; A is closed, therefore $z \in D(A)$ and $w = Az$. Thus $w \in R(A)$. □

Lemma 3. *If the conditions*

$$p(x) \geq C_2 r(x) \geq \delta > 0 \tag{7}$$

and

$$\gamma_{1,\sqrt{pr}} < \infty \tag{8}$$

are fulfilled, then the operator l is continuously invertible. Moreover, for each $y \in D(l)$ the following estimate holds:

$$\left\|(ry')'\right\|_2 + \|\sqrt{pr}y'\|_2 + \|y\|_2 \le C_3\|ly\|_2. \tag{9}$$

Proof of Lemma 3. Let $y \in C_0^{(3)}(\mathbb{R})$. We consider the scalar product $(l_0 y, ry')$ in $L^2(\mathbb{R})$. Integrating by parts, we obtain that

$$(l_0 y, ry') = \left\|(ry')'\right\|_2^2 + \|\sqrt{pr}y'\|_2^2. \tag{10}$$

By Hölder's inequality,

$$(l_0 y, ry') \le \frac{1}{2}\left\|\sqrt{\frac{r}{p}}l_0 y\right\|_2^2 + \frac{1}{2}\|\sqrt{pr}y'\|_2^2. \tag{11}$$

From (10) and (11) we get that

$$\left\|(ry')'\right\|_2^2 + \frac{1}{2}\|\sqrt{pr}y'\|_2^2 \le \frac{1}{2}\left\|\sqrt{\frac{r}{p}}l_0 y\right\|_2^2. \tag{12}$$

If we choose that $g(x) = 1$ and $h(x) = \sqrt{pr}$ in (5), then by condition (8),

$$\|y\|_2 \le 2\gamma_{1,\sqrt{pr}}\|\sqrt{pr}y'\|_2.$$

Taking into account (12), we obtain that

$$\left\|(ry')'\right\|_2^2 + \frac{1}{2}\|\sqrt{pr}y'\|_2^2 + \|y\|_2^2 \le \left(\frac{1}{2} + 4\gamma_{1,\sqrt{pr}}^2\right)\left\|\sqrt{\frac{r}{p}}l_0 y\right\|_2^2.$$

Thus, for $y \in C_0^{(3)}(\mathbb{R})$ we have the estimate (9). Because l is a closed operator, the inequality (9) holds for each $y \in D(l)$.

According to (9), there is an inverse l^{-1} to the operator l, and by Lemma 2, the range $R(l)$ of l is closed. To prove the lemma, it suffices to show that the equality $R(l) = L^2(\mathbb{R})$ holds. We denote $y' = z$ and $Az = -(rz)'' + pz$. Note that by inequality (9) and condition (8), $D(A) \subseteq L^2(\mathbb{R})$, A is a closed operator, and $R(A) = R(l)$ is a closed set. If $R(A) \ne L^2(\mathbb{R})$, then there is a nonzero element $z(x) \in L^2(\mathbb{R})$ such that $z \perp R(A)$. According to the Banach theory of linear operators with a closed range ([18], Chapter 7), z is a generalized solution of the equation

$$-rz'' + pz = 0. \tag{13}$$

By condition (7), the solution to Equation (13) from $L^2(\mathbb{R})$ is only $z = 0$. This is a contradiction, hence $R(l) = L^2(\mathbb{R})$. □

3. Solvability Conditions for the Linear Equation

Theorem 1. *If (7) and*

$$\gamma_{1+|q|,\sqrt{pr}} < \infty \tag{14}$$

are satisfied, then for each $f(x) \in L^2(\mathbb{R})$ there exists a unique solution $y \in L^2(\mathbb{R})$ of Equation (1) and the following inequality holds:

$$\left\|(ry')'\right\|_2 + \frac{1}{2}\|\sqrt{pr}y'\|_2 + \|(1+|q|)y\|_2 \le C\|f\|_2. \tag{15}$$

Proof of Theorem 1. Let $x = bt$, where $b > 0$ and $t \in \mathbb{R}$ is a new variable. Set

$$y(bt) = \tilde{y}(t), r(bt) = \tilde{r}(t), p(bt) = \tilde{p}(t), q(bt) = \tilde{q}(t), b^3 f(bt) = \tilde{f}(t).$$

Substituting this into the Equation (1), we obtain

$$-\left(\tilde{r}(t)\tilde{y}'\right)'' + b^2\tilde{p}(t)\tilde{y}' + b^3\tilde{q}(t)\tilde{y} = \tilde{f}(t). \tag{16}$$

We consider the operator

$$l_{0b}\tilde{y} = -\left(\tilde{r}(t)\tilde{y}'\right)'' + b^2\tilde{p}(t)\tilde{y}' \quad (D(l_{0b}) = C_0^{(3)}(\mathbb{R}))$$

and denote its closure in $L^2(\mathbb{R})$ by l_b. Because the functions $\tilde{r}(t)$ and $b^2\tilde{p}(t)$ satisfy the conditions of Lemma 3, the operator l_b is continuously invertible and

$$\left\|(\tilde{r}\tilde{y}')'\right\|_2 + \left\|b\sqrt{\tilde{p}\tilde{r}}\tilde{y}'\right\|_2 + \|\tilde{y}\|_2 \leq C_1\|l_b\tilde{y}\|_2, \quad \tilde{y} \in D(l_b). \tag{17}$$

It is easy to check that

$$2\gamma_{\tilde{q},\sqrt{\tilde{p}\tilde{r}}} = \frac{1}{b^2}2\gamma_{b^3\tilde{q},b\sqrt{\tilde{p}\tilde{r}}}.$$

By Lemma 1 and (14) we have

$$\left\|b^3\tilde{q}\tilde{y}\right\|_2 \leq 2b^2\gamma_{\tilde{q},\sqrt{\tilde{p}\tilde{r}}}\left\|b\sqrt{\tilde{p}\tilde{r}}\tilde{y}'\right\|_2.$$

If we choose

$$b \leq \frac{1}{2\sqrt{C_1\gamma_{\tilde{q},\sqrt{\tilde{p}\tilde{r}}}}},$$

then by virtue of (17),

$$\left\|b^3\tilde{q}\tilde{y}\right\|_2 \leq \frac{1}{2}\|l_b\tilde{y}\|_2. \tag{18}$$

Here C_1 is a constant from (17). Therefore, there exists a unique solution $\tilde{y}$ to Equation (16) ([19], Chapter 4, Theorem 1.16). From (17) and (18) we deduce that

$$\left\|(\tilde{r}\tilde{y}')'\right\|_2 + \left\|b\sqrt{\tilde{p}\tilde{r}}\tilde{y}'\right\|_2 + \left\|(1+b^3|\tilde{q}|)\tilde{y}\right\|_2 \leq C_2\|l_b\tilde{y}\|_2, \quad \tilde{y} \in D(l_b). \tag{19}$$

By (18), we get that

$$\|l_b\tilde{y}\|_2 \leq \left\|\left(l_b + b^3\tilde{q}E\right)\tilde{y}\right\|_2 + \left\|b^3\tilde{q}\tilde{y}\right\|_2 \leq \left\|\left(l_b + b^3\tilde{q}E\right)\tilde{y}\right\|_2 + \frac{1}{2}\|l_b\tilde{y}\|_2$$

and

$$\|l_b\tilde{y}\|_2 \leq 2\left\|\left(l_b + b^3\tilde{q}E\right)\tilde{y}\right\|_2. \tag{20}$$

The estimates (19) and (20) imply that

$$\left\|(\tilde{r}\tilde{y}')'\right\|_2 + \left\|b\sqrt{\tilde{p}\tilde{r}}\tilde{y}'\right\|_2 + \left\|\left(1+b^3|\tilde{q}|\right)\tilde{y}\right\|_2 \leq C_3\|\tilde{f}\|_2, \quad \tilde{y} \in D(l_b).$$

Returning to the variable x, we obtain estimate (15) for the solution y of Equation (1). □

4. Conditions for the L_2—Maximal Regularity of Solution

We need the following corollary of Theorem 1.

Corollary 1. *Let the functions $r(x)$, $p(x)$, and $q(x)$ satisfy the conditions*

$$0 < \delta \leq C_4 r(x) \leq p(x) \leq C_5 r(x) \tag{21}$$

and

$$\gamma_{1+|q|,\, p} < \infty. \tag{22}$$

Then for each $f(x) \in L^2(\mathbb{R})$, the Equation (1) has a unique solution y, and it is L^2—maximally regular.

Proof of Corollary 1. Under assumption (21), conditions (22) and (14) are equivalent. Therefore, if conditions (21) and (22) are satisfied, then by (1) and (15), we obtain the inequality

$$\left\| (ry')'' \right\|_2 \le C_6 \|f\|_2.$$

According to (15), the solution y satisfies inequality (3). □

Lemma 4. *Suppose that for $r(x)$ and $p(x)$ conditions of Lemma 3 are satisfied, and one of the following conditions (a) and (b) is performed:*

(a) $p(x) \le C_1 r(x)$,

(b)

$$C^{-1} \le \frac{p(x)}{p(\eta)} \le C \ \ (C > 1), \quad \forall x, \eta \in R : |x - \eta| \le 1. \tag{23}$$

Then, for $y \in D(l)$ the following inequality holds:

$$\left\| (ry')'' \right\|_2 + \|py'\|_2 + \|y\|_2 \le C_2 \|ly\|_2. \tag{24}$$

Proof of Lemma 4. If (a) holds, then the conditions of Corollary 1 are satisfied, so the desired result holds. Now assume (b) holds. We denote $v = ry'$, $p_1(x) = \frac{p(x)}{r(x)}$, then ly changes to

$$Tv = -v'' + p_1(x)v, D(T) \subseteq L^2(\mathbb{R})$$

where T is the Sturm-Liouville operator. Using the results of [9] under condition (23), we obtain that for $v \in D(T)$ the following estimate holds:

$$\left\| v'' \right\|_2 + \|p_1 v\|_2 \le C_3 \|Tv\|_2.$$

Hence, taking into account (7) and (8), we obtain (24) for $y \in D(l)$. □

Theorem 2. *Let $r(x)$ and $p(x)$ satisfy the conditions of Lemma 3, and $q(x)$ satisfy (14). Then Equation (1) has a unique solution $y \in L^2(\mathbb{R})$, and for y the following L^2-maximal regularity inequality holds*

$$\left\| (ry')'' \right\|_2 + \|py'\|_2 + \|(1 + |q|)y\|_2 \le C_4 \|f\|_2. \tag{25}$$

Proof of Theorem 2. Condition (22) implies (14). By Theorem 1, Equation (1) is uniquely solvable in sense of Definition 1. Using the method in the proof of Theorem 1, by inequality (24) and Lemma 1, we obtain estimate (25). □

Remark 1. *In Theorem 2 and Corollary 1, some sufficient conditions for the L^2—maximal regularity of the solution y to Equation (1) are obtained. It should be noted that in Corollary 1 such a result was obtained without condition (23) on the oscillation of function $p(x)$. This condition is instead by (21).*

5. The Solvability of the Nonlinear Equation

Now we give one application of Theorem 2 to the coercive solvability of the following third-order nonlinear differential equation:

$$By = -(r(x)y')'' + p(x, y)y' + q(x, y)y = F(x), \tag{26}$$

where $x \in \mathbb{R}$, $r(x)$ is twice continuously differentiable, $p(x, y)$ is continuously differentiable, $q(x, y)$ is a continuous function, and $F(x) \in L^2(\mathbb{R})$. The main feature of Equation (26) is that coefficients p and q can be unbounded.

Definition 2. *A function y is called a solution to Equation (26) if there exists a sequence $\{y_n(x)\}_{n=1}^{\infty}$ of three times continuously differentiable functions such that for any continuous and compactly supported function $\varphi(x)$ the relations $\|\varphi(y_n - y)\|_2 \to 0$, $\|\varphi(By_n - F)\|_2 \to 0 \quad (n \to \infty)$ hold.*

Let us introduce the following notation:

$$\alpha_{g,h}(\tau) = \sup_{x>0} \|g(\cdot, \tau)\|_{L^2(0,x)} \left\|h^{-1}(\cdot, \tau)\right\|_{L^2(x,\infty)},$$

$$\beta_{g,h}(\tau) = \sup_{t<0} \|g(\cdot, \tau)\|_{L^2(t,0)} \left\|h^{-1}(\cdot, \tau)\right\|_{L^2(-\infty,t)},$$

$$\gamma_{g,h} = \max\left(\sup_{\tau\in\mathbb{R}} \alpha_{g,h}(\tau), \sup_{\tau\in\mathbb{R}} \beta_{g,h}(\tau)\right).$$

Theorem 3. *Let*

$$p(x,y) \geq Cr(x) \geq \delta > 0, \gamma_{1+|q|,\sqrt{pr}} < \infty, \tag{27}$$

and one of the following conditions (a) and (b) is performed: (a) for $z \in \mathbb{R}$ there exist the independent of z constant $C_1 > 0$ such that

$$p(x, z) \leq C_1 r(x), \tag{28}$$

(b) for any positive number T

$$\sup_{|x-\eta|\leq 1} \sup_{|\tau_1-\tau_2|\leq T} \frac{p(x, \tau_1)}{p(\eta, \tau_2)} < \infty. \tag{29}$$

Then, for each $F(x) \in L^2(\mathbb{R})$, Equation (26) has a solution $y \in L^2(\mathbb{R})$, and for y the following relation holds:

$$\left\|\left(r(x)y'\right)''\right\|_2 + \|p(x, y)y'\|_2 + \|(1 + |q(x, y)|)y\|_2 < \infty. \tag{30}$$

Proof of Theorem 3. Let $C(\mathbb{R})$ be a space of continuous and bounded functions with norm $\|z\|_{C(\mathbb{R})} = \sup_{x\in\mathbb{R}} |z(x)|$. Let ε and A be some positive numbers. Take v from

$$S_A = \left\{z \in C(\mathbb{R}) : \|z\|_{C(\mathbb{R})} \leq A\right\}.$$

We consider the following linear third-order equation:

$$L_{0,v,\varepsilon} y = -(r(x)y')'' + \left[p(x, v) + \varepsilon\left(1 + x^2\right)^2\right] y' + q(x, v)y = F(x). \tag{31}$$

We denote by $L_{v,\varepsilon}$ the closure in $L^2(\mathbb{R})$ of the differential operator

$$L_{0,v,\varepsilon} y = -(r(x)y')'' + \left[p(x, v) + \varepsilon\left(1 + x^2\right)\right] y' + q(x, v)y \quad (D(L_{0,v,\varepsilon}) = C_0^3(\mathbb{R})).$$

According to (27), (28), the coefficients of Equation (31) satisfy the relations (7), (8), $p(x,v) \leq Cr(x)$, and (22). Let us show that (23) holds. Indeed, for $v \in S_A$ and $x, \eta \in \mathbb{R}$, we have

$$|v(x) - v(\eta)| \leq 2A.$$

Let $\tau_1 = v(x)$ and $\tau_2 = v(\eta)$. Then by (29),

$$\sup_{|x-\eta|\le 1} \frac{p(x, v(x)) + \varepsilon(1+x^2)^2}{p(\eta, v(\eta)) + \varepsilon(1+\eta^2)^2}$$

$$\le \sup_{|x-\eta|\le 1} \sup_{|\tau_1-\tau_2|\le 2A} \frac{p(x, \tau_1)}{p(\eta, \tau_2)} + 9 \le M(A).$$

Thus, all the conditions of Theorem 2 are satisfied. Therefore, there exists a unique solution y to Equation (31) and y satisfies the following inequality:

$$\left\|(r(x)y')''\right\|_2 + \left\|\left[p(x, v) + \varepsilon\left(1+x^2\right)^2\right]y'\right\|_2 + \|(1+|q(x, v)|)y\|_2 \le C_2\|F\|_2. \tag{32}$$

Let $\theta > 0$. By $C^{(\theta)}(\mathbb{R})$ we denote the Hölder space of bounded functions with norm

$$\|\varphi\|_{C^{(\theta)}(\mathbb{R})} = \sup_{x\in\mathbb{R}}\left[|\varphi(x)| + \frac{|\varphi(x+h) - \varphi(x)|}{|h|^\theta}\right].$$

Using well-known embedding theorems and Lemma 1, for y we prove that

$$\|\varepsilon y\|_{C^{(\theta)}(\mathbb{R})} + \sup_{x\in\mathbb{R}}|\varepsilon(1+x^2)y(x)| + \left\|\varepsilon(1+x^2)y\right\|_2 \le C_3\left\|\left[p(x, v) + \varepsilon(1+x^2)^2\right]y'\right\|_2. \tag{33}$$

By (32) and (36), for a solution y to Equation (31), the following estimate holds:

$$\left\|(r(x)y')''\right\|_2 + \left\|\left[p(x, v) + \varepsilon\left(1+x^2\right)^2\right]y'\right\|_2 + \varepsilon\|y\|_{C^{(\theta)}(\mathbb{R})}$$

$$+ \varepsilon\sup_{x\in\mathbb{R}}|(1+x^2)y(x)| + \left\|\left[1 + \varepsilon\left(1+x^2\right) + |q(x, v)|\right]y\right\|_2 \le C_4\|F\|_2. \tag{34}$$

Let us choose the radius A of the ball S_A equal to the right-hand side $C_4\|F\|_2$ of (34). Let $P(v, \varepsilon)$ be the transformation defined in S_A by the formula

$$P(v,\varepsilon) = L_{v,\varepsilon}^{-1} f,$$

where $L_{v,\varepsilon}^{-1}$ is the inverse to the closed operator $L_{v,\varepsilon}$.

According to estimate (34), the operator $P(v,\varepsilon)$ transforms the ball S_A into itself.

The operator $P(v, \varepsilon)$ is compact in $C(\mathbb{R})$. Indeed, by virtue of (34), $P(v, \varepsilon)$ transforms the ball S_A to the set

$$Q_A = \left\{y : \|y\|_W = \left\|(ry')''\right\|_2 + \left\|\left[p(x, v) + \varepsilon\left(1+x^2\right)^2\right]y'\right\|_2\right.$$

$$\left. + \left\|(1 + \varepsilon\left(1+x^2\right) + |q(\cdot, v)|)y\right\|_2 + \varepsilon\|y\|_{C^{(\theta)}(\mathbb{R})} + \sup_{x\in\mathbb{R}}\left|\varepsilon(1+x^2)y(x)\right|\right\}.$$

Let $y \in Q_A$. By virtue of (34),

$$\sup_{|x|\ge N}|y(x)| \le \frac{C_5}{\varepsilon(1+N^2)}. \tag{35}$$

By the Hausdorff theorem, taking into account $y \in C^{(\theta)}(\mathbb{R}) \quad (\theta > 0)$ and (35), we obtain that Q_A is compact in $C(\mathbb{R})$. Therefore, $P(v, \varepsilon)$ is a compact operator.

The continuity of the coefficients of Equation (2) implies that the operator $P(v, \varepsilon)$ continuously depends on $v \in S_A$.

Thus, the operator $P(v, \varepsilon)$ is continuous and compact in the space $C(\mathbb{R})$, and mapping S_A into itself, then, according to the well-known Schauder's theorem [20], there is a fixed point $y_\varepsilon \in S_A$ of the operator $P(v, \varepsilon)$. According to our choice, y_ε is a solution to the equation

$$L_{y,\varepsilon} y = -(r(x)y')'' + \left[p(x, y) + \varepsilon\left(1 + x^2\right)^2\right]y' + q(x, y)y = F(x).$$

By (34), for y_ε we obtain the following inequality:

$$\left\|\left(r(x)y'_\varepsilon\right)''\right\|_2 + \left\|\left[p(x, y_\varepsilon) + \varepsilon\left(1 + x^2\right)^2\right]y'_\varepsilon\right\|_2 + \\ + \left\|\left[1 + \varepsilon(1 + x^2) + |q(x, y_\varepsilon)|\right]y_\varepsilon\right\|_2 \leq C_5\|F\|_2. \quad (36)$$

We choose the sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ of positive numbers such that $\lim\limits_{j\to+\infty} \varepsilon_j = 0$. Let $y_{\varepsilon_j} \in S_A$ is a solution of the following equation:

$$L_{\varepsilon_j} y = -(r(x)y')'' + [p(x, y) + \varepsilon_j(1 + x^2)^2]y' + [1 + \varepsilon_j(1 + x^2) + q(x, y)]y = F(x). \quad (37)$$

For y_{ε_j}, the following estimate holds:

$$\left\|\left(r(x)y'_{\varepsilon_j}\right)''\right\|_2 + \left\|\left[p(x, y_{\varepsilon_j}) + \varepsilon_j(1 + x^2)^2\right]y'_{\varepsilon_j}\right\|_2 \\ + \left\|\left[1 + \varepsilon_j(1 + x^2) + |q(x, y_{\varepsilon_j})|\right]y_{\varepsilon_j}\right\|_2 \leq C_6\|F\|_2. \quad (38)$$

Let $-\infty < a < b < +\infty$. By (38), $y_{\varepsilon_j} \in W_2^1(a, b)$ $(j = 1, 2, \ldots)$. Since $W_2^1(a, b)$ compactly embedded in $L^2(a, b)$, there exists $y \in L^2(a, b)$ such that $\|y_{\varepsilon_j} - y\|_{L^2(a,b)} \to 0$ $(j \to +\infty)$. Hence by (37) and Definition 2, y is a solution to Equation (26). Relation (30) follows from inequality (34). □

Example 1. Consider the following nonlinear equation:

$$-\left((5 + 2\cos 3x)y'\right)'' + \left[\left(5 + 4x^2\right)^4 + 7y^2\right]y' - \left[2x^3 + \sin^7\left(3 + y^4\right)\right]y = F(x). \quad (39)$$

Coefficients of (39) satisfy conditions (27) and (29). Therefore, for each right-hand side $F(x)$ from $L^2(\mathbb{R})$ there exists a solution y of this equation and

$$\left\|\left((5 + 2\cos 3x)y'\right)''\right\|_2 + \left\|\left[\left(5 + 4x^2\right)^4 + 7y^2\right]y'\right\|_2 \\ + \left\|\left[1 + |2x^3 + \sin^7\left(3 + y^4\right)|\right]y\right\|_2 < \infty.$$

6. Conclusions

We have studied the third-order singular linear differential Equation (1) with variable unbounded coefficients and its non-linear generalization (2). They differ from the equations previously studied in [14,15] by the presence of an intermediate coefficient p, which grows rapidly and is not controlled by the coefficients r and q. The correctness of Equation (1) and the L^2-maximal regularity of its generalized solution are proved.

Using this result, we have obtained sufficient conditions for the solvability of a third-order nonlinear Equation (2) with unlimited "coefficients", as well as membership in L^2 of each of terms. Our results extend the results in studies [15] (we removed restrictions of type (4) on the coefficient q) and [16,17] (we cover the case of an unbounded leading coefficient r). In problems of maximal regularity of the solution, there is usually a condition to oscillation of coefficients. In Corollary 1, we discovered that the solution of Equation (1) satisfies the L^2—maximal regularity estimate, although there are no conditions on the oscillation of the functions r, p, and q or their derivatives (see [13,15]).

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