**Rapid Communication**

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# Note on quasivarieties generated by finite pointed abelian groups

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**Abstract:** We prove that a finite pointed abelian group generates a finitely axiomatizable variety that has a finite quasivariety lattice. As a consequence, we obtain that a quasivariety generated by a finite pointed abelian group has a finite basis of quasi-identities. The problems arising from the results obtained are also discussed.

**Keywords:** variety, quasivariety, quasivariety lattice, finitely axiomatizable class, abelian group, pointed group

**MSC 2020:** 06B20, 08B05, 08C15, 20N99

## 1 Introduction

A pointed algebra is obtained from the given algebra by adding a finite set of constants (nullary operations) into the signature. It is also called a pointed enrichment of the given algebra. For example, a pointed abelian group is a pair $(G, g)$ consisting of an abelian group $G$ together with a distinguished element $g$ of $G$. The equational and quasi-equational theories of an algebra and their pointed enrichments can have completely different properties. Indeed, the remarkable theorem of group theory due to Oates and Powell [1] states that every finite group has a finite basis of identities. On the other hand, Bryant [2] constructed a finite pointed group that has no finite basis of identities. For the quasi-equational theories, similar example was found among finite lattices. Namely, there is a finite lattice that has a finite basis of quasi-identities; however, there exists its pointed enrichment that has no finite basis of quasi-identities [3]. Moreover, the aforementioned lattice generates a variety with a finite lattice of quasivarieties, and at the same time, there is a pointed enrichment of it, generating a variety with an infinite lattice of quasivarieties. From the aforementioned case naturally arises the question: Which (quasi-)equational properties of finite algebras (groups, lattices) are preserved with respect to pointed enrichments? (cf. [4]). Note that the finite basis problem for identities of finite pointed groups was discussed in [2] and is still open.

Here, we will examine the aforementioned problem for finite abelian groups and their pointed enrichments. Namely, we establish that a finite pointed abelian group generates a finitely axiomatizable variety that has a finite quasivariety lattice. As a consequence, we obtain that the quasivariety generated by a finite pointed abelian group has a finite basis of quasi-identities. We would like to note that the latter fact was proved in [4], where the authors used a specific property of being a directed representable quasivariety. In our approach, we show the use of a finiteness of quasivariety lattices, which can be applied for a wider class of quasivarieties.

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## 2 Basic definitions and preliminaries

Let us briefly recall some main definitions. For more information on the basic notions of universal algebra and group theory used throughout this article, we refer to [5–7].

A *pointed group* is an algebra $\langle G;\ \cdot, ^{-1}, 1, C\rangle$ of the signature $\langle \cdot, ^{-1}, 1, C\rangle$, where $\langle \cdot\ , ^{-1}, 1\rangle$-reduct is a group and $C$ is a finite set of constants. A pointed group $\langle G; \cdot\ , ^{-1}, 1, C\rangle$ is also called a *pointed enrichment* of the group $\langle G; \cdot\ , ^{-1}, 1\rangle$ and is denoted by $G^c$. A pointed group is *abelian* if its group's reduct is an abelian group.

A *quasivariety* is a class of similar algebras that is closed with respect to subalgebras, direct products, and ultraproducts. Equivalently, a quasivariety is the same thing as a class of similar algebras axiomatized by a set of quasi-identities. A *quasi-identity* means a universal Horn sentence with the non-empty positive part, which is of the form:

$$(\forall \bar{x})[p_1(\bar{x}) \approx q_1(\bar{x}) \wedge \cdots \wedge p_n(\bar{x}) \approx q_n(\bar{x}) \to p(\bar{x}) \approx q(\bar{x})],$$

where $p, q, p_1, q_1, \dots, p_n, q_n$ are the terms of the signature $\sigma$. A *variety* is a quasivariety that is closed under homomorphisms. According to Birkhoff's theorem, a variety is a class of similar algebras axiomatized by a set of identities, where, by an identity, we mean a sentence of the form $(\forall \bar{x})[s(\bar{x}) \approx t(\bar{x})]$ for some terms $s(\bar{x})$ and $t(\bar{x})$.

A quasivariety $\mathcal{R} \subseteq \mathcal{K}$ is *finitely axiomatizable relative to* $\mathcal{K}$ if there is a finite set of quasi-identities $\Sigma$ such that $\mathrm{Mod}(\Sigma) \cap \mathcal{K} = \mathcal{R}$. In this case, $\Sigma$ is a *basis of quasi-identities* of $\mathcal{R}$ relative to $\mathcal{K}$. An algebra has a *finite basis of quasi-identities (identities)* if the quasivariety (variety) generated by this algebra is finitely axiomatizable.

By $\mathbf{Q}(\mathcal{K})$ ($\mathbf{V}(\mathcal{K})$), we denote the smallest quasivariety (variety) containing a class $\mathcal{K}$. If $\mathcal{K}$ is a finite family of finite algebras, then $\mathbf{Q}(\mathcal{K})$ is called finitely generated. In event $\mathcal{K} = \{A_1, \dots, A_n\}$, we write $\mathbf{Q}(A_1, \dots, A_n)$ instead of $\mathbf{Q}(\{A_1, \dots, A_n\})$.

For an algebra $A$, by Con$A$, we denote the congruence lattice of $A$. An algebra $A$ is said to be *subdirectly irreducible* if the least congruence $0_A$ of the congruence lattice Con$A$ is completely meet irreducible. By Birkhoff's representation theorem [8], every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.

For a quasivariety $\mathcal{R}$, by $Lq(\mathcal{R})$, we denote a quasivariety lattice of $\mathcal{R}$, which is the set of all subquasivarieties ordered by the set's inclusion $\subseteq$. Let $\mathcal{K}$ be a subclass of the quasivariety $\mathcal{R}$ and $Lq(\mathcal{R}, \mathcal{K}) = \{\mathcal{M} \cap \mathcal{K} | \mathcal{M} \in Lq(\mathcal{R})\}$. It is easy to see that $Lq(\mathcal{R}, \mathcal{K})$ forms a complete lattice relative to the set inclusion $\subseteq$.

A finite algebra $A$ is *quasicritical* if $A$ is not a subdirect product of its proper subalgebras.

**Lemma 1.** [9, Lemma 9] *Let $\mathcal{K}$ be the class of all quasicritical algebras of the quasivariety $\mathcal{R}$. Then, $Lq(\mathcal{R}) \cong Lq(\mathcal{R}, \mathcal{K})$.*

As a partial case of Theorem 2.13 in [10], we have

**Theorem 2.** [10] *A finite pointed abelian group $A$ is quasicritical if and only if a decomposition of $A$ into a direct sum*

$$A_1 \oplus \ldots \oplus A_m$$

*possesses the following properties: $A_0, \dots, A_m$ are the finite cyclic primary pointed groups of different orders $p_1^{n_1}, \dots, p_m^{n_m}$, and $c^{A_i} = 0^{A_i}$ at most for one $i \neq m$; moreover, if $c^{A_i} = 0^{A_i}$, $i \leq m$, then the order of the pointed group $A_i$ is the greatest among those $A_j$ for which $p_i = p_j$.*

**Theorem 3.** (McKenzie's theorem [11]) *A locally finite congruence modular variety with a finite number of subdirectly irreducible algebras has a finite basis of identities.*

## 3 Identities of pointed abelian groups

By the Oates-Powell theorem, a finite group has a finite basis of identities. The Braynt's example of a pointed group that has no finite basis of identities shows that the Oates-Powell arguments generally do not work in the class of finite pointed groups. However, for pointed abelian groups, we have

**Theorem 4.** *A finite pointed abelian group has a finite basis of identities.*

**Proof.** Since an abelian group and any of its pointed enrichments have the same set of polynomials, by definition of congruence, every equivalence on an abelian group is a congruence if and only if it is a congruence on any (some), its pointed enrichment. It means that an abelian group and any of its pointed enrichment have the same lattices of congruences. Hence, for a finite abelian group $A$ and its pointed enrichment $A^c$, a congruence $\theta$ is meet irreducible in Con$A$ if and only if $\theta$ is meet irreducible in Con$A^c$. Since, for every meet irreducible congruence $\theta$, the quotient group $A/\theta$ is subdirectly irreducible, we obtain that a finite abelian group is subdirectly irreducible if and only if its pointed enrichment is subdirectly irreducible.

Let $B^c \in \mathbf{V}(A^c)$ and $B^c$ be subdirectly irreducible. By Birkhoff's theorem, $B \in \mathbf{HSP}(A^c)$, where $\mathbf{H}$ ($\mathbf{S}$, $\mathbf{P}$) is an operator of taking all homomorphic images (subalgebras, direct products). Thus, there are $C^c \in \mathbf{SP}(A^c)$ and a homomorphism $h : C^c \to B^c$ such that ker$h$ is meet irreducible. Hence, $B$ is subdirectly irreducible in $\mathbf{V}(A)$. Since a variety generated by a finite abelian group has a finite number of finite subdirectly irreducible groups, there is a finite number $n > 0$ such that the power of every subdirectly irreducible group in $\mathbf{V}(A^c)$ is bounded by $n$. Hence, every subdirectly irreducible pointed group in $\mathbf{V}(A^c)$ is bounded by $n$. Since $\mathbf{V}(A^c)$ is locally finite, the number of all subdirectly irreducible pointed groups in $\mathbf{V}(A^c)$ is finite. Since Con$A$ = Con$A^c$ and a variety of abelian groups is congruence permutable, whence congruence modular, every variety of pointed abelian groups is congruence modular.

Thus, we obtain that a variety $\mathbf{V}(A^c)$ is congruence modular and has a finite number of finite subdirectly irreducible pointed abelian groups. By Theorem 3, we obtain that $\mathbf{V}(A^c)$ is finitely axiomatizable. □

## 4 Quasivariety of pointed abelian groups

By the general theory of abelian groups, a quasivariety generated by a finite abelian group is a variety, and it has a finite variety lattice that coincides with its quasivariety lattice. Also, it gives us that a finite abelian group has a finite basis of quasi-identities by the Oates-Powell theorem.

We will establish the similar results for pointed abelian groups.

First, we prove the finiteness of quasivariety lattices.

**Theorem 5.** *A quasivariety generated by a finite set of finite pointed abelian groups has a finite quasivariety lattice.*

**Proof.** We will prove a slightly more general result. Namely, we prove that a finitely generated variety $\mathcal{V}$ of pointed abelian groups has a finite quasivariety lattice. Since, for any subquasivariety $\mathcal{R}$ of $\mathcal{V}$, the quasivariety lattice $Lq(\mathcal{R})$ is a sublattice of $Lq(\mathcal{V})$, we obtain a proof of the theorem.

Let $\mathcal{V}$ be a variety generated by a finite set $\{A_1, \ldots, A_k\}$ of finite pointed abelian groups, i.e., $\mathcal{V} = \mathbf{V}(A_1, \ldots, A_k)$. By Lemma 1, it is enough to show that $\mathcal{V}$ contains a finite number of quasicritical algebras.

The main theorem on the structure of finite abelian groups states that a finite abelian group is isomorphic to a direct sum of some finite primary cyclic groups [7]. By definition, a group's reduct of a pointed abelian group is an abelian group. Hence, as the congruences of an abelian group and its pointed enrichment coincide, we obtain that a finite pointed abelian group is isomorphic to a direct sum of some finite primary pointed cyclic groups, where a primary pointed abelian group is obtained from a primary abelian group by adding a fixed finite number of constants.

Put $n = |A_1| \times \ldots \times |A_k|$. Then, the identity $\forall x[x^n = 1]$ holds on every pointed abelian group from $\mathcal{V}$. Hence, every primary pointed abelian group has at most $n$ elements. Since the number of all primary abelian groups of power at most $n$ is finite, we obtain that the number of all pointed abelian groups of a cardinality at most $n$ belonging to the variety $\mathcal{V}$ is finite too. By Theorem 2, a finite quasicritical pointed abelian group $A$ is a direct sum $B_1 \oplus \ldots \oplus B_m$ of some finite cyclic primary pointed groups $B_0, \ldots, B_m$ of different orders $p_1^{n_1}, \ldots, p_m^{n_m}$. Since the set $S$ of all primary pointed abelian groups is finite in $\mathbf{V}(A_1, \ldots, A_k)$, the set of all different orders $p_1^{n_1}, \ldots, p_m^{n_m}$ has a cardinality at most $2^{|S|}$, when it is finite. Thus, $\mathcal{V}$ contains a finite number of quasicritical pointed abelian groups. □

Furthermore, we need the following fact:

**Lemma 6.** *Suppose that a quasivariety $\mathcal{R}$ has a finite quasivariety lattice. Then, every subquasivariety of $\mathcal{R}$ is finitely axiomatizable relative to $\mathcal{R}$.*

**Proof.** Let $\mathcal{K}$ be a subquasivariety of $\mathcal{R}$ and $Lq(\mathcal{R})$ be finite. And let

$$S = \{\mathcal{M} \in Lq(\mathcal{R}) | \mathcal{M} \nsubseteq \mathcal{K}\}.$$

For every $\mathcal{M} \in S$, there is a quasi-identity $\varphi_{\mathcal{M}}$ such that $\mathcal{K} \vDash \varphi_{\mathcal{M}}$ and $\mathcal{M} \nvDash \varphi_{\mathcal{M}}$. Let $\Sigma_{\mathcal{K}} = \{\varphi_{\mathcal{M}} | \mathcal{M} \in S\}$. Since a quasivariety lattice of $\mathcal{R}$ is finite, the set $S$ is finite, whence $\Sigma_{\mathcal{K}}$ is a finite set of quasi-identities.

We show that $\mathcal{K} = \mathrm{Mod}(\Sigma_{\mathcal{K}}) \cap \mathcal{R}$, which means that $\mathcal{K}$ is finitely axiomatizable with respect to $\mathcal{R}$. Since $\mathcal{K} \vDash \Sigma_{\mathcal{K}}$ and $\mathcal{K} \subseteq \mathcal{R}$, $\mathcal{K} \subseteq \mathrm{Mod}(\Sigma_{\mathcal{K}}) \cap \mathcal{R}$. For inverse inclusion, suppose that $A \vDash \mathrm{Mod}(\Sigma_{\mathcal{K}}) \cap \mathcal{R}$, i.e., $A \in \mathcal{R}$ and $A \vDash \Sigma_{\mathcal{K}}$. These give us $\mathbf{Q}(A) \subseteq \mathcal{R}$ and $\mathbf{Q}(A) \vDash \Sigma_{\mathcal{K}}$. Assume $\mathbf{Q}(A) \nsubseteq \mathcal{K}$. It means $\mathbf{Q}(A) \in S$. Hence, there is a quasi-identity $\varphi_{\mathbf{Q}(A)} \in \Sigma_{\mathcal{K}}$ such that $\mathcal{K} \vDash \varphi_{\mathbf{Q}(A)}$ and $\mathbf{Q(A)} \nvDash \varphi_{\mathbf{Q}(A)}$. It contradicts to our assumption $A \vDash \Sigma_{\mathcal{K}}$. Thus, $\mathbf{Q}(A) \subseteq \mathcal{K}$. Hence, $A \in \mathcal{K}$, i.e., $\mathcal{K} \supseteq \mathrm{Mod}(\Sigma_{\mathcal{K}}) \cap \mathcal{R}$. □

The next result is a consequence of Theorems 4 and 5 and Lemma 6.

**Theorem 7.** *A finite pointed abelian group has a finite basis of quasi-identities.*

**Proof.** Let $A$ be a finite pointed abelian group. Theorems 4 and 5 provide that a variety $\mathbf{V}(A)$ is finitely axiomatizable and has a finite quasivariety lattice. By Lemma 6, a quasivariety $\mathbf{Q}(A)$ is finitely axiomatizable relative to $\mathbf{V}(A)$. Since $\mathbf{V}(A)$ is finitely axiomatizable and $\mathbf{Q}(A)$ is finitely axiomatizable relative to $\mathbf{V}(A)$, according to the compactness theorem, we obtain that $\mathbf{Q}(A)$ is finitely axiomatizable. □

# 5 Conclusion remarks

We would like to note that the results obtained about abelian groups are completely opposite to the results for finite (modular) lattices. Indeed, as we noted in Section 1, a 7-element modular lattice $L$ and its pointed enrichment $L^c$, which were constructed in [3], such that (a) $L$ has a finite basis of quasi-identities but $L^c$ has no one and (b) the quasivariety lattice of $\mathbf{V}(L)$ is finite but the quasivariety lattice of $\mathbf{V}(L^c)$ is infinite. These facts show that the problem "Which (quasi-)equational properties of finite algebras (groups, lattices) are preserved with respect to pointed enrichments?" has a lot of obstacles.

The following two problems come from the well-known result due to Ol'shansky [12]: a finite group has a finite basis of quasi-identities if and only if all of its nilpotent subgroups are abelian. According to this fact and Theorem 7, it is natural to ask:

**Problem 1.** Does there exist a finite pointed group that has no finite basis of quasi-identities and all nilpotent subgroups of its group's reduct are abelian?

And more general:

**Problem 2.** Which finite pointed group (lattice) generates a finitely axiomatizable quasivariety?

As we noted in Section 1, the finite basis problem for finitely generated varieties of pointed groups is discussed in [2].

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