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On one integrable cosmological model of the flat universe in *k*-essence

Abstract. The study of the origin and evolution of our Universe is one of the interesting and actual directions in modern physics and astrophysics. This paper considers the cosmological model of the Universe in the Einstein's theory of gravity and *k*-essence, where the gravitational field interacts in a non-minimal way with the scalar field φ. That is, in action, in the role of the matter field, we consider the special case of the Lagrange function for the essence. The corresponding field equations of the considered model are obtained. Also, particular solutions for the scale factor $a(t)$ were obtained in the form of de Sitter's solution. Two solutions were found, for the potential energy $V(t)$ and the scalar field $\varphi(t)$, and their graphical solutions were also built. The analytical solutions obtained in this work are solutions of the considered integrable systems. These solutions are in good agreement with the available observational data and are able to describe the modern dynamics of the expansion of the Universe.

Key words: cosmological model of the Universe, gravitational theory, *k*-essence, gravitational field, scalar field, expanding Universe, flat Universe, Euler-Lagrange equations, action, metric, integrable cosmological models.

Introduction

The study of the origin and evolution of our Universe is one of the interesting and actual directions in modern physics and astrophysics [1-11]. The general theory of relativity, created in 1916 by A. Einstein, describes the statistical universe. The action for the gravitational field is usually written in a general form: $S = S_{gr.} + S_{mat.}$, where $S_{gr.}$ is the action of the gravitational field, S_{mat} is the action of matter. That is, according to [1], the action for the gravitational field and matter can be written as

$$S = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}R + L_{mat.}\right), \quad (1)$$

where $k = 8\pi G / c^4$, G the gravitational constant, c the speed of light in vacuum, g the determinant of the metric tensor $g_{\mu\nu}$, R the scalar curvature of space and L_{mat} the Lagrange function of matter. In 1922 A.A. Friedman, using action (1) and the metric describing a flat, homogeneous and isotopic Universe, established that the Universe cannot be static, it must expand or contract with time. E. Hubble in 1929, from observational data, established that our Universe is expanding with time at a certain speed.

However, in 1998, a group of astronomers Saul Perlmutor, Adam Reiss and Brian Schmidt, while observing type Ia supernovae, found that the Universe is currently in the stage of accelerated expansion [12, 13]. The main theoretical model for such an expansion is considered to be the model of dark energy, which has negative pressure, and the nature that we still know. To obtain a model of dark energy, as well as to explain the expansion of the Universe in modern cosmology and astrophysics, various generalizations of action (1) are applied or alternative theories of gravity are applied. One of the simplest examples of such a generalization of the gravitational field is the theory of $f(R)$ gravity, where R the scalar of the curvature of space-time is. Also in modern cosmology and astrophysics, various fields of matter are used, such as scalar, phantom, tachyon or spinor fields. For example, the Lagrange function for a scalar field and the Friedman-Robertson-Walker metric is written in the form [14]:

$$L_{mat.} = \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \quad (2)$$

where $0.5\dot{\varphi}^2$ is the kinetic term, and $V(\varphi)$ is the potential energy of the scalar field. This Lagrangian has a canonical form. However, at present, its generalizations have been proposed, known as the k-essence model [15, 16]. The Lagrange function of such a field $L(\varphi, X)$ depends on the variables φ and X, where

$$X = \frac{1}{2} g^{\mu\nu} \frac{\partial\varphi}{\partial x_\mu} \frac{\partial\varphi}{\partial x_\nu}.$$

Action and equations of motion

In this work, we will investigate the model of a flat, homogeneous Universe in the theory of Einstein and k-essence. That is, in action (1), in the role of the matter field, consider the special case of the Lagrange function for the essence. We obtain the corresponding equations of motion, as well as their particular solutions. Let us compare the results obtained with modern observational data.

As mentioned above, we will consider here the cosmological model of a flat Universe in the framework of the general theory of relativity and k essence, where the gravitational field does not interact minimally with the scalar field. The action for such a model is written in the following form [17]:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2} f(\varphi)R + P(\varphi, X)\right], \quad (3)$$

where $u(\varphi)$ is the connection function and $K(\varphi, X)$ is the Lagrange function for the k-essence. Also here we consider the Friedman-Roberson-Walker metric, which describes a flat, homogeneous and isotropic four-dimensional space-time, namely

$$ds^2 = -dt^2 + a(t)^2\left(dx^2 + dy^2 + dz^2\right), \quad (4)$$

where $a(t)$ is a scale factor that depends on time. This metric is in good agreement with modern observational data. For this metric, you can write the following expressions

$$\sqrt{-g} = a^3, \quad R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right), \quad X = -\frac{1}{2}\dot{\varphi}^2.$$

Hereinafter, the dot above the letter will denote the time derivative t. Now it will be possible to write the Lagrange function in action (3) and taking into account the metric (4) as

$$L = 3M_P^2 f a\dot{a}^2 + 3M_P^2 \dot{f} a^2 \dot{a} - a^3 P(\varphi, X). \quad (5)$$

Here and below, the stroke above the letter will denote the derivative with respect to the field.

Next, using the Euler-Lagrange equations and the zero energy condition, we define the equations of motion for the considered model

$$3M_P^2 f\frac{\dot{a}^2}{a^2} + 3M_P^2 f'\dot{\varphi}\frac{\dot{a}}{a} - P_X\dot{\varphi}^2 + P = 0, \quad (6)$$

$$\begin{aligned} &M_P^2 f\frac{\dot{a}^2}{a^2} + 2M_P^2 f\frac{\ddot{a}}{a} + \\ &+ M_P^2\ddot{f} + 2M_P^2\dot{f}\frac{\dot{a}}{a} + P = 0, \end{aligned} \quad (7)$$

$$\begin{aligned} &P_X\ddot{\phi} + P_{XX}\dot{\phi}^2\ddot{\phi} + 3\frac{\dot{a}}{a}P_X\dot{\phi} - \\ &-P' - 3M_P^2 f'\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) = 0. \end{aligned} \quad (8)$$

Also, the system of equations (6) – (8) can be written through the Hubble parameter $H = \dot{a}/a$:

$$3M_P^2 fH^2 + 3M_P^2\dot{f}H = \rho_\varphi, \quad (9)$$

$$M_P^2 f\left(3H^2 + 2\dot{H}\right) + M_P^2\ddot{f} + 2M_P^2\dot{f}H = -p_\varphi, \quad (10)$$

$$\begin{aligned} &\left(P_X + 2P_{XX}X\right)\dot{X} + 6HP_X X - \\ &-P' - 3M_P^2 f'\left(2H^2 + \dot{H}\right) = 0, \end{aligned} \quad (11)$$

where $\rho_\varphi = 2XP_X - P$ and $p_\varphi = P(\varphi, X)$ are the energy density and the k-essence pressure. Hereinafter, the stroke above the letter will denote the derivative with respect to the field φ.

The equation of state parameter for the k-essence can be written in the form

$$\omega = \frac{p_\varphi}{\rho_\varphi} = \frac{P}{2XP_X - P}. \quad (12)$$

The square of the speed of sound is

$$c_s^2 = \frac{p_X}{\rho_X} = \frac{P_X}{2XP_{XX} + P_X}. \quad (13)$$

The system of equations (6)-(8) or (9)-(11) are nonlinear partial differential equations, for the solution of which it is necessary to determine the explicit form of the functions $a(t)$, $f(\varphi)$ and $P(\varphi, X)$. Below we will consider particular solutions of these functions.

Cosmological solutions

We start with the case when $f(\varphi)=1$ and $P(\varphi, X) = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$, then the system of equations (9)-(11) can be rewritten as

$$3H^2 = \frac{1}{M_P^2}\rho_\varphi, \quad (14)$$

$$3H^2 + 2\dot{H} = -\frac{1}{M_P^2}p_\varphi, \quad (15)$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0. \quad (16)$$

Here

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \text{ и } p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

The system of equations (14)-(16) is a standard cosmological model with a scalar field, where the gravitational fields interact non minimally with the scalar field φ.

Now consider the case when

$$f(\varphi) = \varphi, \quad P(\varphi, X) = K(X) - V(\varphi). \quad (17)$$

Then equations (9) – (11) can be rewritten as

$$\begin{gathered} 3M_P^2\phi H^2 + 3M_P^2\dot{\phi}H - \\ -2XK_X + K(X) - V(\phi) = 0, \end{gathered} \quad (18)$$

$$\begin{gathered} M_P^2\phi\left(3H^2 + 2\dot{H}\right) + M_P^2\ddot{\phi} + \\ +2M_P^2\dot{\phi}H + K(X) - V(\phi) = 0, \end{gathered} \quad (19)$$

$$\begin{gathered} \left(K_X + 2XK_{XX}\right)\dot{X} + 6HK_X X + V' - \\ -3M_P^2\left(2H^2 + \dot{H}\right) = 0. \end{gathered} \quad (20)$$

Let's suppose

$$V' - 3M_P^2\left(2H^2 + \dot{H}\right) = 0.$$

Where do we get

$$\frac{1}{3M_P^2}\frac{dV}{d\varphi} = 2H^2 + \dot{H} = n$$

Here n is a certain constant. Accordingly, we obtain a system of differential equations

$$dV = 3nM_P^2 d\varphi,$$

$$a\frac{d^2a}{dt^2} + \left(\frac{da}{dt}\right)^2 - na = 0.$$

Integrating them, we find

$$V(\varphi) = 3nM_P^2\varphi + C_1, \quad (21)$$

$$a(t) = e^{nt}. \quad (22)$$

where C_1 is a certain constant of integration. The particular solution we obtained for the scale factor (22) is the de Sitter solution, which describes the modern dynamics of the expanding Universe. The equation of state parameter for such a solution is equal to $\omega = -1$, which corresponds to the dark energy model. Figure 1 shows the dependence of the scale factor $a(t)$ on time, at. Figure 2 also shows the dependence of the potential energy on the scalar field, at, and. Figure 1 shows the dependence of the scale factor $a(t)$ on time t, at $n = 1$. Figure 2 also shows the dependence of the potential energy $V(\varphi)$ on the scalar field $n=1$, $M_P=1$ and $C_1 = 1$.

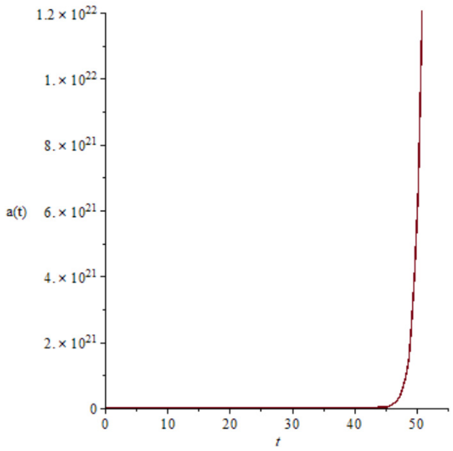

Figure 1 – Dependence of the scale factor $a(t)$ on time t

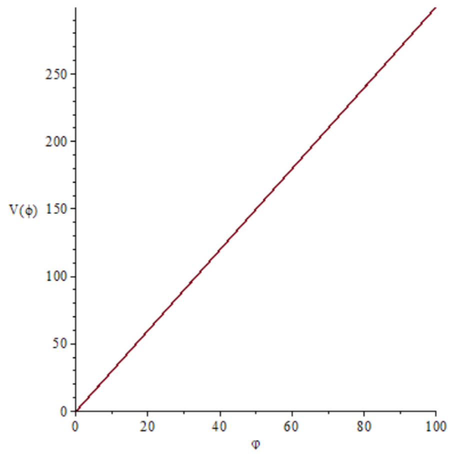

Figure 2 – Dependence of the potential energy $V(\varphi)$ on the scalar field φ

Now equating equations (18) and (19), we obtain

$$\ddot{\varphi}-n\dot{\varphi}+\frac{2}{M_P^2}XK_X=0\,. \qquad (23)$$

Equation (20) can be written as

$$\left(K_X+2XK_{XX}\right)\dot{X}+6nXK_X=0\,. \qquad (24)$$

Here we consider the kinetic Lagrange term for the scalar field $K(X)$ as $K(X)=\varepsilon_i X$, where the ε_i constant can take values $\varepsilon_1=+1$ or $\varepsilon_2=-1$, which correspond to cosmological models of quintessence and phantom field. Then equations (23) and (24) take the form

$$\ddot{\varphi}-n\dot{\varphi}+\frac{\alpha}{M_P^2}\dot{\varphi}^2=0\,.$$

$$\ddot{\varphi}+3n\dot{\varphi}=0\,.$$

Equating these equations, we find

$$\frac{d\varphi}{dt}-4\frac{n}{\alpha}M_P^2=0\,. \qquad (25)$$

By integrating which, we get

$$\varphi(t)=4\frac{n}{\alpha}M_P^2t+C_1\,. \qquad (26)$$

Next, consider the case when

$$f(\varphi)=\varphi,\quad P(\varphi,X)=\alpha X-\frac{1}{2}m\varphi^2\,. \qquad (27)$$

Here $V(\varphi)=\frac{1}{2}m\varphi^2$ is the potential energy of the scalar field. Then equations (8)-(11) can be rewritten as

$$3M_P^2\varphi H^2+3M_P^2\dot{\varphi}H-\frac{1}{2}\alpha\dot{\varphi}^2-\frac{1}{2}m\varphi^2=0, \qquad (28)$$

$$\begin{gathered}3M_P^2\phi H^2+2M_P^2\phi\dot{H}+M_P^2\ddot{\phi}+\\+2M_P^2\dot{\phi}H+\frac{1}{2}\alpha\dot{\phi}^2-\frac{1}{2}m\phi^2=0,\end{gathered} \qquad (29)$$

$$\alpha\dot{\varphi}\ddot{\varphi}+3\alpha\dot{\varphi}^2 H+m\varphi-3M_P^2\left(2H^2+\dot{H}\right)=0. \quad (30)$$

Figure 3 shows the dependence of the potential energy $V(\varphi)$ on the scalar field φ, for the case $V(\varphi)=\frac{1}{2}m\varphi^2$, where $n=1$ and $m=1$.

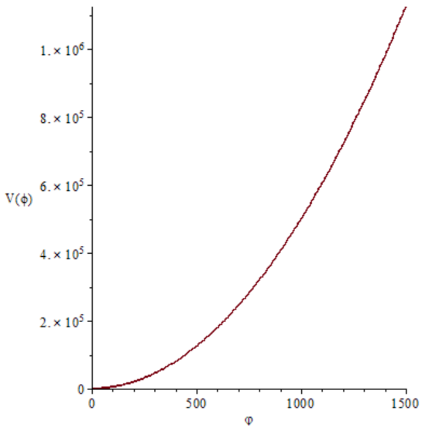

Figure 3 – Dependence of the potential energy V(φ) on the scalar field φ

From equations (28) and (29), we obtain the equation

$$\ddot{\varphi}-H\dot{\varphi}+2\varphi\dot{H}+\frac{\alpha}{M_P^2}\dot{\varphi}^2=0. \quad (31)$$

Substituting solutions (22) into equations (30) and (31), we determine

$$\ddot{\varphi}-n\dot{\varphi}+\frac{\alpha}{M_P^2}\dot{\varphi}^2=0. \quad (32)$$

Integrating this equation, we get

$$\varphi(t)=\frac{M_P^2}{\alpha}\ln\left[\frac{\alpha}{M_P^2}\left(C_1e^{nt}+C_2\right)\right], \quad (33)$$

where C_1 and C_2 are the constants of integration.

Conclusion

In this work, we have considered the cosmological model of the Universe in the framework of the classical theory of gravity and k-essence, where the gravitational field interacts non minimally with the scalar field. Particular solutions are obtained for the scale factor $a(t)$ in the form of the de Sitter solution, and two particular solutions are obtained for the scalar field φ and potential energy $V(\varphi)$. It was found that both are more realistic and are able to describe the modern dynamics of the expansion of the Universe. It was found that $\varphi(t)=\frac{M_P^2}{\alpha}\ln\left[\frac{\alpha}{M_P^2}\left(C_1e^{nt}+C_2\right)\right]$ and $V(\varphi)=\frac{1}{2}m\varphi^2$ are more realistic and are able to describe the modern dynamics of the expansion of the Universe. The analytical solutions obtained here are solutions of the considered integrable systems.

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