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On the Resolvent Existence and the Separability of a Hyperbolic Operator with Fast Growing Coefficients in $L_2(R^2)$

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Abstract. This paper studies the question of the resolvent existence, as well as, the smoothness of elements from the definition domain (separability) of a class of hyperbolic differential operators defined in an unbounded domain with greatly increasing coefficients after a closure in the space $L_2(R^2)$. Such a problem was previously put forward by I.M. Gelfand for elliptic operators.

Here, we note that a detailed analysis shows that when studying the spectral properties of differential operators specified in an unbounded domain, the behavior of the coefficients at infinity plays an important role.

1. Introduction

Consider a differential operator with unbounded coefficients

$$(L+\lambda I)u = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + b(y)u_x + q(y)u + \lambda u \tag{1.1}$$

initially defined on the set $C_0^\infty(R^2)$ of infinitely differentiable and compactly supported functions, where $(x, y) \in \mathbb{R}^2$, $\lambda \geq 0$.

Further it is assumed that the coefficients $b(y), q(y)$ satisfy the condition:

i) $|b(y)| \geq \delta_0 > 0$, $q(y) \geq \delta > 0$ are continuous functions in $\mathbb{R} = (-\infty, \infty)$.

It is easy to verify that the operator $L + \lambda I$ admits a closure in $L_2(R^2)$ which will be denoted by $L + \lambda I$.

We note that a comprehensive bibliography on the existence, uniqueness, and qualitative behavior of solutions of hyperbolic type differential equations is contained, for example, in the papers of J.Hadamard [1], C. Friedrichs [2], S.l. Sobolev [3], L. Garding [4], O.A. Ladyzhenskaya [5], A.V. Bitsadze [6], J.Leray [7], A. Nahushev [8], T. Sh. Kalmenov [9], T.I. Kiguradze [10, 11], A.V. Filinovskii [12] etc.

In these papers, for the differential equations, the Darboux, Goursat and Cauchy problems, periodic and boundary value problems with constant or variable bounded coefficients are studied.

Note that a differential operator of hyperbolic type in the whole space (Euclidean space of dimension) with continuous and bounded coefficients was studied in the paper of M. Nagumo [13].

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As it is well known, a detailed analysis shows that when studying the spectral characteristics of differential operators is given in an unbounded domain, the problem of the coefficients behavior at infinity is an important problem. As J. Leray [7] noted in his work the study of the hyperbolic operators in the whole space is an important and interesting problem.

In this paper, we study in the space $L_2(R^2)$, the question of the existence of a solution to one class of differential operators of hyperbolic type with strongly increasing coefficients.

This paper also considers the question of the elements smoothness from the domain of a singular hyperbolic operator after the closure in $L_2(R^2)$, such a problem was proposed by I.M. Gelfand for elliptic operators [14].

To answer these questions, at first we define:

Definition 1.1. *The hyperbolic operator L is separable if the estimate*

$$\|u_{xx} - u_{yy}\|_2 + \|u_y\|_2 + \|b(y)u_x\|_2 + \|q(y)u\|_2 \leq c \cdot (\|Lu\|_2 + \|u\|_2),$$

holds, where c¿0 does not depend on $u \in \mathbb{D}(L)$, and $\|\cdot\|_2$ is the norm in $L_2(R^2)$.

Recall that in this case, the separability of the operator L and the maximum regularity of solutions of the equation $Lu = f \in L_2(R^2)$ are equivalent [15-17].

For elliptic operators, in the case of unbounded domains, the separability was investigated in papers [18-25].

The main results are presented in the following theorems:

Theorem 1.1. *Let the condition i) be fulfilled. Then there exists a continuous inverse operator $(L + \lambda I)^{-}1$ for $\lambda \geq 0$ defined in the space $L_2(R^2)$.*

Suppose that the coefficients $b(y)$ and $q(y)$ and i) satisfy the conditions:

ii) $\mu = \sup\limits_{|y-t|\leq 1} \frac{b(y)}{b(t)} < \infty$; $\mu = \sup\limits_{|y-t|\leq 1} \frac{q(y)}{q(t)} < \infty$.

iii) $q(y) \leq c_0 \cdot b^2(y),\ y \in \mathbb{R},\ c_0 > 0$ is a constant number, where $\mathbb{R}(-\infty, \infty)$.

Theorem 1.2. *Let the conditions i)-iii) be fulfilled. Then the operator $(L + \lambda I)$ for $\lambda \geq 0$ is separable.*

Example 1.1. *For the operator*

$$Lu = u_{xx} - u_{yy} + e^{100\cdot|y|}u_x + e^{10\cdot|y|} \cdot u,\ u \in \mathbb{D}(L),\ y \in (-\infty, \infty)$$

It is easy to verify, that all conditions of the theorems 1.1-1.2 are satisfied.

Hence, there is a continuous inverse operator for it and the operator is separable, i.e. the estimate

$$\|u_{xx} - u_{yy}\|_2 + \|e^{100\cdot|y|}u_x\|_2 + \|u_y\|_2 + \|e^{10\cdot|y|}u\|_2 \leq c \cdot (\|Lu\|_2 + \|u\|_2),$$

holds, where c¿0 is a constant number.

2. The existence of a resolvent.

2.1. Auxiliary lemmas and estimates.

Consider the operator

$$(l_t + \lambda I)u = -u''(y) + (-t^2 + itb(y) + q(y) + \lambda)u$$

initially defined on the set $C_0^\infty(R)$, $t \in \mathbb{R}$.

It is easy to verify that the operator $l_t + \lambda I$ admits a closure in $L_2(R^2)$, which is also denoted by $l_t + \lambda I$.

Lemma 2.1. *Let the condition i) be fulfilled. Then the estimate*

$$c(\delta) \cdot \|l_t + \lambda I\|_2 \geq (\delta + \lambda)^{\frac{1}{2}} \|u\|_2, \tag{2.1}$$

holds in the space $L_2(R)$ for all $u \in D(l_t)$ when $\lambda \geq 0$, where $\|\cdot\|_2$ is the norm in $L_2(R^2)$, $c(\delta) > 0$.

Proof. Let $u \in C_0^\infty(R)$. Then the equation

$$< (l_t + \lambda I)u, u >= \int_R (-u''(y) + (-t^2 + itb(y) + q(y) + \lambda)u)\bar{u}dy =$$

$$= \int_R |u'(y)|^2 dy + \int_R (-t^2 + itb(y) + q(y) + \lambda)|u|^2 dy, \qquad (2.2)$$

holds, where $< \cdot, \cdot >$ is a scalar product in $L_2(R)$.

From equality (2.2) it follows that

$$| < (l_t + \lambda I)u, u > | \geq \int_R |u'|^2 dy + \int_R (q(y) + \lambda)|u|^2 dy - \int_R t^2 |u|^2 dy.$$

This implies the validity of the following inequalities

$$| < (l_t + \lambda I)u, u > | \geq \int_R |u'|^2 dy - \int_R t^2 |u|^2 dy, \qquad (2.3)$$

$$| < (l_t + \lambda I)u, u > | \geq \int_R (q(y) + \lambda)|u|^2 dy - \int_R t^2 |u|^2 dy. \qquad (2.4)$$

Now, using the Cauchy inequality with $\ll \varepsilon > 0 \gg$ and the condition i), from (2.4), we obtain

$$\frac{1}{2\delta} \|(l_t + \lambda I)u\|_2^2 \geq \frac{1}{2} \int_R (q(y) + \lambda)|u|^2 dy - \int_R t^2 |u|^2 dy, \qquad (2.5)$$

where $\varepsilon = \frac{\delta}{2}$. Next, consider the following functional

$$| < (l_t + \lambda I)u, -itu > | = |it \int_R |u'|^2 + (-t^2 + q(y) + \lambda)|u|^2 dy + \int_R t^2 b(y)|u|^2 dy|. \qquad (2.6)$$

Hence we obtain

$$| < (l_t + \lambda I)u, -itu > | \geq \int_R |t|^2 |b(y)||u|^2 dy. \qquad (2.7)$$

By the condition i) from (2.7) we have

$$| < (l_t + \lambda I)u, -itu > | \geq t^2 \delta_0 \|u\|_2^2. \qquad (2.8)$$

From the last inequality we obtain

$$\|(l_t + \lambda I)u\|_2^2 \geq |t|^2 \delta_0^2 \|u\|_2^2. \qquad (2.9)$$

Now, multiplying by the number $c_0 > 0$ both parts of (2.9) we obtain

$$c_0 \|(l_t + \lambda I)u\|_2^2 \geq c_0 |t|^2 \delta_0^2 \|u\|_2^2. \qquad (2.10)$$

Combining (2.10) and (2.5) and choosing the number $c_0 > 0$ as $|t|^2 \delta_0^2 c_0 - |t|^2 \geq 0$ result in

$$2 \cdot (c_0 + \frac{1}{\delta}) \cdot \|(l_t + \lambda I)u\|_2^2 \geq \int_R (q(y) + \lambda)|u|^2 dy.$$

From the last inequality, by virtue of the condition i), we have

$$c(\delta)\cdot\|(l_t+\lambda I)u\|_2\geq(\delta+\lambda)^{\frac{1}{2}}\cdot\|u\|_2,$$

where $c(\delta)=\sqrt{2}\cdot(c_0+\frac{1}{\delta})^{\frac{1}{2}}$. The inequality (2.1) is proved. Let's take $\{\varphi_j\}$ the set of non-negative functions from $C_0^\infty(R)$ such that

$$\sum_j\varphi_j^2\equiv1,\ \sup\,p\,\varphi_j\subset\triangle_j,\ \bigcup_j\triangle_j=R,$$

where $\triangle_j=(j-1,j+1)$, $j\in Z$.

Remark 2.1. *It is easy to verify that in the system of functions* $\{\varphi_j\}_{j=-\infty}^{j=\infty}$*, supports of functions have no more than a triple intersection, i.e. any point* $y\in R$ *can belong to no more than three segments from the system of segments* $\{sup\,p\,\varphi_j\}$.

Continue $b(y)$, $q(y)$ from $\triangle_j$ for all R so that, their continuation of $b_j(y)$ and $q_j(y)$ were bounded and periodic functions of the same period.

Denote by $l_{t,j,\alpha}+\lambda I$ the closure of operator

$$(l_{t,j,\alpha}+\lambda I)u=-u''(y)+(-t^2+i(tb_j(y)+\alpha)+q_j(y)+\lambda)\cdot u \tag{2.11}$$

defined on $C_0^\infty(R)$, where the sign of a real number α coincides with the sign of the function $b(y)$, i.e. $\alpha\cdot b(y)>0$ at $y\in R$.

Lemma 2.2. *Let the condition i) be fulfilled. Then the estimate*

$$c(\delta)\cdot\|l_{t,j,\alpha}+\lambda I\|_2\geq(\delta+\lambda)^{\frac{1}{2}}\|u\|_2, \tag{2.12}$$

holds in the space $L_2(R)$ *for all* $u\in D(l_{t,j,\alpha})$ *when* $\lambda\geq0$*, where* $\|\cdot\|_2$ *is the norm in* $L_2(R^2)$, $c(\delta)>0$.

Proof. Let $u\in C_0^\infty(R)$. Then

$$|<(l_{t,j,\alpha}+\lambda I)u,u>|\geq|\int_R|u'|^2dy+\int_R(q_j(y)+\lambda)|u|^2dy|-|\int_R t^2|u|^2dy|. \tag{2.13}$$

Hence the following inequalities are obtained

$$\|(l_{t,j,\alpha}+\lambda I)u\|_2\cdot\|u\|_2\geq\int_R|u'|^2dy-\int_R t^2|u|^2dy, \tag{2.14}$$

$$\frac{1}{2\varepsilon}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2+\frac{\varepsilon}{2}\|u\|_2^2\geq\int_R[q_j+\lambda]\cdot|u|^2dy-\int_R t^2|u|^2dy.$$

From the last inequality chosing $\varepsilon=\frac{\delta}{2}$ and considering that $q_j\geq\delta$, we obtain

$$\frac{1}{\delta}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2\geq\frac{1}{2}\int_R[q_j+\lambda]\cdot|u|^2dy-t^2\int_R|u|^2dy. \tag{2.15}$$

Now, the following scalar product is considered

$$|<(l_{t,j,\alpha}+\lambda I)u,-itu>|=|-it(\int_R|u'|^2dy+\int_R[-t^2+q_j(y)+\lambda]|u|^2dy)+\int_R t^2(b_j(y)+\alpha)|u|^2dy|.$$

Hence and by virtue of the condition $\alpha \cdot b(y) > 0$ at $y \in R^n$ we have

$$| < (l_{t,j,\alpha} + \lambda I)u, -itu > | \geq t^2 \int_R (|b_j(y)| + \alpha)|u|^2 dy|.$$

From this inequality we obtain

$$\|(l_{t,j,\alpha} + \lambda I)u\|_2^2 \geq |t|^2(\delta_0 + |\alpha|)^2\|u\|_2^2. \tag{2.16}$$

Further combining inequalities (2.15) and (2.16) and choosing α as $(\delta_0 + |\alpha|)^2 - 1 \geq 0$ we have

$$\frac{1}{\delta}\|(l_{t,j,\alpha} + \lambda I)u\|_2^2 + \|(l_{t,j,\alpha} + \lambda I)u\|_2^2 \geq \frac{1}{2}\int_R [q_j + \lambda] \cdot |u|^2 dy. \tag{2.17}$$

Consequently we have

$$c(\delta) \cdot \|l_{t,j,\alpha} + \lambda I\|_2 \geq (\delta + \lambda)^{\frac{1}{2}}\|u\|_2,$$

where $c(\delta) = \sqrt{2(\frac{1}{\delta} + 1)}$. Lemma 2.2 is proved.

Lemma 2.3. *The operator $l_{t,j,\alpha} + \lambda I$ has the continuous inverse $(l_{t,j,\alpha} + \lambda I)^{-1}$ for $\lambda \geq 0$ defined in the whole space $L_2(R)$.*

Proof. From the estimate (2.12) it follows that to prove lemma 2.3 and it is need to show, that the range of values $R(l_{t,j,\alpha} + \lambda I)$ of the operator $l_{t,j,\alpha} + \lambda I$ coincides with the whole space $L_2(R)$. And also from the estimate (2.16) it follows that $\|(l_{t,j,\alpha} + \lambda I)^{-1}\|_2 \to 0$ for $|t| \to \infty$. Hence, it suffices to prove Lemma 2.3 for finite $t \neq 0$. Let's prove it by contradiction.

Let there be an element $v \in L_2(R), v \neq 0$ such that $< (l_{t,j,\alpha} + \lambda I)u, v >= 0$ for any $u \in D(l_{t,j,\alpha})$. From the last equality we obtain

$$(l_{t,j,\alpha} + \lambda I)^* v = -v'' + (-t^2 - itb_j(y) + q_j(y) + \lambda)v = 0 \tag{2.18}$$

in terms of distributions. Since $b_j(y)v,\ q_j(y)v \in L_2(R)$, then from (2.18) it follows that $v'' \in L_2(R)$ at the finite t, i.e. $v \in W_2^2(R)$. Now if the inequality

$$\|(l_{t,j,\alpha} + \lambda I)^* v_n\|_2 \geq c\|v_n\|_2 \tag{2.19}$$

holds for any $v_n \in C_0^R$, c is a constant number, then it is also true for $v \in W_2^2(R)$. Really, for $v \in W_2^2(R)$ there is a sequence $\{v_n\} \in C_0^\infty(R)$ converting to $v(y)$ in a norm $L_2(R)$. It is easy to prove that the inequality (2.19) is true for every $v_n(y)$. It is proved in the same way as inequality (2.12) in Lemma 2.2. Passing in it to the limit by $n \to \infty$, we obtain, as it is easily seen, inequality (2.19) for $v(y)$ (this procedure is shortly called the closure of the inequality (2.19) in the norm $L_2(R)$), i.e.

$$\|(l_{t,j,\alpha} + \lambda I)^* v\|_2 \geq c\|v\|_2$$

Since $(l_{t,j,\alpha} + \lambda I)^* = 0$, then from the last inequality it follows that $v \equiv 0$. Lemma 2.3 is proved.

Lemma 2.4. *Let the condition i) be fulfilled and $\lambda \geq 0$. Then the following inequalities are true:*
a)$\|(l_{t,j,\alpha} + \lambda I)^{-1}\|_{2\to 2} \leq \frac{c}{(\delta+\lambda)^{\frac{1}{2}}}, c = c(\delta) > 0;$
b)$\|\frac{d}{dy}(l_{t,j,\alpha} + \lambda I)^{-1}\|_{2\to 2} \leq \frac{c}{(\delta+\lambda)^{\frac{1}{4}}}, c > 0.$

Proof. Estimate a) follows the estimate (2.12). Also from the estimate (2.12) it follows that the inequality

$$\frac{c(\delta)}{(\delta + \lambda)^{\frac{1}{2}}}\|(l_{t,j,\alpha} + \lambda I)u\|_2 \geq \|u\|_2. \tag{2.20}$$

From the inequality (2.14) taking into account the inequality (2.20), it is found that

$$\frac{c(\delta)}{(\delta+\lambda)^{\frac{1}{2}}}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2 \geq \int\limits_R |u'|^2 dy - t^2\int\limits_R |u|^2 dy. \tag{2.21}$$

Now, by multiplying both sides of the inequality (2.16) by the number $\frac{1}{(\delta+\lambda)^{\frac{1}{2}}}$,the following inequality is found:

$$\frac{1}{\sqrt{\delta+\lambda}}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2 \geq \frac{|t|^2(\delta_0+|\alpha|)^2}{(\delta+\lambda)^{\frac{1}{2}}}\|u\|_2^2. \tag{2.22}$$

Now, by combining (2.21) and (2.22) we get

$$\frac{c(\delta)+1}{(\delta+\lambda)^{\frac{1}{2}}}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2 \geq \int\limits_R |u'|^2 dy + t^2\int\limits_R \left(\frac{(\delta_0+|\alpha|)^2}{(\delta+\lambda)^{\frac{1}{2}}}-1\right)|u|^2 dy.$$

Hence chosing α as $\frac{(\delta_0+|\alpha|)^2}{(\delta+\lambda)^{\frac{1}{2}}}-1\geq 0$. From the last inequality the following estimate is found

$$\frac{c}{(\delta+\lambda)^{\frac{1}{2}}}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2 \geq \|u'\|_2^2,$$

$c=c(\delta)+1$. The estimate b) is proved. Lemma 2.4 is completely proved.

Denote by $l_{t,\alpha}+\lambda I$ closure in $L_2(R)$ differential expression

$$(l_{t,\alpha}+\lambda I)u = -u''(y) + (-t^2 + it(b(y)+\alpha) + q(y)+\lambda)\cdot u$$

define on the set $C_0^\infty(R)$.

Lemma 2.5. *Let the condition i) be fulfilled and $\lambda\geq 0$. Then the inequalities*

$$\|(l_{0,\alpha}+\lambda I)u\|_2 \geq (\delta+\lambda)\|u\|_2, \tag{2.23}$$

$$\|(l_{t,\alpha}+\lambda I)u\|_2 \geq |t|(\delta_0+|\lambda|)\|u\|_2, t\neq 0. \tag{2.24}$$

hold for $u\in D(l_{t,\alpha}+\lambda I)$.

Proof. The inequalities (2.23) and (2.24) are proved by using functionals $<(l_{0,\alpha}+\lambda I)u,u>$, $<(l_{t,\alpha}+\lambda I)u,u>, u\in C_0^\infty(R)$. Suppose

$$K_{\lambda,\alpha}f = \sum_{\{j\}}\varphi_j(l_{t,j,\alpha}+\lambda I)^{-1}\varphi_j f, \tag{2.25}$$

where $f\in L_2(R), \{\varphi_j\}$ is a set of functions in $C_0^\infty(R)$ such that $\sum\limits_{\{j\}}\varphi_j^2\equiv 1, sup\, p\varphi_j\subset\triangle_j$, $\bigcup\limits_{\{j\}}\Delta_j=\mathbb{R},\ \ \Delta_j=(j-1,j+1),\ \ j\in Z,\ \ l_{t,j,\alpha}+\lambda I$ is the operator from Lemma 2.2.

It is easy to check that

$$(l_{t,j,\alpha}+\lambda I)K_{\lambda,\alpha}f = f - B_{\lambda,\alpha}f, \tag{2.26}$$

where

$$B_{\lambda,\alpha}f = \sum_{\{j\}}\varphi_j''(l_{t,j,\alpha}+\lambda I)^{-1}f + 2\sum_{\{j\}}\varphi_j'\frac{d}{dy}(l_{t,j,\alpha}+\lambda I)^{-1}\varphi_j f.$$

Lemma 2.6. *Let the condition i) be fulfilled. Then there is a number $\lambda_0>0$ such that*

$$\|B_{\lambda,\alpha}\|_{2\to 2} < 1 \ \ for\ \ all\ \ \lambda\geq\lambda_0.$$

Proof. Let $f \in C_0^\infty(R)$. Subsequently, taking into account that on the interval $\Delta_j (j \in Z)$ only if the functions $\varphi_{j-1},\ \varphi_{j+1}$ are different from zero, we obtain

$$\|B_{\lambda,\alpha} f\|_2^2 = \int_{-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} \varphi_j^{''} (l_{t,j\alpha} + \lambda I)^{-1} \varphi_j f + 2 \sum_{\{j\}} \varphi_j^{'} \frac{d}{dy} (l_{t,j,\alpha} + \lambda I)^{-1} \varphi_j f \right)^2 dy \leq$$

$$\leq \sum_{j=-\infty}^{\infty} \int_{\Delta_j} \left| \sum_{k=j-1}^{j+1} \left(\varphi_k^{''} (l_{t,k,\alpha} + \lambda I)^{-1} \varphi_k f + 2 \varphi_k^{'} \frac{d}{dy} (l_{t,k,\alpha} + \lambda I)^{-1} \varphi_k f \right) \right|^2 dy.$$

Hence, taking into account the previous inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ and estimates a), b) of Lemma 2.4, we obtain

$$\|B_{\lambda,\alpha} f\|_2^2 \leq c_0 \left[\frac{c}{(\delta + \lambda)} + \frac{c}{(\delta + \lambda)^{\frac{1}{2}}} \right] \cdot \|f\|_2^2,$$

where $c_0 = max \left\{ |\varphi_j^{'}|,\ |\varphi_j^{''}| \right\}$, and the constant c from Lemma 2.4.

Consequently, it is easy to find such a number $\lambda_0 > 0$, that with $\lambda \geq \lambda_0$ $\|B_{\lambda,\alpha}\|_{2\to 2} < 1$. Lemma 2.6 is proved.

Lemma 2.7. *Let the condition i) be fulfilled and $\lambda \geq \lambda_0 > 0$. Then the $l_{t,\alpha} + \lambda I$ boundedly invertible, and for the inverse operator $(l_{t,\alpha} + \lambda I)^{-1}$ fulfilled the equation:*

$$(l_{t,\alpha} + \lambda I)^{-1} = K_{\lambda,\alpha} (I - B_{\lambda,\alpha} f)^{-1} \tag{2.27}$$

Proof. The proof of the lemma follows from the representation (2.26) taking into account of Lemmas 2.5 and 2.6.

Now, consider the question of the invertibility of the initial operator $l_t + \lambda I$. To do this, consider the following equation

$$(l_t + \lambda I) u = -u^{''}(y) + (-t^2 + itb(y) + q(y) + \lambda) u = f(x) \tag{2.28}$$

where $f(x) \in L_2(R)$.

Definition 2.1. *The solution of the equation (2.28) is the function $u \in L_2(R)$, for which there is a sequence $\{u_n\}_{n=1}^\infty \subset C_0^\infty(R)$ such that:*

$$\|u_n - u\|_2 \to 0,\ \ \|(l_t + \lambda I) u_n - f\|_2 \to 0 \ \ as\ \ n \to \infty.$$

This shows that the inverse operator $(l_t + \lambda I)^{-1}$ coincides with the closure in $L_2(R)$ of the operator $l_t + \lambda I$ defined on $C_0^\infty(R)$.

Lemma 2.8. *Let the condition i) be fulfilled and $\lambda \geq \lambda_0 > 0$. Then the $l_t + \lambda I$ is boundedly invertible, and for the inverse operator $(l_t + \lambda I)^{-1}$ it has fulfilled the equation*

$$(l_t + \lambda I)^{-1} f = (l_{t,\alpha} + \lambda I)^{-1} (I - A_{\lambda,\alpha})^{-1} f, \tag{2.29}$$

where, $f \in L_2(R)$, $\|A_{\lambda,\alpha}\|_{2\to 2} < 1$.

Proof. Let be $t \neq 0$. The equation

$$(l_t + \lambda I) u = -u^{''}(y) + (-t^2 + itb(y) + q(y) + \lambda) u = f \tag{2.30}$$

can be rewriten in the form

$$v - A_{\lambda,\alpha} v = f, \tag{2.31}$$

where $v = (l_{t,\alpha} + \lambda I)u$, $A_{\lambda,\alpha} = it\alpha(l_{t,\alpha} + \lambda I)^{-1}$, $i^2 = -1$. Lemma 2.5 implies that:

$$\|A_{\lambda,\alpha}v\|_2 \leq \frac{|t| \cdot |\alpha|}{|t|(\delta_0 + |\alpha|)} < 1. \tag{2.32}$$

From (2.30)-(2.32) it follows that:

$$u = (l_t + \lambda I)^{-1} f = (l_{t,\alpha} + \lambda I)^{-1}(I - A_{\lambda,\alpha})^{-1} f$$

at $t \neq 0$.

It is known that for $t = 0$ the operator is essentially self-adjoint [26], and for all $u \in D(l_0)$ the estimate is valid

$$\|(l_0 + \lambda I)u\|_2 \geq (\delta + \lambda)\|u\|_2.$$

It follows that the operator $l_0 + \lambda I$ has a bounded inverse $(l_0 + \lambda I)^{-1}$ defined on all $L_2(R)$. Lemma 2.8 is proved.

Lemma 2.9. *Let the condition i) be fulfilled and $\lambda \geq 0$. Then for all $u \in D(l_t)$ the following estimates are true:*

$$\|(l_0 + \lambda I)u\|_2 \geq \delta\|u\|_2,$$

$$\|(l_t + \lambda I)u\|_2 \geq |t|\delta_0\|u\|_2, \quad t \neq 0.$$

Proof. Lemma 2.9 is proved the same as lemma 2.5.

The following lemma is known [27]:

Lemma 2.10. *Let the operator $l_t + \lambda_0 I$ $(\lambda_0 \geq 0)$ be boundedly invertible in $L_2(R)$ and if $\lambda \in [0, \lambda_0]$ the estimate*

$$\|(l_t + \lambda I)u\|_2 \geq c \cdot \|u\|_2,$$

holds for all $u \in D(l_t + \lambda I)$, $c > 0$ - constant number.

Then the operator $l_t : L_2(R) \to L_2(R)$ is also boundedly invertible.

From Lemmas 2.8-2.10, the following lemma is easily derived:

Lemma 2.11. *Let the condition i) be fulfilled and $\lambda \geq 0$. Then the operator $l_t + \lambda I$ boundedly invertible in the space $L_2(R)$.*

Proof of the theorem 1.1. To prove Theorem 1.1, the following lemma shoued be firstly proved for operator (1.1).

Lemma 2.12. *Let the condition i) be fulfilled and $\lambda \geq 0$. Then the estimate*

$$\|(L + \lambda I)u\|_2 \geq c \cdot \|u\|_2 \tag{2.33}$$

holds for any $u \in D(L + \lambda I)$, were $c > 0$ is a constant number.

Proof. Estimate (2.33) we prove first for real-valued functions. Since the coefficients of the operator (1.1) are real, the estimate (2.33) is valid for complex-valued functions. Let $u \in C_0^\infty$. Consider the scalar product:

$$< (L + \lambda I)u, u >= \int_{R^2} [u_{xx} - u_{yy} + (b(y)u_x + q(y) + \lambda)u]u dx dy.$$

By using the method of integration by parts and using the Cauchy inequality with ≪ $\varepsilon > 0$ ≫, we obtain

$$\frac{1}{2\delta}\|(L + \lambda I)u\|_2^2 \geq \|u_y\|_2^2 + \left(\frac{\delta}{2} + \lambda\right) \cdot \|u\|_2^2 - \|u_x\|_2^2. \tag{2.34}$$

Further, consider the expression

$$< (L+\lambda I)u, u_x >= \int_{R^2} u_{xx}u_x dxdy - \int_{R^2} u_{yy}u_x dxdy + \int_{R^2} b(y)u_x^2 dxdy + \int_{R^2} (q(y)+\lambda)uu_x dxdy$$

Applying the method of integration by parts for each term the following is obtained

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_x u_{xx} dxdy = 0;$$

$$I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_{yy} u_x dxdy = 0;$$

$$I_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (q(y)+\lambda) uu_x dxdy = 0.$$

Given this and applying the Cauchy inequality with $\ll \varepsilon > 0 \gg$, we have

$$\frac{1}{2\varepsilon}\|(L+\lambda I)u\|_2^2 \geq \left(\min_{y\in R}|b(y)| - \frac{\varepsilon}{2}\right)\cdot \|u_x\|_2^2.$$

Choosing, $\varepsilon = \delta_0$ we find:

$$\frac{1}{\delta_0}\|(L+\lambda I)u\|_2^2 \geq \delta_0 \cdot \|u_x\|_2^2. \tag{2.35}$$

Further, multiplying both sides of the inequality (2.35) by $\frac{1}{\delta_0}$ and combining with (2.34), we obtain

$$\left(\frac{1}{2\delta} + \frac{1}{\delta_0^2}\right)\cdot \|(L+\lambda I)u\|_2^2 \geq \|u_y\|_2^2 + \left(\frac{\delta}{2}+\lambda\right)\cdot \|u\|_2^2 - \|u_x\|_2^2 + \|u_x\|_2^2$$

Hence

$$\left(\frac{1}{2\delta} + \frac{1}{\delta_0^2}\right)\cdot \|(L+\lambda I)u\|_2^2 \geq \left(\frac{\delta}{2}+\lambda\right)\cdot \|u\|_2^2 \geq \frac{\delta}{2}\cdot \|u\|_2^2. \tag{2.36}$$

The last inequality proves Lemma 2.12.

Now, the existence of the inverse operator $(L+\lambda I)^{-1}$ of $L+\lambda I$ should be shown.

Definition 2.2. *The solution of the equation* $(L+\lambda I)u = f$ *is the function* $u \in L_2(R^2)$, *for which there is a sequence* $\{u_n\}_{n=1}^{\infty} \subset C_0^{\infty}(R^2)$ *such that*

$$\|u_n - u\|_2 \to 0, \ \ \|(L+\lambda I)u_n - f\|_2 \to 0 \ \ at \ \ n\to\infty.$$

This shows that the inverse operator $(L+\lambda I)^{-1}$ coinsides with the closure in $L_2(R^2)$ of the operator $L+\lambda I$ defined on $C_0^{\infty}(R^2)$.

Consider the equation:

$$(L+\lambda I)u = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + (q(y)+\lambda)u = f \in C_0^{\infty}(R^2).$$

By applying the Fourier transformation by x, the following equation is obtained

$$(l_t + \lambda I)\tilde{u} = -\tilde{u}''(y) + (-t^2 + itb(y) + q(y) + \lambda)\tilde{u} = \tilde{f}(t,y),$$

where $\tilde{u}, \tilde{f}$ are Fourier transform with respect to the variable x of functions $u(x,y)$ and $f(x,y)$.

In the following the Fourier transformation is denoted by $F_{x\to t}$, and the Fourier inversion is denoted by $F^{-1}_{t\to x}$.

From Lemma 2.11 and using the properties of the Fourier transform we have

$$u(x,y) = (L+\lambda I)^{-1} f = F^{-1}_{t\to x}(l_t+\lambda I)^{-1}\tilde{f}. \tag{2.37}$$

The last equality by virtue of the continuity of the operator $(l_t+\lambda I)^{-1}$ and the Fourier transformation holds for all $f(x,y)\in L_2(R^2)$. Uniqueness follows from Lemmas 2.12. Theorem 1.1 is proved.

3. About the separability of the operator $L+\lambda I$.

In this section, we carry out a series of statements and estimates that reduce the separability of an operator with unbounded coefficients to the case of an operator with periodic coefficients.

Lemma 3.1. *Let the condition i)-iii) be fulfilled. Then the following inequalities are true*

$$\|(l_{t,j,\alpha}+\lambda I)^{-1}\|_{2\to 2} \le \frac{1}{|t|\cdot|b(\tilde{y}_j)|},\quad t\neq 0; \tag{3.1}$$

$$\|(l_{t,j,\alpha}+\lambda I)^{-1}\|_{2\to 2} \le \frac{1}{q(\overline{y}_j)+\lambda},\quad t\neq 0, \tag{3.2}$$

where $|b(\tilde{y}_j)| = \min\limits_{y\in\overline{\triangle}_j}|b(y)|$, $|q(\overline{y}_j)| = \min\limits_{y\in\overline{\triangle}_j}|q(y)|$, $c>0$.

Proof. For any $u\in C_0^\infty(R^2)$ we have

$$< (l_{t,j,\alpha}+\lambda I)u,u >= \int\limits_R \left(|u'|^2+(-t^2+q_j(y)+\lambda)|u|^2\right)dy + \int\limits_R it(b_j(y)+\alpha)|u|^2dy. \tag{3.3}$$

Hence, taking into account the condition i) and using the Cauchi-Bunyakovsky inequality, we find

$$\|(l_{t,j,\alpha}+\lambda I)u\|_2^2 \ge |t|^2\cdot\left(|b_j(\tilde{y})|+|\alpha|\right)^2\|u\|_2^2 \ge |t|^2\cdot|b_j(\tilde{y})|^2\|u\|_2^2. \tag{3.4}$$

Further, from inequality (3.4), according to the definition of the norm of the operator, we obtain

$$\|(l_{t,j,\alpha}+\lambda I)u\|_{2\to 2} \le \frac{1}{|t|\cdot|b(\tilde{y}_j)|},\quad t\neq 0.$$

Inequality (3.1) is proved.

From inequality (3.3), by virtue of the Cauchy inequality with $\ll \varepsilon>0\gg$, we obtain

$$\frac{1}{2\left(q(\overline{y}_j)+\lambda\right)}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2+\frac{q(\overline{y}_j)+\lambda}{2}\|u\|_2^2 \ge \int\limits_R\left[|u'|^2+(q_j(y)+\lambda)|u|^2\right]dy - \int\limits_R t^2|u|^2,$$

where $\varepsilon = q(\overline{y}_j)+\lambda$. Hence

$$\frac{1}{2\left(q(\overline{y}_j)+\lambda\right)}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2 \ge \int\limits_R|u'|^2dy+\frac{q(\overline{y}_j)+\lambda}{2}\int\limits_R|u|^2dy-\int\limits_R t^2|u|^2dy. \tag{3.5}$$

We divide both sides of the inequality (3.4) by $2\left(q(\overline{y}_j)+\lambda\right)$:

$$\frac{1}{2\left(q(\overline{y}_j)+\lambda\right)}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2 \geq \frac{\left[|t|\cdot\left(|b(\tilde{y}_j)|+|\alpha|\right)\right]^2}{2\left(q(\overline{y}_j)+\lambda\right)}\|u\|_2^2. \tag{3.6}$$

Since $\overline{y}_j$, $\tilde{y}_j \in \triangle_j$, then from the condition ii) we have

$$\mu_1^{-1} \leq \frac{q(\overline{y}_j)}{q(\tilde{y}_j)} \leq \mu_1, \quad \mu_1 \geq 1.$$

By virtue of the last inequality, it is easy to verify the validity of the following inequality

$$\frac{1}{2\left(q(\overline{y}_j)+\lambda\right)}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2 \geq \frac{|t|^2(|b(\tilde{y}_j)|+|\alpha|)^2}{2\mu_1\left(q(\tilde{y}_j)+\lambda\right)}\|u\|_2^2. \tag{3.7}$$

Multiplying both sides of (3.7) by the number c¿0 and combining with inequality (3.5) we get

$$\frac{c+1}{2(q(\overline{y}_j)+\lambda)}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2 \geq \|u'\|_2^2 + \frac{q(\overline{y}_j)+\lambda}{2}\|u\|_2^2 + t^2\int_R \left[\frac{c\left(|b(\tilde{y}_j)|+|\alpha|\right)^2}{2\mu_1\left(q(\tilde{y}_j)+\lambda\right)}-1\right]\cdot|u|^2dy. \tag{3.8}$$

Hence taking into account the condition iii) and choosing α and c so that $\frac{c\cdot\left(|b(\tilde{y}_j)|+|\alpha|\right)^2}{2\mu_1\left(q(\tilde{y}_j)+\lambda\right)}-1 \geq 0$, we get

$$\frac{c+1}{2(q(\overline{y}_j)+\lambda)}\|(l_{t,j,\alpha}+\lambda I)u\|_2^2 \geq \|u'\|_2^2 + \frac{q(\overline{y}_j)+\lambda}{2}\|u\|_2^2.$$

Hence we find

$$c_1\cdot\|(l_{t,j,\alpha}+\lambda I)u\|_2^2 \geq (q(\overline{y}_j)+\lambda)^2\|u\|_2^2, \tag{3.9}$$

where $c_1 = c+1$.

From (3.9), according to the definition of the norm, we have:

$$\|(l_{t,j,\alpha}+\lambda I)^{-1}\|_{2\to 2} \leq \frac{c_1}{q(\overline{y}_j)+\lambda}.$$

Lemma 3.1 is proved.

Lemma 3.2. *Let the conditions of the lemma 3.1 be fulfilled and $\lambda > 0$ be such that $\|B_{\lambda,\alpha}\|_{2\to 2} < 1$. Then the following estimate is fair*

$$\|\rho(y)|t|^\alpha(l_{t,\alpha}+\lambda I)^{-1}\|_2^2 \leq c(\lambda)\sup_{\{j\}}\|\rho(y)|t|^\alpha\varphi_j(l_{t,j,\alpha}+\lambda I)^{-1}\|_2^2, \tag{3.10}$$

where $\alpha = 0, 1$, $\rho(y)$ - continuous function in R.

Proof. From representation (2.27) it is clear that the operator $\rho(y)|t|^\alpha(l_{t,\alpha}+\lambda I)^{-1}$ is bounded (or unbounded) together with the operator $\rho(y)|t|^\alpha K_{\lambda,\alpha}(I-B_{\lambda,\alpha})^{-1}$. Therefore, the estimate of the operator's norm $\rho(y)|t|^\alpha K_{\lambda,\alpha}(I-B_{\lambda,\alpha})^{-1}$ will be considered. For any $f \in L_2(R)$ we have:

$$\|\rho(y)|t|^\alpha(l_{t,\alpha}+\lambda I)^{-1}f\|_2^2 = \left\|\rho(y)|t|^\alpha\sum_{\{j\}}\varphi_j(l_{t,j,\alpha}+\lambda I)^{-1}\varphi_j(I-B_{\lambda,\alpha})^{-1}f\right\|_2^2 \leq$$

$$\leq \sum_{\{j\}} \int\limits_{j-1}^{j+1} \left| \sum_{\{j\}} \rho(y)|t|^{\alpha} \varphi_j (l_{t,j,\alpha} + \lambda I)^{-1} \varphi_j (I - B_{\lambda,\alpha})^{-1} f \right|^2 dy.$$

It is not difficult to verify that on $\overline{\triangle}_j = [j-1, j+1]$ is only $\varphi_{j-1}, \ \varphi_j, \ \varphi_{j+1} \neq 0$, considering this and the well-known inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ we have:

$$\|\rho(y)|t|^{\alpha}(l_{t,\alpha} + \lambda I)^{-1} f\|_2^2 \leq \sum_{\{j\}} \int\limits_{j-1}^{j+1} \left| \sum_{k=j-1}^{j+1} \rho(y)|t|^{\alpha} \varphi_k (l_{t,j,\alpha} + \lambda I)^{-1} \varphi_k (I - B_{\lambda,\alpha})^{-1} f \right|^2 dy \leq$$

$$\leq 3 \sum_{\{j\}} \int\limits_{\triangle_j} \sum_{k=j-1}^{j+1} \left| \rho(y)|t|^{\alpha} \varphi_k (l_{t,k,\alpha} + \lambda I)^{-1} \varphi_k (I - B_{\lambda,\alpha})^{-1} f \right|^2 dy \leq$$

$$\leq 9 \sum_{\{j\}} \left\| \rho(y)|t|^{\alpha} \varphi_j (l_{t,j,\alpha} + \lambda I)^{-1} \varphi_j (I - B_{\lambda,\alpha})^{-1} f \right\|^2 dy =$$

$$= 9 \sup_{\{j\}} \left\| \rho(y)|t|^{\alpha} \varphi_j (l_{t,j,\alpha} + \lambda I)^{-1} \right\|_2^2 \cdot \left\| (I - B_{\lambda,\alpha})^{-1} f \right\|_2^2 \leq$$

$$\leq 9 \cdot c(\lambda) \sup_{\{j\}} \left\| \rho(y)|t|^{\alpha} \varphi_j (l_{t,j,\alpha} + \lambda I)^{-1} \right\|_2^2 \cdot \left\| f \right\|_2^2,$$

where $c(\lambda) = \|(I - B_{\lambda,\alpha})^{-1}\|_{2\to 2}^2, \sum\limits_{\{j\}} \varphi_j^2 \equiv 1$.

Hence we have

$$\|\rho(y)|t|^{\alpha}(l_{t,\alpha} + \lambda I)^{-1}\|_2^2 \leq 9 \cdot c(\lambda) \sup_{\{j\}} \left\| \rho(y)|t|^{\alpha} \varphi_j (l_{t,j,\alpha} + \lambda I)^{-1} \right\|_2^2.$$

Lemma 3.2 is proved.

Lemma 3.3. *Let the conditions of the lemma 3.2 be fulfilled. Then the following estimates are true:*
a) $\|q(y)(l_{t,\alpha} + \lambda I)^{-1}\|_{2\to 2} \leq c_1 < \infty$;
b) $\|itb(y)(l_{t,\alpha} + \lambda I)^{-1}\|_{2\to 2} \leq c_2 < \infty$;
c) $\|\frac{d}{dy}(l_{t,\alpha} + \lambda I)^{-1}\|_{2\to 2} \leq c_3 < \infty$.

Proof. According to Lemma 3.2, we have:

$$\|q(y)(l_{t,\alpha} + \lambda I)^{-1}\|_{2\to 2} \leq c(\lambda) \sup_{\{j\}} \left\| q(y)|t|^{\alpha} \varphi_j (l_{t,j,\alpha} + \lambda I)^{-1} \right\|_{2\to 2}.$$

Hence and from (3.1) we get

$$\|q(y)(l_{t,\alpha} + \lambda I)^{-1}\|_{2\to 2} \leq c(\lambda) \sup_{\{j\}} \left\| q(y) \varphi_j (l_{t,j,\alpha} + \lambda I)^{-1} \right\|_{2\to 2} \leq$$

$$\leq c(\lambda) \frac{\max\limits_{y\in\triangle_j} |q(y)\varphi_j|}{|q(\overline{y}_j)|} \leq c(\lambda) \frac{\max\limits_{y\in\triangle_j} |q(y)|}{\min\limits_{y\in\triangle_j} q(y)} \leq c(\lambda) \sup_{|y-t|\leq 1} \frac{q(y)}{q(t)} \leq c_1 < \infty.$$

The inequality a) of Lemma 3.3. is proved.

Further, using inequality (3.1) we get

$$\|itb(y)(l_{t,\alpha} + \lambda I)^{-1}\|_{2\to 2} \leq c(\lambda) \sup_{\{j\}} \left\| itb(y) \varphi_j (l_{t,j,\alpha} + \lambda I)^{-1} \right\|_{2\to 2} \leq$$

$$\leq c(\lambda) \sup_{|y-t|\leq 1} \frac{b(y)}{b(t)} \leq c_2 < \infty.$$

Inequality b) of Lemma 3.3. is proved.

Now, we prove the inequality c) of Lemma 3.3. From the representation (2.27) we have

$$\left\| \frac{d}{dy}(l_{t,\alpha} + \lambda I)^{-1} f \right\|_2^2 = \left\| \frac{d}{dy} K_{\lambda,\alpha}(I - B_{\lambda,\alpha})^{-1} f \right\|_2^2 \leq$$

$$\leq \left\| \sum_{\{j\}} \varphi_j^{'}(l_{t,j,\alpha} + \lambda I)^{-1} \varphi_j (I - B_{\lambda,\alpha})^{-1} f + \sum_{\{j\}} \varphi_j \frac{d}{dy}(l_{t,j,\alpha} + \lambda I)^{-1} \varphi_j (I - B_{\lambda,\alpha})^{-1} f \right\|_2^2 \leq$$

$$\leq c_0 \cdot c(\lambda) \cdot \sup_{\{j\}} \left\| (l_{t,j,\alpha} + \lambda I)^{-1} \right\|_2^2 \cdot \|f\|_2^2 + c_0 \cdot \sup_{\{j\}} \left\| \frac{d}{dy}(l_{t,j,\alpha} + \lambda I)^{-1} \right\|_2^2 \cdot \left\| f \right\|_2^2, \tag{3.11}$$

where $c_0 = max\left(\sup_{y\in R} |\varphi^{'}(y)|, \ \sup_{y\in R} \varphi_j(y) \right)$, $c(\lambda = \|(I - B_{\lambda,\alpha})^{-1}\|_{2\to 2}^2$.

From (3.11), according to Lemma 2.4, we find

$$\left\| \frac{d}{dy}(l_{t,\alpha} + \lambda I)^{-1} f \right\|_2^2 \leq c_3 \cdot \|f\|_2^2,$$

where $c_3 = c_0 \cdot c(\lambda) \cdot \left(\frac{c(\delta)}{(\delta+\lambda)} + \frac{c(\delta)}{(\delta+\lambda)^{\frac{1}{2}}} \right)$.

The inequality c) of Lemma 3.3 is proved.

Lemma 3.4. *Let the conditions of the lemma 3.3 be fulfilled. Then the following estimates are true*

a) $\|q(y)(l_t + \lambda I)^{-1}\|_{2\to 2} \leq c_4 < \infty$;

b) $\|itb(y)(l_t + \lambda I)^{-1}\|_{2\to 2} \leq c_5 < \infty$;

c) $\|\frac{d}{dy}(l_t + \lambda I)^{-1}\|_{2\to 2} \leq c_6 < \infty$.

Proof. From the representation (2.29) we have:

$$\left\| q(y)(l_t + \lambda I)^{-1} f \right\|_2^2 = \left\| q(y)(l_{t,\alpha} + \lambda I)^{-1}(I - A_{\lambda,\alpha})^{-1} f \right\|_2^2 \leq$$

$$\leq \left\| q(y)(l_{t,\alpha} + \lambda I)^{-1} \right\|_{2\to 2}^2 \cdot \left\| (I - A_{\lambda,\alpha})^{-1} f \right\|_2^2. \tag{3.12}$$

Hence, according to Lemma 3.3 we find

$$\left\| q(y)(l_t + \lambda I)^{-1} f \right\|_2^2 \leq c_1 \left\| (I - A_{\lambda,\alpha})^{-1} \right\|_{2\to 2} \cdot \|f\|_2. \tag{3.13}$$

Since $\left\| (I - A_{\lambda,\alpha})^{-1} \right\|_{2\to 2} \leq c_0 < \infty$, then (3.13) it follows that

$$\left\| q(y)(l_t + \lambda I)^{-1} f \right\|_{2\to 2} \leq c_1 \cdot c_0 < c_4 < \infty.$$

Now, we prove the inequality b) of Lemma 3.4. Using the inequality b) of Lemma 3.3, we have

$$\left\| itb(y)(l_t + \lambda I)^{-1} f \right\|_2^2 \leq \left\| itb(y)(l_{t,\alpha} + \lambda I)^{-1} \right\|_{2\to 2}^2 \cdot \left\| (I - A_{\lambda,\alpha})^{-1} \right\|_{2\to 2}^2 \cdot \|f\|_2^2.$$

Hence we have $\left\| itb(y)(l_t + \lambda I)^{-1} \right\|_{2\to 2} \leq c_2 \cdot c_0 < c_5 < \infty$.

The inequality b) of Lemma 3.4 is proved.

Further, using the clause *b*) of Lemma 3.3, we obtain

$$\left\|\frac{d}{dy}(l_t+\lambda I)^{-1}f\right\|_2^2 \le \left\|\frac{d}{dy}(l_{t,\alpha}+\lambda I)^{-1}\right\|_{2\to 2}^2 \cdot \left\|(I-A_{\lambda,\alpha})^{-1}\right\|_{2\to 2}^2 \cdot \|f\|_2^2 \le c_3\cdot c_0\cdot \|f\|_2^2$$

Lemma 3.4 is proved.

Proof of Theorem 1.2. According to Theorem 1.1, the inverse operator $(L+\lambda I)^{-1}$ to the operator $L+\lambda I$ has the form

$$u(x,\ y) = (L+\lambda I)^{-1}f = F^{-1}_{t\to y}(l_t+\lambda I)^{-1}\tilde{f}. \tag{3.14}$$

From (3.14) using the properties of the Fourier transform we obtain

$$b(y)u_x = b(y)D_x(L+\lambda I)^{-1}f = b(y)\frac{\partial}{\partial x}F^{-1}_{t\to x}(l_t+\lambda I)^{-1}\tilde{f} =$$

$$= b(y)\frac{\partial}{\partial x}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(l_t+\lambda I)^{-1}\tilde{f}(t,y)e^{ixt}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}b(y)it(l_t+\lambda I)^{-1}\tilde{f}(t,y)e^{ixt}dt =$$

$$F^{-1}_{t\to x}(itb(y))(l_t+\lambda I)^{-1}\tilde{f}(t,y).$$

Hence and from the properties of the Fourier transform we find

$$\|b(y)u_x\|_2^2 = \left\|b(y)D_x(L+\lambda I)^{-1}f\right\|_2^2 =$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left|itb(y)(l_t+\lambda I)^{-1}\tilde{f}(t,y)\right|^2 dy\right)dt = \int_{-\infty}^{\infty}\left\|itb(y)(l_t+\lambda I)^{-1}\tilde{f}(t,y)\right\|_2^2 dt \le$$

$$\le \int_{-\infty}^{\infty}\left\|itb(y)(l_t+\lambda I)^{-1}\right\|_{2\to 2}^2\cdot\left\|\tilde{f}(t,y)\right\|_2^2 dt \le \sup_{t\in R}\left\|itb(y)(l_t+\lambda I)^{-1}\right\|_{2\to 2}^2\cdot\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left|\tilde{f}(t,y)\right|^2 dy\right)dt =$$

$$= \sup_{t\in R}\left\|itb(y)(l_t+\lambda I)^{-1}\right\|_2^2\cdot\|f(x,y)\|_2^2.$$

From the last inequality and the inequality b) of lemma 3.4 it follows that

$$\|b(y)u_x\|_2 = c_5\|(L+\lambda I)u\|_2, \tag{3.15}$$

where $(L+\lambda I)u = f$. Using clause *a*) of Lemma 3.4 we also find that

$$\|q(y)u\|_2^2 = \int_{-\infty}^{\infty}\left\|q(y)(l_t+\lambda I)^{-1}\tilde{f}(t,y)\right\|_2^2 dt \le$$

$$\le \sup_{t\in R}\left\|q(y)(l_t+\lambda I)^{-1}\right\|_{2\to 2}^2\cdot\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left|\tilde{f}(t,y)\right|^2 dy\right)dt \le c_4\cdot\|(L+\lambda I)u\|_2, \tag{3.16}$$

where $(L+\lambda I)u = f$.

Using representation (3.14) and the properties of the Fourier transform, we obtain

$$\|u_y\|_2^2 \le \int_{-\infty}^{\infty}\left\|\frac{\partial}{\partial y}(l_t+\lambda I)^{-1}\right\|_{2\to 2}^2\cdot\left\|\tilde{f}(t,y)\right\|_2^2 dt \le \sup_{t\in R}\left\|\frac{\partial}{\partial y}(l_t+\lambda I)^{-1}\right\|_{2\to 2}^2\cdot\int_{-\infty}^{\infty}\left\|\tilde{f}(t,y)\right\|_2^2 dt$$

Considering the inequality c) of Lemma 3.4 from the last inequality, we obtain

$$||u_y||_2 \leq c_6 \cdot ||(L + \lambda I)u||_2, \tag{3.17}$$

where $(L + \lambda I)u = f$.

Using inequalities (3.15)-(3.17), we obtain the following inequality

$$||u_{xx} - u_{yy}|| = \left\|(L + \lambda I)u - b(y)u_x - q(y)u - \lambda u\right\|_2 \leq$$

$$||(L + \lambda I)u||_2 + ||b(y)u_x||_2 + ||q(y)u||_2 + \lambda||u||_2 \leq c(\lambda)||(L + \lambda I)u||_2, \tag{3.18}$$

where $c(\lambda) > 0$-constant number independent of $u(x, y)$.

Inequalities (3.15)-(3.18) imply that for any $u(x, y) \in D(L)$

$$||u_{xx} - u_{yy}|| + ||b(y)u_x||_2 + ||q(y)u||_2 + ||u_y||_2 \leq c \cdot ||(L + \lambda I)u||_2,$$

$c > 0$-constant number. Theorem 1.2 is proved

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