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BOUNDEDNESS AND COMPACTNESS OF A CERTAIN CLASS OF MATRIX OPERATORS WITH VARIABLE LIMITS OF SUMMATION

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Abstract. Necessary and sufficient conditions for the boundedness and compactness of the matrix operator of the form $(Af)_n = \sum_{k=\alpha(n)}^{\beta(n)} a_{n,k} f_k$, from $l_{p,v}$ to $l_{q,u}$ when $1 < p \leq q < \infty$ are given.

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1 Introduction

Let $1 < p, q < \infty, \frac{1}{p} + \frac{1}{p'} = 1$. Let $\{\omega_i\}_{i=1}^{\infty}$, $\{u_i\}_{i=1}^{\infty}$ be non-negative, $\{v_i\}_{i=1}^{\infty}$ positive sequences of real numbers, which will be referred to as weights. Let $l_{p,v}$ be the space of sequences $f = \{f_i\}_{i=1}^{\infty}$, for which the following norm is finite:

$$\|f\|_{p,v} := \left(\sum_{i=1}^{\infty} |f_i v_i|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

In this paper we consider the problems of boundedness and compactness of matrix operators of the following form

$$(Af)_n = \sum_{k=\alpha(n)}^{\beta(n)} a_{n,k} f_k, \quad n \geq 1 \tag{1.1}$$

from $l_{p,v}$ to $l_{q,u}$, where $(a_{n,k})$ is a non-negative matrix of operator A, which satisfy the following Oinarov's discrete general condition: there exists $d \geq 1$, a sequence of positive numbers $\{\omega_i\}_{i=1}^{\infty}$ and a non-negative matrix $(b_{i,j})$, such that the inequalities

$$\frac{1}{d}(b_{n,k}\omega_m + a_{k,m}) \leq a_{n,m} \leq d(b_{n,k}\omega_m + a_{k,m}) \tag{1.2}$$

holds for all $1 \leq k \leq n$, $\alpha(n) \leq m \leq \beta(k)$, where $\alpha(n)$, $\beta(n)$ are sequences of the natural numbers such that:

$$\begin{array}{ll} (i) & \alpha(n) \text{ and } \beta(n) \text{ are strictly increasing sequences;} \\ (ii) & \alpha(1) = \beta(1) = 1 \text{ and } \alpha(n) < \beta(n), \text{ for } n \geq 2. \end{array} \tag{1.3}$$

Note that from (1.3) it follows $n \leq \alpha(n) < \beta(n)$ for $n \geq 2$.

An analogue of this question for continuous operators has been studied in a series of papers [9], [12]-[15].

When $a_{n,k} = 1$, operator (1.1) coincides with the discrete Hardy type operator with variable limits of summation of the following form

$$(Hf)_n = \sum_{k=\alpha(n)}^{\beta(n)} f_k, \quad n \geq 1, \tag{1.4}$$

its boundedness from $l_{p,v}$ to $l_{q,u}$ was studied in [1], [2].

When $\alpha(n) = 1$, $\beta(n) = n$, $\forall n \in \mathbb{N}$ in (1.4) we obtain the discrete Hardy operator, which is investigated in detail in [3], [5], [6]. References about generalizations of the original forms of the discrete and continuous Hardy inequalities can be found in various books, see e.g [8]. In [10], [11], [16] necessary and sufficient conditions for the boundedness of the matrix operator (1.1) have been obtained under the different assumptions for the entries of the matrix $(a_{n,k})$, when $\alpha(n) = 1$, $\beta(n) = n$, $\forall n \in \mathbb{N}$.

We note that from (1.2) it easily follows that

$$a_{k,m} \leq d a_{n,m}, \tag{1.5}$$

$$b_{n,k}\omega_m \leq d a_{n,m} \tag{1.6}$$

for $1 \leq k \leq n$, $\alpha(n) \leq m \leq \beta(k)$.

In the sequel we suppose that the symbol $M \ll K$ means $M \leq cK$, where a positive constant c may depend only on parameters such as p, q and d. If $M \ll K \ll M$, then we write $M \approx K$.

2 Main results

Let $s \in \mathbb{N}$. We assume $\Omega(s) := \{n \in \mathbb{N} : \alpha(n) \leq s\}$. Note that $\Omega(s) \neq \varnothing$ since at the least $1 \in \Omega(s)$. For all $s \in \mathbb{N}$ we denote $\alpha^{-1}(s) := \max \Omega(s)$. Hence it follows that

$$\alpha^{-1}(\alpha(s)) = s, \quad \alpha(\alpha^{-1}(s)) \leq s.$$

Let $m \in \mathbb{N}$. From the condition (1.3) it follows that $\Omega_1 := \{s \in \mathbb{N} : m \leq s \leq \alpha^{-1}(\beta(m))\} \neq \varnothing$ since $m \in \Omega_1$.

Our first result reads as follows.

Theorem 2.1. *Let $1 < p \leq q < \infty$. Let the entries of matrix $(a_{n,k})$ satisfy condition (1.2). Then operator (1.1) is bounded from $l_{p,v}$ to $l_{q,u}$ if and only if $F = F_1 + F_2 < \infty$, where*

$$F_1 = \sup_{m \geq 1} \sup_{m \leq s \leq \alpha^{-1}(\beta(m))} \left(\sum_{n=m}^{s} u_n^q \right)^{\frac{1}{q}} \left(\sum_{k=\alpha(s)}^{\beta(m)} a_{m,k}^{p'} v_k^{-p'} \right)^{\frac{1}{p'}},$$

$$F_2 = \sup_{m \geq 1} \sup_{m \leq s \leq \alpha^{-1}(\beta(m))} \left(\sum_{n=m}^{s} u_n^q b_{n,m}^q \right)^{\frac{1}{q}} \left(\sum_{k=\alpha(s)}^{\beta(m)} \omega_k^{p'} v_k^{-p'} \right)^{\frac{1}{p'}}.$$

Moreover, $\|A\|_{l_{p,v} \to l_{q,u}} \approx F$.

Proof. Necessity. Suppose that operator (1.1) is bounded from $l_{p,v}$ to $l_{q,u}$ that equivalently means the validity of the following inequality

$$\left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=\alpha(n)}^{\beta(n)} a_{n,k} f_k \right)^q \right)^{\frac{1}{q}} \leq \|A\| \left(\sum_{n=1}^{\infty} f_n^p v_n^p \right)^{\frac{1}{p}}, \quad \forall f \geq 0. \tag{2.1}$$

Here and in the sequel $\|A\| \equiv \|A\|_{l_{p,v}\to l_{q,u}}$.

Let $m \in \mathbb{N}$ and $m \le s \le \alpha^{-1}(\beta(m))$. Then we take the following test sequence $\bar{f}_i = \chi_{[\alpha(s),\beta(m)]}(i) a_{m,i}^{p'-1} v_i^{-p'}$, where $\chi_{[\alpha(s),\beta(m)]}(i) = \begin{cases} 1, i \in [\alpha(s),\beta(m)]; \\ 0, i \notin [\alpha(s),\beta(m)]. \end{cases}$

Substituting the test sequence in the right-hand side of (2.1) we have

$$\|\bar{f}\|_{p,v} = \left(\sum_{i=1}^{\infty} \bar{f}_i^p v_i^p\right)^{\frac{1}{p}} = \left(\sum_{i=\alpha(s)}^{\beta(m)} a_{m,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p}}, \tag{2.2}$$

Substituting the test sequence in the left-hand side of inequality (2.1) and using (1.5) we obtain that

$$\|A\bar{f}\|_{q,u} = \left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=\alpha(n)}^{\beta(n)} a_{n,k}\bar{f}_k\right)^q\right)^{\frac{1}{q}} \ge \left(\sum_{n=m}^{s} u_n^q \left(\sum_{k=\alpha(s)}^{\beta(m)} a_{n,k} a_{m,k}^{p'-1} v_k^{-p'}\right)^q\right)^{\frac{1}{q}}$$

$$\ge \frac{1}{d}\left(\sum_{k=\alpha(s)}^{\beta(m)} a_{m,k}^{p'} v_k^{-p'}\right)\left(\sum_{n=m}^{s} u_n^q\right)^{\frac{1}{q}}. \tag{2.3}$$

From (2.1), (2.2) and (2.3) it follows that

$$\left(\sum_{k=\alpha(s)}^{\beta(m)} a_{m,k}^{p'} v_k^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{n=m}^{s} u_n^q\right)^{\frac{1}{q}} \ll \|A\|$$

for all $m, s \ge 1$ such that $m \le s \le \alpha^{-1}(\beta(m))$. Therefore

$$F_1 \ll \|A\|. \tag{2.4}$$

Now we assume that $\tilde{f}_i = \chi_{[\alpha(s),\beta(m)]}(i)\omega_i^{p'-1} v_i^{-p'}$, and we apply the test sequence to (2.1). For the right-hand side of (2.1) it yelds that

$$\|\tilde{f}\|_{p,v} = \left(\sum_{i=1}^{\infty} \tilde{f}_i^p v_i^p\right)^{\frac{1}{p}} = \left(\sum_{i=\alpha(s)}^{\beta(m)} \omega_i^{p'} v_i^{-p'}\right)^{\frac{1}{p}}. \tag{2.5}$$

Substituting $\tilde{f}$ in the left-hand side of inequality (2.1) and using (1.6) we find that

$$\|A\tilde{f}\|_{q,u} = \left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=\alpha(n)}^{\beta(n)} a_{n,k}\tilde{f}_k\right)^q\right)^{\frac{1}{q}} \ge \left(\sum_{n=m}^{s} u_n^q \left(\sum_{k=\alpha(s)}^{\beta(m)} a_{n,k} \omega_k^{p'-1} v_k^{-p'}\right)^q\right)^{\frac{1}{q}}$$

$$\ge \frac{1}{d}\left(\sum_{n=m}^{s} u_n^q b_{n,m}^q\right)^{\frac{1}{q}} \left(\sum_{k=\alpha(s)}^{\beta(m)} \omega_k^{p'} v_k^{-p'}\right). \tag{2.6}$$

Since $m, s \ge 1$ are arbitrary, such that $m \le s \le \alpha^{-1}(\beta(m))$, then (2.1), (2.5) and (2.6) imply that

$$F_2 \ll \|A\|. \tag{2.7}$$

Sufficiency. Let $F < \infty$ and $0 \le f \in l_{p,v}$. To prove the boundedness of operator (1.1) we use the discrete case of the block-diagonal method (see [2]). The continuous analogue of this method is called the Batuev-Stepanov block-diagonal method [4]. For given sequences $\alpha(n), \beta(n)$ which satisfy (1.3) we select the sequences of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ and $\{n'_k\}_{k\in\mathbb{N}}$ the following way

$$n_1 = 1, \quad n'_k = \alpha^{-1}\left(\beta(n_k)\right) \quad \text{and} \quad n'_k + 1 = n_{k+1}, \quad k \geq 1.$$

Obviously that $n'_1 = 1$ and

$$\alpha(n'_k) = \alpha\left(\alpha^{-1}(\beta(n_k))\right) \leq \beta(n_k) < \alpha(n_{k+1}). \tag{2.8}$$

Splitting the set $\mathbb{N}$ into the sequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{n'_k\}_{k\in\mathbb{N}}$ we have

$$\|Af\|^q_{q,u} = \sum_{n=1}^{\infty} u_n^q \left(\sum_{i=\alpha(n)}^{\beta(n)} a_{n,i} f_i\right)^q = \sum_k \sum_{n=n_k}^{n'_k} u_n^q \left(\sum_{i=\alpha(n)}^{\beta(n)} a_{n,i} f_i\right)^q =$$

(use the relations $\alpha(n_k) \leq \alpha(n) \leq \alpha(n'_k) \leq \beta(n_k)$)

$$\approx \sum_k \sum_{n=n_k}^{n'_k} u_n^q \left(\sum_{i=\alpha(n)}^{\beta(n_k)} a_{n,i} f_i + \sum_{i=\beta(n_k)}^{\beta(n)} a_{n,i} f_i\right)^q \approx \sum_k \sum_{n=n_k}^{n'_k} u_n^q \left(\sum_{i=\alpha(n)}^{\beta(n_k)} a_{n,i} f_i\right)^q$$

$$+ \sum_k \sum_{n=n_k}^{n'_k} u_n^q \left(\sum_{i=\beta(n_k)}^{\beta(n)} a_{n,i} f_i\right)^q = \sum_k \sum_{n=n_k}^{n'_k} u_n^q (T_k f)^q_k + \sum_k \sum_{n=n_k}^{n'_k} u_n^q (S_k f)^q_n =$$

$$= \sum_k \|T_k f\|^q_{l_{q,u[n_k,n'_k]}} + \sum_k \|S_k f\|^q_{l_{q,u[n_k,n'_k]}}$$

$$\leq \sum_k \|T_k\|^q \|f\|^q_{l_{p,v}[\alpha(n_k),\beta(n_k)]} + \sum_k \|S_k\|^q \|f\|^q_{l_{p,v}[\beta(n_k),\beta(n'_k)]}$$

$$\leq \left(\sup_k \|T_k\| + \sup_k \|S_k\|\right)^q \|f\|^q_{p,v}$$

Hence

$$\|A\|_{l_{p,v}\to l_{q,u}} \ll \left(\sup_k \|T_k\| + \sup_k \|S_k\|\right). \tag{2.9}$$

Therefore, for the proof of the boundedness of the operator A we need to prove the boundedness of the operators T_k and S_k.

Now we consider the operator T_k. Using that $a_{n,i} \approx b_{n,n_k}\omega_i + a_{n_k,i}$ if $1 \leq n_k \leq n \leq n'_k$, $\alpha(n) \leq i \leq \beta(n_k)$ we have that

$$(T_k f)_n = \sum_{i=\alpha(n)}^{\beta(n_k)} a_{n,i} f_i \approx \sum_{i=\alpha(n)}^{\beta(n_k)} (b_{n,n_k}\omega_i + a_{n_k,i}) f_i$$

$$= b_{n,n_k} \sum_{i=\alpha(n)}^{\beta(n_k)} \omega_i f_i + \sum_{i=\alpha(n)}^{\beta(n_k)} a_{n_k,i} f_i = (T_{k,1} f)_n + (T_{k,2} f)_n$$

and

$$\|T_k f\|_{l_{q,u[n_k,n'_k]}} \approx \|T_{k,1} f\|_{l_{q,u[n_k,n'_k]}} + \|T_{k,2} f\|_{l_{q,u[n_k,n'_k]}} \tag{2.10}$$

From (2.10) we have that $\|T_k\| \leq \|T_{k,1}\| + \|T_{k,2}\|$, where $\|T_{k,i}\|$ is the norm of the operator $T_{k,i}$: $l_{p,v}[\alpha(n_k), \beta(n_k)] \to l_{q,u}[n_k, n'_k]$, $i = 1, 2$.

The values $\|T_{k,1}\|$, $\|T_{k,2}\|$ are the best constants in the following inequalities, respectively.

$$\left(\sum_{n=n_k}^{n'_k} b^q_{n,n_k} u^q_n \left(\sum_{i=\alpha(n)}^{\beta(n_k)} \omega_i f_i\right)^q\right)^{\frac{1}{q}} \le \|T_{k,1}\| \left(\sum_{i=\alpha(n_k)}^{\beta(n_k)} (v_i f_i)^p\right)^{\frac{1}{p}}, \tag{2.11}$$

$$\left(\sum_{n=n_k}^{n'_k} u^q_n \left(\sum_{i=\alpha(n)}^{\beta(n_k)} a_{n_k,i} f_i\right)^q\right)^{\frac{1}{q}} \le \|T_{k,1}\| \left(\sum_{i=\alpha(n_k)}^{\beta(n_k)} (v_i f_i)^p\right)^{\frac{1}{p}}. \tag{2.12}$$

Let $w_i = 1$, $i \in [n_k, n'_k]$, $w_i = 0$, $i \notin [n_k, n'_k]$ and $d_i = 1$, $i \in [\alpha(n_k), \beta(n_k)]$, $d_i = 0$, $i \notin [\alpha(n_k), \beta(n_k)]$. We consider the inequality

$$\left(\sum_{n=1}^{\infty} b^q_{n,n_k} u^q_n w^q_n \left(\sum_{i=\alpha(n)}^{\infty} d_i \omega_i f_i\right)^q\right)^{\frac{1}{q}} \le C \left(\sum_{i=1}^{\infty} (v_i f_i)^p\right)^{\frac{1}{p}}. \tag{2.13}$$

If the inequality (2.13) holds with the constant C, then the inequality (2.11) holds with the estimate $\|T_{k,1}\| \le C$. Since the inequality (2.13), in fact is the Hardy inequality with a lower variable limit, then by Theorem 1 in [2] and using (2.8) we obtain

$$\|T_{k,1}\| \ll \sup_{n\ge1} \left(\sum_{j=1}^{n} w^q_j u^q_j b^q_{j,n_k}\right)^{\frac{1}{q}} \left(\sum_{j=\alpha(n)}^{\infty} d^{p'}_j \omega^{p'}_j v^{-p'}_j\right)^{\frac{1}{p'}}$$

$$= \sup_{n_k \le n \le \alpha^{-1}(\beta(n_k))} \left(\sum_{j=n_k}^{n} u^q_j b^q_{j,n_k}\right)^{\frac{1}{q}} \left(\sum_{j=\alpha(n)}^{\beta(n_k)} \omega^{p'}_j v^{-p'}_j\right)^{\frac{1}{p'}}.$$

Similarly for (2.12) we obtain

$$\|T_{k,2}\| \ll \sup_{n_k \le n \le \alpha^{-1}(\beta(n_k))} \left(\sum_{j=n_k}^{n} u^q_j\right)^{\frac{1}{q}} \left(\sum_{j=\alpha(n)}^{\beta(n_k)} a^{p'}_{n_k,j} v^{-p'}_j\right)^{\frac{1}{p'}}.$$

Then on the basis (2.10) we have

$$\|T_k\| \ll \sup_{n_k \le n \le \alpha^{-1}(\beta(n_k))} \left(\sum_{j=n_k}^{n} u^q_j b^q_{j,n_k}\right)^{\frac{1}{q}} \left(\sum_{j=\alpha(n)}^{\beta(n_k)} \omega^{p'}_j v^{-p'}_j\right)^{\frac{1}{p'}}$$

$$+ \sup_{n_k \le n \le \alpha^{-1}(\beta(n_k))} \left(\sum_{j=n_k}^{n} u^q_j\right)^{\frac{1}{q}} \left(\sum_{j=\alpha(n)}^{\beta(n_k)} a^{p'}_{n_k,j} v^{-p'}_j\right)^{\frac{1}{p'}}.$$

Hence

$$\sup_{k\ge1} \|T_k\| \le \sup_{k\ge1} \sup_{n_k \le n \le \alpha^{-1}(\beta(n_k))} \left(\sum_{j=n_k}^{n} u^q_j b^q_{j,n_k}\right)^{\frac{1}{q}} \left(\sum_{j=\alpha(n)}^{\beta(n_k)} \omega^{p'}_j v^{-p'}_j\right)^{\frac{1}{p'}}$$

$$+\sup_{k\geq 1}\sup_{n_k\leq n\leq\alpha^{-1}(\beta(n_k))}\left(\sum_{j=n_k}^{n}u_j^q\right)^{\frac{1}{q}}\left(\sum_{j=\alpha(n)}^{\beta(n_k)}a_{n_k,j}^{p'}v_j^{-p'}\right)^{\frac{1}{p'}}$$

$$\leq\sup_{m\geq 1}\sup_{m\leq n\leq\alpha^{-1}(\beta(m))}\left(\sum_{j=m}^{n}u_j^q b_{j,m}^q\right)^{\frac{1}{q}}\left(\sum_{j=\alpha(n)}^{\beta(m)}\omega_j^{p'}v_j^{-p'}\right)^{\frac{1}{p'}}$$

$$+\sup_{m\geq 1}\sup_{m\leq n\leq\alpha^{-1}(\beta(m))}\left(\sum_{j=m}^{n}u_j^q\right)^{\frac{1}{q}}\left(\sum_{j=\alpha(n)}^{\beta(m)}a_{m,j}^{p'}v_j^{-p'}\right)^{\frac{1}{p'}}=F_1+F_2. \tag{2.14}$$

Now we estimate $\|S_k\|$, $k\in\mathbb{N}$. The value $\|S_k\|$ is the best constant in the following inequality

$$\left(\sum_{n=n_k}^{n_k'}u_n^q\left(\sum_{i=\beta(n_k)}^{\beta(n)}a_{n,i}f_i\right)^q\right)^{\frac{1}{q}}\leq\|S_k\|\left(\sum_{i=\beta(n_k)}^{\beta(n_k')}f_i^p v_i^p\right)^{\frac{1}{p}},\quad\forall f\geq 0.$$

Here after replacing $i=\beta(j)$ we have

$$\left(\sum_{n=n_k}^{n_k'}u_n^q\left(\sum_{j=n_k}^{n}\tilde{a}_{n,j}\tilde{f}_j\right)^q\right)^{\frac{1}{q}}\leq\|S_k\|\left(\sum_{j=n_k}^{n_k'}\tilde{f}_j^p\tilde{v}_j^p\right)^{\frac{1}{p}},$$

where $\tilde{f}_j=f_{\beta(j)}$, $\tilde{v}_j=v_{\beta(j)}$ and $\tilde{a}_{n,j}:=a_{n,\beta(j)}$. From (1.2) we have $a_{n,i}\approx b_{n,m}\omega_i+a_{m,i}$ when $1\leq m\leq n$ and $\alpha(n)\leq i\leq\beta(m)$. Then $\tilde{a}_{n,j}\approx b_{n,m}\tilde{\omega}_j+\tilde{a}_{m,j}$ for $n\geq m\geq j$, satisfies the assumption 1.1 in [11]. Then by Theorem 2.1 in [11] we have that

$$\|S_k\|\approx\sup_{n_k\leq m\leq n_k'}\left(\sum_{n=m}^{n_k'}b_{n,m}^q u_n^q\right)^{\frac{1}{q}}\left(\sum_{j=n_k}^{m}\tilde{\omega}_j^{p'}\tilde{v}_j^{-p'}\right)^{\frac{1}{p'}}+\sup_{n_k\leq m\leq n_k'}\left(\sum_{n=m}^{n_k'}u_n^q\right)^{\frac{1}{q}}\left(\sum_{j=n_k}^{m}\tilde{a}_{m,j}^{p'}\tilde{v}_j^{-p'}\right)^{\frac{1}{p'}}.$$

Making replacement $\beta(j)=i$ and using (2.8) we obtain

$$\|S_k\|\ll\sup_{n_k\leq m\leq n_k'}\left(\sum_{n=m}^{n_k'}b_{n,m}^q u_n^q\right)^{\frac{1}{q}}\left(\sum_{i=\alpha(n_k')}^{\beta(m)}\omega_i^{p'}v_i^{-p'}\right)^{\frac{1}{p'}}$$

$$+\sup_{n_k\leq m\leq n_k'}\left(\sum_{n=m}^{n_k'}u_n^q\right)^{\frac{1}{q}}\left(\sum_{i=\alpha(n_k')}^{\beta(m)}a_{m,i}^{p'}v_i^{-p'}\right)^{\frac{1}{p'}}$$

and

$$\sup_{k\geq 1}\|S_k\|\ll\sup_{k\geq 1}\sup_{n_k\leq m\leq n_k'}\left(\sum_{n=m}^{n_k'}b_{n,m}^q u_n^q\right)^{\frac{1}{q}}\left(\sum_{i=\alpha(n_k')}^{\beta(m)}\omega_i^{p'}v_i^{-p'}\right)^{\frac{1}{p'}}$$

$$+\sup_{k\geq 1}\sup_{n_k\leq m\leq n_k'}\left(\sum_{n=m}^{n_k'}u_n^q\right)^{\frac{1}{q}}\left(\sum_{i=\alpha(n_k')}^{\beta(m)}a_{m,i}^{p'}v_i^{-p'}\right)^{\frac{1}{p'}}.$$

Hence using that $m \le n'_k \le \alpha^{-1}(\beta(m))$ we have

$$\sup_{k\ge1}\|S_k\| \le \sup_{m\ge1}\sup_{m\le n'_k\le\alpha^{-1}(\beta(m))}\left(\sum_{n=m}^{n'_k} b^q_{n,m}u^q_n\right)^{\frac1q}\left(\sum_{i=\alpha(n'_k)}^{\beta(m)} \omega_i^{p'}v_i^{-p'}\right)^{\frac1{p'}}$$

$$+\sup_{m\ge1}\sup_{m\le n'_k\le\alpha^{-1}(\beta(m))}\left(\sum_{n=m}^{n'_k} u^q_n\right)^{\frac1q}\left(\sum_{i=\alpha(n'_k)}^{\beta(m)} a^{p'}_{m,i}v_i^{-p'}\right)^{\frac1{p'}}$$

$$\le \sup_{m\ge1}\sup_{m\le s\le\alpha^{-1}(\beta(m))}\left(\sum_{n=m}^{s} b^q_{n,m}u^q_n\right)^{\frac1q}\left(\sum_{i=\alpha(s)}^{\beta(m)} \omega_i^{p'}v_i^{-p'}\right)^{\frac1{p'}}$$

$$+\sup_{m\ge1}\sup_{m\le s\le\alpha^{-1}(\beta(m))}\left(\sum_{n=m}^{s} u^q_n\right)^{\frac1q}\left(\sum_{i=\alpha(s)}^{\beta(m)} a^{p'}_{m,i}v_i^{-p'}\right)^{\frac1{p'}} \le F_1+F_2. \tag{2.15}$$

From (2.9), (2.14) and (2.15) it follows that $\|A\| \ll F_1+F_2 = F < +\infty$ □

Now we state our compactness result for operator (1.1) from $l_{p,v}$ to $l_{q,u}$.

Theorem 2.2. *Let $1 < p \le q < \infty$ and the elements of the matrix $(a_{n,k})$ satisfy condition (1.2). Then operator (1.1) is compact from $l_{p,v}$ to $l_{q,u}$ if and only if*

$$\lim_{m\to\infty}(F_1)_m = 0, \tag{2.16}$$

$$\lim_{m\to\infty}(F_2)_m = 0, \tag{2.17}$$

where

$$(F_1)_m = \sup_{m\le s\le\alpha^{-1}(\beta(m))}\left(\sum_{n=m}^{s} u^q_n\right)^{\frac1q}\left(\sum_{k=\alpha(s)}^{\beta(m)} a^{p'}_{m,k}v_k^{-p'}\right)^{\frac1q};$$

$$(F_2)_m = \sup_{m\le s\le\alpha^{-1}(\beta(m))}\left(\sum_{n=m}^{s} u^q_n b^q_{n,m}\right)^{\frac1q}\left(\sum_{k=\alpha(s)}^{\beta(m)} \omega_k^{p'}v_k^{-p'}\right)^{\frac1q}.$$

Proof. Necessity. Let operator (1.2) be compact. For all $m, s\in\mathbb{N} : m \le s \le \alpha^{-1}(\beta(m))$ we define the following sequence: $\tilde g = \{\tilde g_k\}_{k=1}^\infty : \tilde g_k = \frac{\tilde f_k}{\|\tilde f\|_{p,v}}$, where

$$\tilde f_k = \begin{cases} a^{p'-1}_{m,k}v_k^{-p'}, & \alpha(s)\le k\le\beta(m),\\ 0, & k>\beta(m),\quad k<\alpha(s).\end{cases}$$

It is obvious that $\|\tilde g\| = 1$. Since operator (1.2) is compact from $l_{p,v}$ to $l_{q,u}$, it yelds that the set $\{uA\varphi, \|\varphi\|_{p,v} = 1\}$ is precompact in l_q. Therefore by using the criterion of precompactness of sets in l_p [7] we conclude that

$$\lim_{m\to\infty}\sup_{\|\varphi\|_{p,v}=1}\left(\sum_{n=m}^{\infty} u^q_n(A\varphi)^q_n\right)^{\frac1q} = 0. \tag{2.18}$$

Using (1.5) we have that

$$\sup_{\|\varphi\|_{p,v}=1}\left(\sum_{n=m}^{\infty}u_n^q(A\varphi)_n^q\right)^{\frac{1}{q}}\geq\left(\sum_{n=m}^{\infty}u_n^q(A\tilde{g})_n^q\right)^{\frac{1}{q}}$$

$$=\left(\sum_{n=m}^{\infty}u_n^q\left(\sum_{k=\alpha(n)}^{\beta(n)}a_{n,k}\tilde{g}_k\right)^q\right)^{\frac{1}{q}}\geq\frac{1}{d}\left(\sum_{n=m}^{s}u_n^q\left(\sum_{k=\alpha(s)}^{\beta(m)}a_{m,k}\frac{a_{m,k}^{p'-1}v_k^{-p'}}{\|\tilde{f}\|_{p,v}}\right)^q\right)^{\frac{1}{q}}$$

$$=\frac{1}{d}\left(\sum_{n=m}^{s}u_n^q\right)^{\frac{1}{q}}\left(\sum_{k=\alpha(s)}^{\beta(m)}a_{m,k}^{p'}v_k^{-p'}\right)^{\frac{1}{p'}}$$

for all $m,s\in\mathbb{N}:1\leq m\leq s\leq\alpha^{-1}(\beta(m))$.

Hence

$$\sup_{\|\varphi\|_{p,v}=1}\left(\sum_{n=m}^{\infty}u_n^q(A\varphi)_n^q\right)^{\frac{1}{q}}\gg\sup_{m\leq s\leq\alpha^{-1}(\beta(m))}\left(\sum_{n=m}^{s}u_n^q\right)^{\frac{1}{q}}\left(\sum_{k=\alpha(s)}^{\beta(m)}a_{m,k}^{p'}v_k^{-p'}\right)^{\frac{1}{p'}}=(F_1)_m. \tag{2.19}$$

From (2.18) and (2.19) (2.16) follows.

To prove (2.17) for all $1\leq m\leq s\leq\alpha^{-1}(\beta(m))$ we introduce the following sequence $\bar{g}=\{\bar{g}_k\}_{k=1}^{\infty}$: $\bar{g}_k=\frac{\bar{f}_k}{\|\bar{f}\|_{p,v}}$, where

$$\bar{f}_k=\begin{cases}\omega_k^{p'-1}v_k^{-p'}, & \alpha(s)\leq k\leq\beta(m),\\ 0, & k<\alpha(s),\quad k>\beta(m).\end{cases}$$

Using (1.6) in (2.18) we get that

$$\sup_{\|\varphi\|_{p,v}=1}\left(\sum_{n=m}^{\infty}u_n^q(A\varphi)_n^q\right)^{\frac{1}{q}}\geq\left(\sum_{n=m}^{\infty}u_n^q\left(\sum_{k=\alpha(n)}^{\beta(n)}a_{n,k}\bar{g}_k\right)^q\right)^{\frac{1}{q}}$$

$$\geq\frac{1}{d}\left(\sum_{n=m}^{\infty}u_n^q\left(\sum_{k=\alpha(s)}^{\beta(m)}b_{n,m}\omega_k\frac{\bar{f}_k}{\|\bar{f}\|_{p,v}}\right)^q\right)^{\frac{1}{q}}$$

$$=\frac{1}{d}\left(\sum_{n=m}^{\infty}b_{n,m}^q u_n^q\right)^{\frac{1}{q}}\left(\sum_{k=\alpha(s)}^{\beta(m)}\omega_k^{p'}v_k^{-p'}\right)^{\frac{1}{p'}}$$

for all $1\leq m\leq s\leq\alpha^{-1}(\beta(m))$. Hence

$$\sup_{\|\varphi\|_{p,v}=1}\left(\sum_{n=m}^{\infty}u_n^q\,(A\varphi)_n^q\right)^{\frac{1}{q}}\gg\sup_{m\leq s\leq\alpha^{-1}(\beta(m))}\left(\sum_{n=m}^{s}b_{n,m}^q u_n^q\right)^{\frac{1}{q}}\left(\sum_{k=\alpha(s)}^{\beta(m)}\omega_k^{p'}v_k^{-p'}\right)^{\frac{1}{p'}}=(F_2)_m. \tag{2.20}$$

From (2.18) and (2.20) (2.17) follows.

Sufficiency. Assume that (2.16) and (2.17) hold. Then by Theorem 2.1 operator (1.1) is bounded from $l_{p,v}$ to $l_{q,u}$. Therefore, the set $\{uAf,\|f\|_{p,v}\leq 1\}$ is bounded in l_q. Let us show that this set is precompact in l_q. By the criterion of precompactness of sets in l_q, the bounded set $\{uAf, \|f\|_{p,v} \leq 1\}$ is compact in l_q if

$$\lim_{r\to\infty} \sup_{\|f\|_{p,v}\leq 1} \left(\sum_{n=r}^{\infty} u_n^q |(Af)_n|^q \right)^{\frac{1}{q}} = 0. \tag{2.21}$$

Then by Theorem 2.1 we have that

$$\sup_{\|f\|_{p,v}\leq 1} \left(\sum_{n=r}^{\infty} u_n^q |(Af)_n|^q \right)^{\frac{1}{q}} \ll F(r), \tag{2.22}$$

where $F(r) = F_1(r) + F_2(r)$,

$$F_1(r) = \sup_{m\geq r} \sup_{m\leq s\leq \alpha^{-1}(\beta(m))} \left(\sum_{n=m}^{s} u_n^q \right)^{\frac{1}{q}} \left(\sum_{k=\alpha(s)}^{\beta(m)} a_{m,k}^{p'} v_k^{-p'} \right)^{\frac{1}{p'}} = \sup_{m\geq r} (F_1)_m, \tag{2.23}$$

$$F_2(r) = \sup_{m\geq r} \sup_{m\leq s\leq \alpha^{-1}(\beta(m))} \left(\sum_{n=m}^{s} u_n^q b_{n,m}^q \right)^{\frac{1}{q}} \left(\sum_{k=\alpha(s)}^{\beta(m)} \omega_k^{p'} v_k^{-p'} \right)^{\frac{1}{p'}} = \sup_{m\geq r} (F_2)_m. \tag{2.24}$$

From (2.16), (2.17), (2.23) and (2.24) we obtain that

$$\lim_{r\to\infty} F_1(r) = \lim_{r\to\infty} \sup_{m\geq r} (F_1)_m = \overline{\lim_{r\to\infty}} (F_1)_r = \lim_{r\to\infty} (F_1)_r = 0,$$

$$\lim_{r\to\infty} F_2(r) = \lim_{r\to\infty} \sup_{m\geq r} (F_2)_m = \overline{\lim_{r\to\infty}} (F_2)_r = \lim_{r\to\infty} (F_2)_r = 0.$$

Hence, by using (2.22) we obtain (2.21).

□

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