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# Weighted differential inequality and oscillatory properties of fourth order differential equations

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## Abstract

In this paper, we investigate the oscillatory properties of two fourth order differential equations in dependence on boundary behavior of its coefficients at infinity. These properties are established based on two-sided estimates of the least constant of a certain weighted differential inequality.

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**Keywords:** Fourth order differential equation; Oscillation; Nonoscillation; Variational principle; Weighted inequality

## 1 Introduction

Let  $I = (0, \infty)$  and  $1 < p < \infty$ . Let  $u$  be a positive function continuous on  $I$ . Suppose that  $v$  is a positive function sufficiently times continuously differentiable on the interval  $I$ .

In the paper, we investigate the oscillatory properties of the following half-linear and linear fourth order differential equations:

$$(v(t)y''(t)|y'(t)|^{p-2})'' - u(t)y(t)|y(t)|^{p-2} = 0, \quad t \in I, \quad (1)$$

and

$$(v(t)y''(t))'' - u(t)y(t) = 0, \quad t \in I. \quad (2)$$

One of the directions of the qualitative theory of differential equations is the investigation of their oscillatory properties, which have important applications in physics, technology, medicine, biology, and in other scientific applications. Therefore, the oscillatory properties of various models described by linear, quasilinear, and nonlinear differential equations, including delay differential equations, are intensely studied. Recently, [14, 15], and [16] have investigated the oscillatory properties of second order impulsive differential equations, the research of which has been significantly developed in recent decades. Most of the results regarding the oscillatory properties of differential equations relate to second

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order equations. In particular, there are fairly simple methods for establishing the oscillatory properties of second order linear and half-linear equations given in a symmetric form (see, e.g., [3]). At present, the oscillatory properties of such equations are studied mainly through three methods. The first method considers the equation as a perturbation of an Euler-type equation, the solutions of which are known. The second method is based on reducing of the equation to a Hamiltonian system. The third method, called the variational principle, is based on establishing inequality (3). The first and second methods are difficult to extend to equations of the fourth or higher order. Therefore, in the recent works [4, 5, 17], and [18], to overcome these difficulties that arise when equations are fourth or higher order, at least one of the coefficients has to be taken as a power function. The third (variational) method is more flexible in its extension to fourth or higher order equations. The key idea of this method is to characterize a suitable inequality and estimate its least constant. Thus, in the papers [1, 7, 8], and [13] restrictions on the coefficients have been removed using the variational method.

In this paper, we also use the variational method: based on the variational Lemma A (see Sect. 4), we equivalently relate nonoscillation of equations (1) and (2) with the value of the least constant  $C$  in inequality (3), then we obtain estimates for the least constant  $C$  in inequality (3), and from the obtained estimates, in terms of the coefficients, we derive the oscillatory properties of equations (1) and (2). One of the main and most technically difficult problems in the theory of inequalities is finding of the exact values of their least constants. Unfortunately, in this paper we have not found the exact value of the least constant in inequality (3), we have been able to obtain its two-sided estimates, which is currently the best possible.

Let  $T \geq 0$ . Denote by  $W_{p,v}^2(T, \infty)$  the space of functions  $f : (T, \infty) \rightarrow \mathbb{R}$  having generalized derivatives up to the second order on the interval  $(T, \infty)$ , for which  $\|f''\|_{p,v} < \infty$ , where  $\|g\|_{p,v} = (\int_T^\infty v(t)|g(t)|^p dt)^{\frac{1}{p}}$  is the standard norm of the weighted space  $L_{p,v}(T, \infty)$ .

Let  $\dot{M}_p(T, \infty) = \{f \in W_p^2(T, \infty) : \text{supp} f \subset (T, \infty) \text{ and } \text{supp} f \text{ is compact}\}$ , where  $W_p^2(T, \infty)$  is the Sobolev space.

By the conditions on the function  $v$ , we have that  $\dot{M}_p(T, \infty) \subset W_{p,v}^2(T, \infty)$ . Denote by  $\dot{W}_{p,v}^2(T, \infty)$  the closure of the set  $\dot{M}_p(T, \infty)$  with respect to the norm  $\|f''\|_{p,v}$ .

Let us consider the following second order differential inequality:

$$\int_T^\infty u(t)|f(t)|^2 dt \leq C_T \int_T^\infty v(t)|f''(t)|^2 dt, \quad f \in \dot{W}_{2,v}^2(T, \infty). \tag{3}$$

On the basis of Lemma 3.1 of the work [7], we have the following statement connecting the oscillatory properties of equation (2) to the least constant  $C_T$  in inequality (3).

**Lemma 1** *Let  $C_T$  be the least constant in (3).*

- (i) *Equation (2) is nonoscillatory if and only if there exists a number  $T > 0$  such that  $0 < C_T \leq 1$ .*
- (ii) *Equation (2) is oscillatory if and only if for any number  $T > 0$  we have that  $C_T > 1$ .*

In this paper, we establish an analogue of Lemma 1 also for equation (1).

Since from Lemma 1 it follows that the oscillatory properties of equations (1) and (2) depend on the least constant  $C_T$  in (3), we first find two-sided estimates of  $C_T$  of independent interest. Then, on the basis of the obtained estimates, we study the oscillatory properties of equations (1) and (2).

This paper is organized as follows. Section 2 contains all the auxiliary statements necessary to prove the main results. In Sect. 3, we find two-sided estimates of the least constant  $C_T$ . In Sect. 4, on the basis of the obtained results, we get nonoscillation conditions of equations (1) and (2), and then oscillation conditions of equation (2). Section 5 contains criteria of strong nonoscillation and strong oscillation of equation (2).

In the sequel,  $\chi_{(a,b)}(\cdot)$  stands for the characteristic function of the interval  $(a, b) \subset I$ . Moreover,  $p' = \frac{p}{p-1}$ .

### 2 Auxiliary statements

Let  $0 \leq a < b \leq \infty$ . In the book [9], there is the following statement.

**Theorem A** *Let  $1 < p \leq q < \infty$ .*

(i) *The inequality*

$$\left( \int_a^b u(x) \left( \int_a^x f(t) dt \right)^q dx \right)^{\frac{1}{q}} \leq C \left( \int_a^b v(t) |f(t)|^p dt \right)^{\frac{1}{p}}, \quad f \geq 0, \tag{4}$$

*holds if and only if*

$$A^+ = \sup_{a < z < b} \left( \int_z^b u(x) dx \right)^{\frac{1}{q}} \left( \int_a^z v^{1-p'}(t) dt \right)^{\frac{1}{p'}} < \infty,$$

*in addition,  $A^+ \leq C \leq p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A^+$ , where  $C$  is the least constant in (4).*

(ii) *The inequality*

$$\left( \int_a^b u(x) \left( \int_x^b f(t) dt \right)^q dx \right)^{\frac{1}{q}} \leq C \left( \int_a^b v(t) |f(t)|^p dt \right)^{\frac{1}{p}}, \quad f \geq 0, \tag{5}$$

*holds if and only if*

$$A^- = \sup_{a < z < b} \left( \int_a^z u(x) dx \right)^{\frac{1}{q}} \left( \int_z^b v^{1-p'}(t) dt \right)^{\frac{1}{p'}} < \infty,$$

*in addition,  $A^- \leq C \leq p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A^-$ , where  $C$  is the least constant in (5).*

We also need a statement from the works [6] and [12]. Let  $0 < \tau \leq \infty$  and

$$B_1(\tau) = \sup_{0 < z < \tau} \left( \int_z^\tau u(x) dx \right)^{\frac{1}{q}} \left( \int_0^z (z-t)^{p'} v^{1-p'}(t) dt \right)^{\frac{1}{p'}},$$

$$B_2(\tau) = \sup_{0 < z < \tau} \left( \int_z^\tau (x-z)^q u(x) dx \right)^{\frac{1}{q}} \left( \int_0^z v^{1-p'}(t) dt \right)^{\frac{1}{p'}},$$

$$B(\tau) = \max\{B_1(\tau), B_2(\tau)\}.$$

**Theorem B** ([6]) *Let  $1 < p \leq q < \infty$ . Then inequality*

$$\left( \int_0^\tau \left( u(x) \int_0^x (x-t) f(t) dt \right)^q dx \right)^{\frac{1}{q}} \leq C \left( \int_0^\tau v(t) |f(t)|^p dt \right)^{\frac{1}{p}}, \quad f \geq 0, \tag{6}$$

holds if and only if  $B(\tau) < \infty$ ; in addition,  $B(\tau) \leq C \leq 8p^{\frac{1}{q}}(p')^{\frac{1}{p'}} B(\tau)$ , where  $C$  is the least constant in (6).

For  $f \in W_{p,v}^2(I)$ , we assume that  $\lim_{t \rightarrow 0^+} f(t) = f(0)$ ,  $\lim_{t \rightarrow 0^+} f'(t) = f'(0)$ ,  $\lim_{t \rightarrow \infty} f(t) = f(\infty)$  and  $\lim_{t \rightarrow \infty} f'(t) = f'(\infty)$ .

Assume that

$$L_2W = \{f \in W_{p,v}^2(I) : f(0) = f'(0) = 0\},$$

$$L_2R_1W = \{f \in W_{p,v}^2(I) : f(0) = f'(0) = f'(\infty) = 0\}.$$

From the results of the paper [10] we have one more statement.

**Lemma C** *Let  $1 < p < \infty$ .*

(i) *If the conditions*

$$\int_0^1 v^{1-p'}(t) dt < \infty \quad \text{and} \quad \int_1^\infty v^{1-p'}(t) dt = \infty \tag{7}$$

*hold, then*

$$\mathring{W}_{p,v}^2(I) = L_2W. \tag{8}$$

(ii) *If the conditions*

$$\int_0^\infty v^{1-p'}(t) dt < \infty \quad \text{and} \quad \int_1^\infty t^{p'} v^{1-p'}(t) dt = \infty \tag{9}$$

*hold, then*

$$\mathring{W}_{p,v}^2(I) = L_2R_1W. \tag{10}$$

Let us note that the second conditions in (7) and (9) cover all the possible singularities of the function  $v$  at infinity.

### 3 Inequality (3)

The problem of studying inequality (3) is of independent interest, since it can be applied to study the spectral properties of fourth order differential operators, as well as to obtain a priori estimates for differential equations and considered as an embedding to solve various problems of analysis. Therefore, we investigate it in the following more general form

$$\left( \int_0^\infty u(t) |f(t)|^q dt \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty v(t) |f''(t)|^p dt \right)^{\frac{1}{p}}, \quad f \in \mathring{W}_{p,v}^2(I). \tag{11}$$

In the paper [11], inequality (11) was studied for different zero boundary conditions at the endpoints of  $I$ . However, the obtained estimates of the least constant in (11) are somewhat cumbersome, and they are not suitable for applying them to establishing the oscillatory

properties of equations (1) and (2). In this paper, we modify the method of studying inequality (3) given in [11] and find two-sided estimates of the least constant  $C$  in (11) suitable for establishing simple conditions of the oscillatory properties of equations (1) and (2).

Inequality (11) under conditions (7) is well known. Since due to (8) inequality (11) is equivalent to inequality (6), for  $\tau = \infty$ , estimates of the least constant in (11) have the form

$$B(\infty) \leq C \leq 8p^{\frac{1}{q}}(p')^{\frac{1}{p'}}B(\infty). \tag{12}$$

Now, we consider inequality (11) under conditions (9). Let  $0 < \tau \leq \infty$  and

$$\begin{aligned} B_3(\tau) &= \left( \int_{\tau}^{\infty} u(t) dt \right)^{\frac{1}{q}} \left( \int_0^{\tau} (\tau - s)^{p'} v^{1-p'}(s) ds \right)^{\frac{1}{p'}}, \\ F_1(\tau) &= \sup_{z > \tau} \left( \int_z^{\infty} u(t) dt \right)^{\frac{1}{q}} \left( \int_{\tau}^z (s - \tau)^{p'} v^{1-p'}(s) ds \right)^{\frac{1}{p'}}, \\ F_2(\tau) &= \sup_{z > \tau} \left( \int_{\tau}^z (t - \tau)^q u(t) dt \right)^{\frac{1}{q}} \left( \int_z^{\infty} v^{1-p'}(s) ds \right)^{\frac{1}{p'}}, \\ \mathcal{B}(\tau) &= \max\{B(\tau), B_3(\tau)\}, \\ F(\tau) &= \max\{F_1(\tau), F_2(\tau)\}, \\ \mathcal{BF}(\tau) &= \max\{\mathcal{B}(\tau), F(\tau)\}, \\ \mathcal{BF} &= \inf_{\tau \in I} \mathcal{BF}(\tau). \end{aligned}$$

Let  $v^{1-p'} \in L_1(I)$ , then for any  $\tau \in I$  there exists  $k_{\tau} > 0$  such that

$$\int_0^{\tau} v^{1-p'}(t) dt = k_{\tau} \int_{\tau}^{\infty} v^{1-p'}(t) dt, \tag{13}$$

where  $k_{\tau}$  is increasing in  $\tau$ ,  $\lim_{\tau \rightarrow 0^+} k_{\tau} = 0$  and  $\lim_{\tau \rightarrow \infty} k_{\tau} = \infty$ . Moreover, for  $k_{\tau_1} = 1$ , we have that

$$\int_0^{\tau_1} v^{1-p'}(t) dt = \int_{\tau_1}^{\infty} v^{1-p'}(t) dt. \tag{14}$$

Equality (13) is used below to prove the main theorem of this section. The arbitrariness of the parameter  $\tau : 0 < \tau < \infty$  allows to get the required estimates of the least constant  $C$  in (11) in contrast to what was in [11], where the parameter  $\tau$  is fixed and equal to  $\tau_1$ .

**Theorem 1** *Let  $1 < p \leq q < \infty$  and condition (9) hold. Then, for the least constant  $C$  in inequality (11), the following estimates*

$$4^{-\frac{1}{p}} \mathcal{BF} \leq C \leq 11p^{\frac{1}{q}}(p')^{\frac{1}{p'}} \mathcal{BF}, \tag{15}$$

$$\sup_{\tau \in I} (1 + k_{\tau}^{1-p})^{-\frac{1}{p}} F(\tau) \leq C \leq 11p^{\frac{1}{q}}(p')^{\frac{1}{p'}} F(\tau_0) \tag{16}$$

hold, where

$$\tau_0 = \sup\{\tau \in I : \mathcal{B}(\tau) < F(\tau)\}. \tag{17}$$

*Remark 1* It is obvious that the function  $F(\tau)$  does not increase in  $\tau \in I$ . Suppose that  $\lim_{\tau \rightarrow 0^+} F(\tau) = D$ . Below, in the proof of Theorem 1, both for  $D = \infty$  and  $D < \infty$ , we prove the existence of a nonempty neighborhood of zero, where  $F(\tau) > \mathcal{B}(\tau)$ . In addition, for  $D < \infty$ , we get  $\lim_{\tau \rightarrow 0^+} \mathcal{B}(\tau) = 0$ . Therefore, there is finite  $\tau_0 > 0$  in (17).

*Proof Sufficiency.* By Lemma C, from conditions (9), we have that (10) holds. Hence, for  $f \in \dot{W}_{p,\nu}^2(I)$  and  $\tau \in I$ , we get

$$\begin{aligned} f(x) &= \chi_{(0,\tau)}(x) \int_0^x (x-s)f''(s) ds + \chi_{(\tau,\infty)}(x) \left( \int_0^\tau (\tau-s)f''(s) ds \right. \\ &\quad \left. - \int_\tau^x (s-\tau)f''(s) ds - (x-\tau) \int_x^\infty f''(s) ds \right). \end{aligned} \tag{18}$$

As in Theorem 2.1 of [11], replacing (18) into the left-hand side of (11), then using Minkowski’s inequality for sums, applying Theorems B and A and the inverse Hölder inequality, we have

$$\begin{aligned} \left( \int_0^\infty u(t)|f(t)|^q dt \right)^{\frac{1}{q}} &\leq (8p^{\frac{1}{q}}(p')^{\frac{1}{p'}} \mathcal{B}(\tau) + B_3(\tau)) \left( \int_0^\tau v(t)|f''(t)|^p dt \right)^{\frac{1}{p}} \\ &\quad + p^{\frac{1}{q}}(p')^{\frac{1}{p'}} (F_1(\tau) + F_2(\tau)) \left( \int_\tau^\infty v(t)|f''(t)|^p dt \right)^{\frac{1}{p}} \\ &\leq 11p^{\frac{1}{q}}(p')^{\frac{1}{p'}} \mathcal{B}F(\tau) \left( \int_0^\infty v(t)|f''(t)|^p dt \right)^{\frac{1}{p}}. \end{aligned} \tag{19}$$

Since the left-hand side of (19) does not depend on  $\tau \in I$ , we find the right-hand estimate of (15).

The function  $F(\tau)$  does not increase in  $\tau \in I$ . Let us show that

$$\lim_{\tau \rightarrow 0^+} F(\tau) > \lim_{\tau \rightarrow 0^+} \sup \mathcal{B}(\tau). \tag{20}$$

If  $\lim_{\tau \rightarrow 0^+} F(\tau) = \infty$ , then the validity of (20) is obvious. Let  $\lim_{\tau \rightarrow 0^+} F(\tau) < \infty$ . Then from  $\lim_{\tau \rightarrow 0^+} F_2(\tau) < \infty$  it follows that  $t^q u(t) \in L_1(0, 1)$ . Hence,  $\lim_{\tau \rightarrow 0^+} B_2(\tau) = 0$ . From the estimates

$$\begin{aligned} B_1(\tau) &< \left( \int_0^\tau t^q u(t) dt \right)^{\frac{1}{q}} \left( \int_0^\tau v^{1-p'}(t) dt \right)^{\frac{1}{p'}}, \\ B_3(\tau) &< \tau \left( \int_\tau^N t^q u(t) dt \right)^{\frac{1}{q}} \left( \int_0^\tau v^{1-p'}(t) dt \right)^{\frac{1}{p'}} \\ &\quad + \left( \int_N^\infty u(t) dt \right)^{\frac{1}{q}} \left( \int_0^\tau (\tau-s)^{p'} v^{1-p'}(s) ds \right)^{\frac{1}{p'}}, \quad N > \tau > 0, \end{aligned}$$

we have  $\lim_{\tau \rightarrow 0^+} B_1(\tau) = \lim_{\tau \rightarrow 0^+} B_3(\tau) = 0$ , i.e.,  $\lim_{\tau \rightarrow 0^+} \mathcal{B}(\tau) = 0$ . Consequently, (20) holds. Therefore, there exists finite  $\tau_0$  in (17) and the right-hand side of (16) holds.

Necessity. We use the ideas and methods applied to prove Theorem 2.1 of [11]. For  $f \in \dot{W}_{p,v}^2(I) = L_2R_1W$ , we assume that  $g = f''$ . Then from (10) it follows that the condition  $f \in L_2R_1W$  is equivalent to the condition  $g \in \tilde{L}_{p,v}(I) = \{g \in L_{p,v}(I) : \int_0^\infty g(s) ds = 0\}$  (see [11, Theorem 2.1]).

Let  $0 < \tau < \infty$ . We consider two sets

$$\mathcal{L}_1 = \{g \in L_{p,v}(0, \tau) : g \geq 0\}, \quad \mathcal{L}_2 = \{g \in L_{p,v}(\tau, \infty) : g \leq 0\}.$$

As in [11], for any  $g_1 \in \mathcal{L}_1$  and  $g_2 \in \mathcal{L}_2$ , we respectively construct functions  $g_2 \in \mathcal{L}_2$  and  $g_1 \in \mathcal{L}_1$  such that  $g(t) = g_1(t)$  for  $0 < t \leq \tau$  and  $g(t) = g_2(t)$  for  $t > \tau$  belong to  $\tilde{L}_{p,v}(I)$ .

By the condition of Theorem 1, we have that  $v^{1-p'} \in L_1(I)$ . Therefore, for any  $\tau \in I$ , there exists  $k_\tau$  such that (13) holds. We define a strictly decreasing function  $\rho : (0, \tau) \rightarrow (\tau, \infty)$  from the equalities

$$\begin{aligned} \int_0^s v^{1-p'}(t) dt &= k_\tau \int_{\rho(s)}^\infty v^{1-p'}(t) dt, \quad s \in (0, \tau); \\ \int_0^{\rho^{-1}(s)} v^{1-p'}(t) dt &= k_\tau \int_s^\infty v^{1-p'}(t) dt, \quad s \in (\tau, \infty), \end{aligned} \tag{21}$$

where  $\rho^{-1}$  is the inverse function to the function  $\rho$ .

The function  $\rho$  is locally absolutely continuous on  $(0, \tau)$ . Moreover,  $\rho(\tau) = \tau$  and  $\lim_{s \rightarrow 0^+} \rho(s) = \infty$ . From the second equality of (21) it follows that  $\rho^{-1}$  is also locally absolutely continuous on  $(\tau, \infty)$ . By differentiation of (21), we get

$$\begin{aligned} \frac{1}{k_\tau} &= \frac{v^{1-p'}(\rho(s))}{v^{1-p'}(s)} |\rho'(s)|, \quad s \in (0, \tau); \\ k_\tau &= \frac{v^{1-p'}(\rho^{-1}(s))}{v^{1-p'}(s)} |(\rho^{-1}(s))'|, \quad s \in (\tau, \infty). \end{aligned} \tag{22}$$

For  $g_1 \in \mathcal{L}_1$ , we assume that

$$g_2(t) = -k_\tau g_1(\rho^{-1}(t)) \frac{v^{1-p'}(t)}{v^{1-p'}(\rho^{-1}(t))}, \quad t > \tau. \tag{23}$$

Changing the variables  $\rho^{-1}(t) = s$  and using the first equality in (22), we find that

$$\int_\tau^\infty v(t) |g_2(t)|^p dt = k_\tau^{p-1} \int_0^\tau v(s) |g_1(s)|^p ds < \infty, \tag{24}$$

i.e.,  $g_2 \in \mathcal{L}_2$ .

Similarly, for  $g_2 \in \mathcal{L}_2$ , assuming

$$g_1(t) = -\frac{1}{k_\tau} g_2(\rho(t)) \frac{v^{1-p'}(t)}{v^{1-p'}(\rho(t))}, \quad 0 < t \leq \tau, \tag{25}$$

we get that  $g_1 \in \mathcal{L}_1$  and (24) holds.

In both cases, assuming  $g(t) = g_1(t)$  for  $0 < t \leq \tau$  and  $g(t) = g_2(t)$  for  $t > \tau$ , we have

$$\begin{aligned} \int_0^\infty v(t)|g(t)|^p dt &= (1 + k_\tau^{p-1}) \int_0^\tau v(t)|g_1(t)|^p dt \\ &= (1 + k_\tau^{1-p}) \int_\tau^\infty v(t)|g_2(t)|^p dt < \infty, \end{aligned} \tag{26}$$

i.e.,  $g \in L_{p,v}(I)$ .

From the condition  $v^{1-p'} \in L_1(I)$  it follows that  $g \in L_1(I)$ . For any  $\tau \in I$ , integrating both sides of (23) from  $\tau$  to  $\infty$  and both sides of (25) from 0 to  $\tau$ , we establish that  $\int_\tau^\infty g(t) dt = -\int_0^\tau g(t) dt$ , i.e.,  $\int_0^\infty g(t) dt = 0$ . Hence, we constructed the function  $g \in \tilde{L}_{p,v}$  from the functions  $g_1 \in \mathcal{L}_1$  and  $g_2 \in \mathcal{L}_2$ . Substituting the constructed function in (11) and using (18), we obtain

$$\begin{aligned} &\left( \int_0^\tau u(t) \left( \int_0^t (t-s)g_1(s) ds \right)^q dt + \int_\tau^\infty u(t) \left( \int_0^\tau (\tau-s)g_1(s) ds + \int_\tau^t (s-\tau)|g_2(s)| ds \right. \right. \\ &\quad \left. \left. + (t-\tau) \int_t^\infty |g_2(s)| ds \right)^q dt \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty v(t)|g(t)|^p dt \right)^{\frac{1}{p}}, \end{aligned} \tag{27}$$

where all the terms in the left-hand side are nonnegative.

Let the function  $g \in \tilde{L}_{p,v}(I)$  be constructed from the function  $g_1 \in \mathcal{L}_1$ . Then from (26) and (27) we have

$$\begin{aligned} &\left( \int_0^\tau u(t) \left( \int_0^t (t-s)g_1(s) ds \right)^q dt \right)^{\frac{1}{q}} \leq C(1 + k_\tau^{p-1})^{\frac{1}{p}} \left( \int_0^\tau v(t)|g_1(t)|^p dt \right)^{\frac{1}{p}}, \\ &\left( \int_\tau^\infty u(t) dt \right)^{\frac{1}{q}} \int_0^\tau (\tau-s)g_1(s) ds \leq C(1 + k_\tau^{p-1})^{\frac{1}{p}} \left( \int_0^\tau v(t)|g_1(t)|^p dt \right)^{\frac{1}{p}}. \end{aligned}$$

Due to the arbitrariness of the function  $g_1 \in \mathcal{L}_1$ , by Theorem B, from the inverse Hölder inequality the latter gives

$$B(\tau) \leq C(1 + k_\tau^{p-1})^{\frac{1}{p}}, \quad B_3(\tau) \leq C(1 + k_\tau^{p-1})^{\frac{1}{p}}.$$

The last two estimates imply that

$$B(\tau) \leq C(1 + k_\tau^{p-1})^{\frac{1}{p}}. \tag{28}$$

Similarly, for the function  $g \in \tilde{L}_{p,v}(I)$  constructed from the function  $g_2 \in \mathcal{L}_2$ , due to (26) and (27), we obtain

$$F(\tau) \leq C(1 + k_\tau^{1-p})^{\frac{1}{p}}. \tag{29}$$

From (28) and (29) we deduce that

$$BF = \inf_{\tau \in I} \max \{ B(\tau), F(\tau) \} \leq C \inf_{\tau \in I} [ \max(1 + k_\tau^{p-1})(1 + k_\tau^{1-p}) ]^{\frac{1}{p}} \leq 4^{\frac{1}{p}} C.$$

This gives the left-hand estimate of (15). Moreover, from (29) we also deduce the left-hand estimate of (16). The proof of Theorem 1 is complete. □



#### 4 Oscillatory properties of equations (1) and (2)

Two points  $t_1$  and  $t_2$  of the interval  $I$  such that  $t_1 \neq t_2$  are called conjugate with respect to equation (1)((2)), if there exists a nonzero solution  $y$  of equation (1)((2)) having zeros of multiplicity two  $y^{(i)}(t_1) = y^{(i)}(t_2) = 0, i = 0, 1$ , at these points  $t_1$  and  $t_2$ .

Equation (1) ((2)) is called oscillatory (at infinity) if for any  $T > 0$  there exist conjugate points with respect to equation (1)((2)) to the right of  $T$ . Otherwise, equation (1)((2)) is called nonoscillatory.

Let us begin with equation (1). From Theorem 9.4.4 of the book [3], where the variational method of nonoscillation is established for half-linear higher order equation, we have the following statement.

**Lemma A** *Let  $1 < p < \infty$ . If for some  $T > 0$  the inequality*

$$\int_T^\infty [v(t)|f''(t)|^p - u(t)|f(t)|^p] dt > 0 \tag{30}$$

*holds for all nonzero  $f \in \dot{M}_p(T, \infty)$ , then equation (1) is nonoscillatory.*

Due to the compactness of the set  $\text{supp} f$  for  $f \in \dot{M}_p(T, \infty)$ , inequality (30) coincides with the inequality

$$\int_T^\infty u(t)|f(t)|^p dt < \int_T^\infty v(t)|f''(t)|^p dt, \quad f \in \dot{M}_p(T, \infty). \tag{31}$$

From (31) and the density of  $\dot{M}_p(T, \infty)$  in  $\dot{W}_{p,v}^2(T, \infty)$  we have the following lemma.

**Lemma 2** *Let  $1 < p < \infty$ . If for some  $T > 0$  the inequality*

$$\int_T^\infty u(t)|f(t)|^p dt \leq C_T \int_T^\infty v(t)|f''(t)|^p dt, \quad f \in \dot{W}_{p,v}^2(T, \infty), \tag{32}$$

*holds with the least constant  $C_T : 0 < C_T < 1$ , then equation (1) is nonoscillatory.*

**Theorem 2** *Let  $1 < p < \infty$  and (7) hold. If*

$$\limsup_{z \rightarrow \infty} \int_z^\infty u(t) dt \left( \int_0^z (z-t)^{p'} v^{1-p'}(t) dt \right)^{p-1} \leq \frac{(p-1)^{p-1}}{(8p)^p}, \tag{33}$$

$$\limsup_{z \rightarrow \infty} \int_z^\infty (t-z)^p u(t) dt \left( \int_0^z v^{1-p'}(t) dt \right)^{p-1} \leq \frac{(p-1)^{p-1}}{(8p)^p}, \tag{34}$$

*then equation (1) is nonoscillatory.*

*Proof* From (33) and (34), in view of the upper limit definition, there exist  $T_1 > 0$  and  $T_2 > 0$  such that

$$(B_1(T_1, \infty))^p = \sup_{z > T_1} \int_z^\infty u(t) dt \left( \int_{T_1}^z (z-t)^{p'} v^{1-p'}(t) dt \right)^{p-1} < \frac{(p-1)^{p-1}}{(8p)^p},$$

$$(B_2(T_2, \infty))^p = \sup_{z > T_2} \int_z^\infty (t-z)^p u(t) dt \left( \int_{T_2}^z v^{1-p'}(t) dt \right)^{p-1} < \frac{(p-1)^{p-1}}{(8p)^p}.$$

These inequalities hold for  $T = \max\{T_1, T_2\}$ , hence

$$(8p^{\frac{1}{p}}(p')^{\frac{1}{p'}}\tilde{B}(T, \infty))^p < 1, \tag{35}$$

where  $\tilde{B}(T, \infty)$  is equal to  $B(\infty)$  from (12) for  $p = q$  and the interval  $(T, \infty)$  instead of the interval  $(0, \infty)$ . Then from (35) and (12) it follows that  $C_T < 1$ , and by Lemma 2 equation (1) is nonoscillatory. The proof of Theorem 2 is complete.  $\square$

**Theorem 3** *Let  $1 < p < \infty$  and (9) hold. If*

$$\limsup_{z \rightarrow \infty} \int_z^\infty u(t) dt \left( \int_0^z s^{p'} v^{1-p'}(s) ds \right)^{p-1} \leq \frac{(p-1)^{p-1}}{(11p)^p}, \tag{36}$$

$$\limsup_{z \rightarrow \infty} \int_0^z t^p u(t) dt \left( \int_z^\infty v^{1-p'}(s) ds \right)^{p-1} \leq \frac{(p-1)^{p-1}}{(11p)^p}, \tag{37}$$

then equation (1) is nonoscillatory.

*Proof* From (36) and (37), as in Theorem 2, there exist  $T_1 > 0$  and  $T_2 > 0$  such that

$$(\tilde{F}_1(T_1))^p = \sup_{z > T_1} \int_z^\infty u(t) dt \left( \int_{T_1}^z (s - T_1)^{p'} v^{1-p'}(s) ds \right)^{p-1} < \frac{(p-1)^{p-1}}{(11p)^p},$$

$$(\tilde{F}_2(T_2))^p = \sup_{z > T_2} \int_{T_2}^z (t - T_2)^p u(t) dt \left( \int_z^\infty v^{1-p'}(s) ds \right)^{p-1} < \frac{(p-1)^{p-1}}{(11p)^p},$$

where  $\tilde{F}_1(T_1)$  and  $\tilde{F}_2(T_2)$  are, respectively, equal to  $F_1(T_1)$  and  $F_2(T_2)$  from Theorem 1 for  $p = q$ . It is obvious that the last two inequalities hold for  $T = \max\{T_1, T_2\}$ , hence  $(\tilde{F}(\tau))^p = \max\{(\tilde{F}_1(T))^p, (\tilde{F}_2(T))^p\} < \frac{(p-1)^{p-1}}{(11p)^p}$  or

$$(11p^{\frac{1}{p}}(p')^{\frac{1}{p'}}\tilde{F}(T))^p < 1. \tag{38}$$

Since  $\tilde{F}(\tau)$  does not increase, then from (38) we get

$$(11p^{\frac{1}{p}}(p')^{\frac{1}{p'}}\tilde{F}(\tau_0))^p < 1, \tag{39}$$

where  $\tau_0$  is defined by (17) for  $\tau \in (T, \infty)$ . Then from (16) it follows that  $C_T < 1$  for the least constant  $C_T$  in (32). Therefore, by Lemma 2, equation (1) is nonoscillatory. The proof of Theorem 3 is complete.  $\square$

Now, we turn to equation (2).

**Theorem 4** *Let (7) hold for  $p = 2$ .*

(i) *If*

$$\limsup_{z \rightarrow \infty} \int_z^\infty u(t) dt \int_0^z (z-t)^2 v^{-1}(t) dt \leq \frac{1}{16^2},$$

$$\limsup_{z \rightarrow \infty} \int_z^\infty (t-z)^2 u(t) dt \int_0^z v^{-1}(t) dt \leq \frac{1}{16^2},$$

then equation (2) is nonoscillatory.

(ii) If

$$\limsup_{z \rightarrow \infty} \int_z^\infty u(t) dt \int_0^z (z-t)^2 v^{-1}(t) dt > 1 \tag{40}$$

or

$$\limsup_{z \rightarrow \infty} \int_z^\infty (t-z)^2 u(t) dt \int_0^z v^{-1}(t) dt > 1, \tag{41}$$

then equation (2) is oscillatory.

*Proof* The statement of part (i) follows from the statement of Theorem 2 for  $p = 2$ . Let us prove part (ii). Let (40) hold. Then, by the upper limit definition, there exists a sequence  $\{z_n\}_{n=1}^\infty \subset I$  such that  $\lim_{n \rightarrow \infty} z_n = \infty$  and

$$\lim_{n \rightarrow \infty} \int_{z_n}^\infty u(t) dt \int_0^{z_n} (z_n-t)^2 v^{-1}(t) dt > 1. \tag{42}$$

It is easy to see that

$$\lim_{z \rightarrow \infty} \left( \int_T^z (z-t)^2 v^{-1}(t) dt \right)^{-1} \int_0^z (z-t)^2 v^{-1}(t) dt = 1$$

for any  $T > 0$ . Then from (42) it follows that

$$\begin{aligned} 1 &< \lim_{\substack{n \rightarrow \infty \\ z_n > T}} \int_{z_n}^\infty u(t) dt \int_T^{z_n} (z_n-t)^2 v^{-1}(t) dt \\ &\leq \sup_{z > T} \int_z^\infty u(t) dt \int_T^z (z-t)^2 v^{-1}(t) dt = (\overline{B}_1(T, \infty))^2, \end{aligned} \tag{43}$$

where  $\overline{B}_1(T, \infty)$  is equal to  $B_1(\infty)$  from (12) for  $p = q = 2$  and the interval  $(T, \infty)$  instead of the interval  $(0, \infty)$ . Then from (43) and (12) it follows that  $C_T > 1$  for any  $T > 0$ . Therefore, by Lemma 1, equation (2) is oscillatory. If (41) holds, the proof is similar, so we omit the details. The proof of Theorem 4 is complete.  $\square$

**Theorem 5** Let (9) hold for  $p = 2$ .

(i) If

$$\begin{aligned} \limsup_{z \rightarrow \infty} \int_z^\infty u(t) dt \int_0^z s^2 v^{-1}(s) ds &\leq \frac{1}{22^2}, \\ \limsup_{z \rightarrow \infty} \int_0^z t^2 u(t) dt \int_z^\infty v^{-1}(s) ds &\leq \frac{1}{22^2}, \end{aligned}$$

then equation (2) is nonoscillatory.

(ii) If

$$\limsup_{z \rightarrow \infty} \int_z^\infty u(t) dt \int_0^z s^2 v^{-1}(s) ds > 1 \tag{44}$$

or

$$\limsup_{z \rightarrow \infty} \int_0^z t^2 u(t) dt \int_z^\infty v^{-1}(s) ds > 1, \tag{45}$$

then equation (2) is oscillatory.

*Proof* The statement of part (i) follows from the statement of Theorem 3 for  $p = 2$ . Let us prove part (ii). Let (45) hold. Then there exists a sequence  $\{z_n\}_{n=1}^\infty \subset I$  such that  $\lim_{n \rightarrow \infty} z_n = \infty$  and

$$\limsup_{z \rightarrow \infty} \int_0^z t^2 u(t) dt \int_z^\infty v^{-1}(s) ds = \lim_{n \rightarrow \infty} \int_0^{z_n} t^2 u(t) dt \int_{z_n}^\infty v^{-1}(s) ds. \tag{46}$$

Since for any  $\tau > 0$  we have that

$$\lim_{z \rightarrow \infty} \left( \int_\tau^z (t - \tau)^2 u(t) dt \right)^{-1} \int_0^z t^2 u(t) dt = \lim_{z \rightarrow \infty} \frac{1}{1 - \frac{\tau}{z}} = 1,$$

then from (46) it follows that

$$\begin{aligned} \limsup_{z \rightarrow \infty} \int_0^z t^2 u(t) dt \int_z^\infty v^{-1}(s) ds &= \lim_{\substack{n \rightarrow \infty \\ z_n > \tau}} \int_\tau^{z_n} (t - \tau)^2 u(t) dt \int_z^\infty v^{-1}(s) ds \\ &\leq \inf_{\tau > 0} \sup_{z > \tau} \int_\tau^z (t - \tau)^2 u(t) dt \int_z^\infty v^{-1}(s) ds \\ &\leq \lim_{\tau \rightarrow \infty} \sup_{z > \tau} \int_\tau^z (t - \tau)^2 u(t) dt \int_z^\infty v^{-1}(s) ds \\ &= \lim_{\tau \rightarrow \infty} (\bar{F}_2(\tau))^2, \end{aligned} \tag{47}$$

where  $\bar{F}_2(\tau)$  is equal to  $F_2(\tau)$  for  $p = q = 2$ . Similarly,

$$\limsup_{z \rightarrow \infty} \int_z^\infty u(t) dt \int_0^z s^2 v^{-1}(s) ds \leq \lim_{\tau \rightarrow \infty} (\bar{F}_1(\tau))^2, \tag{48}$$

where  $\bar{F}_1(\tau)$  is  $F_1(\tau)$  for  $p = q = 2$ .

Consider the interval  $(T, \infty)$  for arbitrary  $T > 0$ . Then the left-hand side of estimate (16) for  $p = q = 2$  has the form

$$\sup_{\tau > T} (1 + k_\tau^{-1})^{-1} (\bar{F}(\tau))^2 \leq C_T, \tag{49}$$

where  $\bar{F}(\tau) = \max\{\bar{F}_1(\tau), \bar{F}_2(\tau)\}$  and  $k_\tau$  is defined on the interval  $(T, \infty)$ . Since  $\lim_{\tau \rightarrow \infty} k_\tau = \infty$ , then

$$\begin{aligned} \sup_{\tau > T} (1 + k_\tau^{-1})^{-1} (\bar{F}(\tau))^2 &\geq \lim_{\tau \rightarrow \infty} (1 + k_\tau^{-1})^{-1} (\bar{F}(\tau))^2 \\ &= \lim_{\tau \rightarrow \infty} (\bar{F}(\tau))^2 \geq \lim_{\tau \rightarrow \infty} (\bar{F}_2(\tau))^2. \end{aligned} \tag{50}$$

Hence, from (45), (47), (50), and (49) it follows that  $C_T > 1$  for any  $T > 0$ . Therefore, by Lemma 1 equation (2) is oscillatory. If (44) holds, the proof is similar, so we omit the details. The proof of Theorem 5 is complete.  $\square$

### 5 Strong oscillation and nonoscillation of equation (2)

Let us consider equation (2) with the parameter  $\lambda > 0$ :

$$(v(t)y''(t))'' - \lambda u(t)y(t) = 0, \quad t \in I. \tag{51}$$

By Lemma A, if for some  $T > 0$  the inequality

$$\int_T^\infty [v(t)|f''(t)|^p - \lambda_0 u(t)|f(t)|^p] dt > 0, \quad f \in \dot{M}_p(T, \infty), \tag{52}$$

holds, then equation (51) is nonoscillatory for  $\lambda = \lambda_0$ . It is obvious that if inequality (52) holds for  $\lambda_0$ , then it holds for any  $\lambda < \lambda_0$ , i.e., equation (51) is nonoscillatory for any  $\lambda < \lambda_0$ . Inversely, if inequality (52) does not hold, then equation (51) is oscillatory for  $\lambda = \lambda_0$ . Hence, it is oscillatory for any  $\lambda > \lambda_0$ . Therefore, we can find  $\lambda_0$  called the critical constant of oscillation such that equation (51) is nonoscillatory for any  $\lambda < \lambda_0$  and oscillatory for any  $\lambda > \lambda_0$ . In the case when the critical constant does not exist, equation (51) is either nonoscillatory or oscillatory for all  $\lambda > 0$ . If equation (51) is nonoscillatory for all  $\lambda > 0$ , it is called strong nonoscillatory. If equation (51) is oscillatory for all  $\lambda > 0$ , it is called strong oscillatory.

Inequality (3) on the interval  $(T, \infty)$  for equation (51) has the form

$$\lambda \int_T^\infty u(t)|f(t)|^2 dt \leq \lambda C_T \int_T^\infty v(t)|f''(t)|^2 dt, \quad f \in \dot{W}_{2,v}^2(T, \infty), \tag{53}$$

with the least constant  $\lambda C_T$ , where  $C_T$  is the least constant in (3).

**Lemma 3** *Let  $C_T$  be the least constant in (3).*

- (i) *Equation (51) is strong nonoscillatory if and only if  $\lim_{T \rightarrow \infty} C_T = 0$ .*
- (ii) *Equation (51) is strong oscillatory if and only if for any number  $T > 0$  we have that  $C_T = \infty$ .*

*Proof* Part (i). Let equation (51) be nonoscillatory for  $\lambda > 0$ . Then, by Lemma 1, there exists  $T_1 > 0$  such that  $0 < \lambda C_{T_1} \leq 1$ . Moreover, equation (51) has no conjugate points on the interval  $(T_1, \infty)$ . Hence, for any  $T > T_1$ , equation (51) has no conjugate points on the interval  $(T, \infty)$  and  $0 < \lambda C_T \leq 1$ . Assume that  $T_\lambda = \inf\{T > 0 : \lambda C_T \leq 1\}$ . Then  $\lambda C_{T_\lambda} \leq 1$ . Let equation (51) be strong nonoscillatory. Then, by Lemma 1, for any  $\lambda$  there exists  $T_\lambda > 0$  such that  $0 < \lambda C_{T_\lambda} \leq 1$  or  $C_{T_\lambda} \leq \frac{1}{\lambda}$ . This gives that  $\lim_{\lambda \rightarrow \infty} C_{T_\lambda} = 0$ . Let  $0 < \lambda_1 < \lambda_2$  and  $\lambda_2 C_{T_{\lambda_2}} \leq 1$ . Then  $\lambda_1 C_{T_{\lambda_2}} \leq 1$ . Therefore,  $T_{\lambda_1} \leq T_{\lambda_2}$  and  $T_\lambda$  does not decrease in  $\lambda > 0$ . Hence, there exists  $\lim_{\lambda \rightarrow \infty} T_\lambda = T_\infty$ . If  $T_\infty < \infty$ , then  $\lim_{\lambda \rightarrow \infty} C_{T_\lambda} = C_{T_\infty} = 0$ . Then, due to inequality (3), this fact holds if  $u(t) = 0$  for  $t > T_\infty$ . The obtained contradiction yields that  $T_\infty = \infty$ . Thus,  $\lim_{\lambda \rightarrow \infty} C_{T_\lambda} = \lim_{T \rightarrow \infty} C_T = 0$ .

Inversely, let  $\lim_{T \rightarrow \infty} C_T = 0$ . Then, for any  $\lambda > 0$ , there exists  $T(\lambda) > 0$  such that  $C_{T(\lambda)} \leq \frac{1}{\lambda}$  or  $\lambda C_{T(\lambda)} \leq 1$ . Therefore, by Lemma 1, equation (51) is nonoscillatory for any  $\lambda > 0$ , i.e., it is strong nonoscillatory.

Part (ii). Let equation (51) be strong oscillatory. Then, by Lemma 1, for any  $\lambda > 0$  and  $T > 0$ , we have that  $\lambda C_T > 1$ , which yields  $C_T > \frac{1}{\lambda}$  and  $C_T \geq \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} = \infty$  for all  $T > 0$ .

Inversely, if  $C_T = \infty$  for any  $T > 0$ , we have that  $\lambda C_T = \infty > 1$  for any  $\lambda > 0$  and  $T > 0$ . Then, by Lemma 1, equation (51) is oscillatory for any  $\lambda > 0$ , i.e., it is strong oscillatory. The proof of Lemma 3 is complete.  $\square$

Now, on the basis of Lemma 3, we can establish criteria of strong nonoscillation and strong oscillation of equation (51).

Criteria of strong nonoscillation and strong oscillation of equation (51) under conditions (7) for  $p = 2$  are corollaries of the results of the work [13]. Therefore, we investigate equation (51) under conditions (9) for  $p = 2$ .

**Theorem 6** *Let (9) hold for  $p = 2$ .*

(i) *Equation (51) is strong nonoscillatory if and only if*

$$\lim_{z \rightarrow \infty} \int_z^\infty u(t) dt \int_0^z s^2 v^{-1}(s) ds = 0, \tag{54}$$

$$\lim_{z \rightarrow \infty} \int_0^z t^2 u(t) dt \int_z^\infty v^{-1}(s) ds = 0. \tag{55}$$

(ii) *Equation (51) is strong oscillatory if and only if*

$$\limsup_{z \rightarrow \infty} \int_z^\infty u(t) dt \int_0^z s^2 v^{-1}(s) ds = \infty \tag{56}$$

or

$$\limsup_{z \rightarrow \infty} \int_0^z t^2 u(t) dt \int_z^\infty v^{-1}(s) ds = \infty. \tag{57}$$

*Proof* Part (i). Let equation (51) be strong nonoscillatory. Then, by Lemma 3, we have that  $\lim_{T \rightarrow \infty} C_T = 0$ . From the left-hand side estimate of (16), for any  $T > 0$  and  $\tau \in (T, \infty)$ , we have

$$(1 + k_\tau^{-1})^{-1} (\bar{F}(\tau))^2 \leq C_T. \tag{58}$$

We choose  $\tau = \tau_1$  such that  $\int_T^{\tau_1} v^{-1}(s) ds = \int_{\tau_1}^\infty v^{-1}(s) ds$ . Then  $k_{\tau_1} = 1$  and  $T < \tau_1 < \infty$ . Therefore,  $\lim_{T \rightarrow \infty} \tau_1 = \infty$  and from (58) it follows that  $0 = \lim_{T \rightarrow \infty} C_T \geq \frac{1}{2} \times \lim_{\tau \rightarrow \infty} (\bar{F}(\tau))^2$ . Hence,  $\lim_{\tau \rightarrow \infty} \bar{F}_i(\tau) = 0, i = 1, 2$ . Then from (48) and (47) we get that (54) and (55) hold, respectively.

Inversely, let (54) and (55) hold. Then we have

$$\begin{aligned} \limsup_{z \rightarrow \infty} \int_z^\infty u(t) dt \int_0^z s^2 v^{-1}(s) ds &= \limsup_{\tau \rightarrow \infty} \sup_{z > \tau} \int_z^\infty u(t) dt \int_0^z s^2 v^{-1}(s) ds \\ &\geq \limsup_{\tau \rightarrow \infty} \sup_{z > \tau} \int_z^\infty u(t) dt \int_\tau^z (s - \tau)^2 v^{-1}(s) ds \\ &= \lim_{\tau \rightarrow \infty} (\bar{F}_1(\tau))^2. \end{aligned} \tag{59}$$

Similarly, we find that

$$\limsup_{z \rightarrow \infty} \int_0^z t^2 u(t) dt \int_z^\infty v^{-1}(s) ds \geq \lim_{\tau \rightarrow \infty} (\bar{F}_2(\tau))^2. \tag{60}$$

Therefore, from (54) and (55) it follows that

$$\lim_{\tau \rightarrow \infty} (\bar{F}(\tau))^2 = 0. \tag{61}$$

For  $p = q = 2$ , from the right-hand side estimate of (16), we have

$$C_T \leq (22\bar{F}(\tau_0))^2, \tag{62}$$

where  $\tau_0$  is defined by (17) for  $\tau > T$  and  $T < \tau_0 < \infty$ . Then from (61) and (62) we obtain

$$\lim_{T \rightarrow \infty} C_T \leq 22^2 \lim_{T \rightarrow \infty} (\bar{F}(\tau_0))^2 = 22^2 \lim_{\tau \rightarrow \infty} (\bar{F}(\tau))^2 = 0.$$

Hence,  $\lim_{T \rightarrow \infty} C_T = 0$ , and by Lemma 3 equation (51) is strong nonoscillatory.

Part (ii). Let equation (51) be strong oscillatory. Then, by Lemma 3, we have that  $C_T = \infty$  for any  $T > 0$ . Therefore, from (62) we derive that  $\bar{F}(\tau_0) = \infty$  for any  $\tau_0 = \tau(T)$ . Since  $\bar{F}(\tau)$  does not increase, then  $\lim_{\tau \rightarrow \infty} \bar{F}(\tau) = \infty$ .

If  $\lim_{\tau \rightarrow \infty} \bar{F}_1(\tau) = \infty$ , then from (59) we have that (56) holds. If  $\lim_{\tau \rightarrow \infty} \bar{F}_2(\tau) = \infty$ , then from (60) we have that (57) holds.

Inversely, let (57) hold. Then from (47) it follows that  $\lim_{\tau \rightarrow \infty} \bar{F}_2(\tau) = \infty$ . Then  $\lim_{\tau \rightarrow \infty} \bar{F}(\tau) = \infty$ . Since  $\bar{F}(\tau)$  does not increase, then  $\bar{F}(\tau) = \infty$  for any  $\tau \in (T, \infty)$  and  $T > 0$ . Hence, from (49) we obtain that  $C_T = \infty$  for all  $T > 0$ . Thus, by Lemma 3, equation (51) is strong oscillatory. If (56) holds, the proof is similar, so we omit the details. The proof of Theorem 6 is complete.  $\square$

Let positive functions  $a$  and  $b$  belong to  $C^n(I)$ . In the oscillation theory of differential equations there is known the reciprocity principle [2]: the equation  $(-1)^n(a(t)y^{(n)}(t))^{(n)} = b(t)y(t)$  is nonoscillatory if and only if the equation  $(-1)^n(\frac{1}{b(t)}y^{(n)}(t))^{(n)} = \frac{1}{a(t)}y(t)$  is nonoscillatory.

Now, let us assume that together with the function  $v$  the function  $u$  is also sufficiently times continuously differentiable on the interval  $I$ . Then, by the reciprocity principle, equation (51) is nonoscillatory if and only if the equation

$$(u^{-1}(t)y''(t))'' - \lambda v^{-1}(t)y(t) = 0, \quad t \in I, \tag{63}$$

is nonoscillatory. The statement equivalent to the above statement is as follows: equation (51) is oscillatory if and only if equation (63) is oscillatory. Thus, on the basis of the reciprocity principle, from Theorem 6 we have the following theorem.

**Theorem 7** *Let  $u \in L_1(I)$  and  $t^2u(t) \notin L_1(1, \infty)$ .*

(i) *Equation (51) is strong nonoscillatory if and only if*

$$\lim_{z \rightarrow \infty} \int_z^\infty v^{-1}(t) dt \int_0^z s^2 u(s) ds = 0,$$

$$\lim_{z \rightarrow \infty} \int_0^z t^2 v^{-1}(t) dt \int_z^\infty u(s) ds = 0.$$

(ii) Equation (51) is strong oscillatory if and only if

$$\lim_{z \rightarrow \infty} \sup \int_z^\infty v^{-1}(t) dt \int_0^z s^2 u(s) ds = \infty$$

or

$$\lim_{z \rightarrow \infty} \sup \int_0^z t^2 v^{-1}(t) dt \int_z^\infty u(s) ds = \infty.$$

**Remark 2** The oscillatory properties in Sects. 4 and 5 are studied under the assumption that the function  $v^{-1}$  is nonsingular at zero, i.e.,  $v^{-1} \in L_1(0, 1)$ . This assumption is posed to get simple integral expressions. We consider the oscillatory properties of equations at infinity. Therefore, the oscillation and nonoscillation of the equations are determined by the behavior of their coefficients in the neighborhood of infinity so that their changes at the finite part do not affect the results. For example, if in Theorems 2–5 instead of zero we put any number greater than zero, then the values of the limits in these theorems do not change. If we do not pose the above assumption, we can take any other point of the interval  $I$ . For example, if we take 1, then the integral from 0 to  $z$  is replaced by the integral from 1 to  $z$  and  $t^2$  is replaced by  $(t - 1)^2$ .

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The authors declare that they have no competing interests.

##### Authors' contributions

All three authors have on equal level discussed and posed the research questions in this paper. AK has substantially helped to prove the main results and to type the manuscript. OR is the main author concerning the proofs of the main results. YS has put the results into a more general frame and instructed how to write the paper in this final form. All authors read and approved the final manuscript.

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