mathematics

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Weighted Second-Order Differential Inequality on Set of Compactly Supported Functions and Its Applications

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Abstract: In the paper, we establish the oscillatory and spectral properties of a class of fourth-order differential operators in dependence on integral behavior of its coefficients at zero and infinity. In order to obtain these results, we investigate a certain weighted second-order differential inequality of independent interest.

Keywords: weighted inequality; fourth-order differential operator; oscillation; non-oscillation; spectrum discreteness; spectrum positive definiteness; nuclear operator

MSC: 34C10; 47B25; 26D10

Citation: Kalybay, A.; Oinarov, R.; Sultanaev, Y. Weighted Second-Order Differential Inequality on Set of Compactly Supported Functions and Its Applications. *Mathematics* **2021**, *9*, 2830. https://doi.org/10.3390/math9212830

Academic Editor: Ioannis K. Argyros

Received: 30 August 2021
Accepted: 4 November 2021
Published: 8 November 2021

1. Introduction

Let $I = (0, \infty)$ and $1 < p, q < \infty$. Let r, v, and u be positive functions, such that r is continuously differentiable, u and v are locally summable on the interval I. In addition, let $r^{-1} \equiv \frac{1}{r} \in L_1^{loc}(I)$, $v^{-p'} \in L_1^{loc}(I)$, and $p' = \frac{p}{p-1}$.

We consider the inequality

$$\left(\int_0^\infty |u(t)f(t)|^q dt \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty |v(t)D_r^2 f(t)|^p dt \right)^{\frac{1}{p}}, f \in C_0^\infty(I), \tag{1}$$

where $D_r^2 f(t) = \frac{d}{dt} r(t) \frac{df(t)}{dt}$ and $C_0^\infty(I)$ is the set of compactly supported functions infinitely time continuously differentiable on I. Assume that $D_r^1 f(t) = r(t)\frac{df(t)}{dt}$.

Let $W_{p,v}^2(r) \equiv W_{p,v}^2(r, I) \equiv W_{p,v}^2(I)$ be a set of functions $f : I \to R$, which together with functions $D_r^1 f(t)$ have generalized derivatives on the interval I, with the finite norm

$$\|f\|_{W_{p,v}^2(r)} = \|vD_r^2 f\|_p + |D_r^1 f(1)| + |f(1)|, \tag{2}$$

where $\|\cdot\|_p$ is the standard norm of the space $L_p(I)$.

By the assumptions on the functions r and v, we have that $C_0^\infty(I) \subset W_{p,v}^2(r)$. Denote by $\mathring{W}_{p,v}^2(r) \equiv \mathring{W}_{p,v}^2(r, I)$ the closure of the set $C_0^\infty(I)$ with respect to norm (2). Then, inequality (1) is equivalent to inequality

$$\left(\int_0^\infty |u(t)f(t)|^q dt \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty |v(t)D_r^2 f(t)|^p dt \right)^{\frac{1}{p}}, f \in \mathring{W}_{p,v}^2(r), \tag{3}$$

In addition, the least constants in (1) and (3) coincide.

Let us note that inequality (3) is equivalent to the inequality in the form

$$\left(\int_0^\infty |u(t)f(t)|^q dt\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty |\rho(t)f''(t)+w(t)f'(t)|^p dt\right)^{\frac{1}{p}}.$$

First, we investigate inequality (3). Then, we apply the obtained results to study the oscillatory properties of the fourth-order differential equation

$$D_r^2(v(t)D_r^2 y(t)) - u(t)y(t) = 0, t > 0, \tag{4}$$

and the spectral properties of the differential operator L generated by the differential expression

$$Ly(t) = \frac{1}{u(t)} D_r^2(v(t)D^2 y(t)). \tag{5}$$

Relations (3)–(5) for $r = 1$ have the forms

$$\left(\int_0^\infty |u(t)f(t)|^q dt\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty |v(t)f''(t)|^p dt\right)^{\frac{1}{p}}, \tag{6}$$

$$(v(t)y''(t))'' - u(t)y(t) = 0, \tag{7}$$

$$Ly(t) = \frac{1}{u(t)}(v(t)y''(t))'', \tag{8}$$

respectively. Criteria for the validity of inequality (6) under various boundary conditions on the function f are given in [1,2]. Following the ideas and research methods of [2], we find characterizations of inequality (3) in terms different from those in [2], which are convenient for studying the oscillatory properties of Equation (4) and the spectral properties of operator (5).

There is a series of works that investigate equations in form (7) and operators in form (8) associated with these equations. In these works, the oscillatory properties of the fourth and higher-order equations are studied by three methods. The first method considers the equations as perturbations of Euler-type equations with known solutions. The second method is based on the reduction of the equations to Hamiltonian systems. The third method, applied to the symmetric equations only, studies their oscillatory properties by the variational principle, which requires establishing inequality (6). In the first method, at least one of the coefficients of the equations must be a power function. In the second method, oscillation conditions are found in an implicit form containing the principle solutions of the Hamiltonian systems, the finding of which is a difficult problem. To avoid this difficulty, one or both coefficients of the equation have been taken as power functions. In the third method, due to the lack of characterizations of inequality (6) in general form, one of the coefficients of the equations has also been taken as a power function. In the papers [3–6], published in recent years, the oscillatory properties of the equations in form (7) have been established by the above three methods under the restriction that at least one of the coefficients is a power function. In the paper [7], the restrictions on the coefficients have been removed. However, the results in [7], being cumbersome, do not reveal how the behavior of each of the coefficients affects the oscillatory properties of the equations at zero and at infinity. The presented paper focuses on overcoming these problems.

Property BD (see [8]), i.e., boundedness from below and discreteness of the operator L generated by differential expression (5), is connected with the non-oscillation of differential Equation (4), and the estimate of the first eigenvalue of the operator L follows from the estimate of the least constant in inequality (3). In turn, since differential Equation (4) is symmetric, by the variational principle (see [9]), the oscillatory properties of differential Equation (4) are connected with inequality (3). Thus, in the paper, we discuss three interconnected problems, which we investigate depending on the degree of singularity of the functions $\upsilon^{-p'}$ and r^{-1} at zero and at infinity. We say that the functions $\upsilon^{-p'}$ and r^{-1} are strongly singular if they satisfy the conditions of statement (i), weakly singular if they satisfy the conditions of statement (ii) or (iii), and regular if they satisfy the conditions of statement (iv) of Theorems 4 and 5 from Section 2 at infinity (at zero). Usually, the problem is studied in the case when one endpoint of the interval is regular and the other endpoint is singular. For example, if the functions $\upsilon^{-p'}$ and r^{-1} are strongly singular at infinity and regular at zero, then, in general, the functions $f \in \mathring{W}^2_{p,\upsilon}(r)$ have no boundary values at infinity, and have two boundary values at zero $f(0) = D^1_r f(0) = 0$. In this case, inequality (3) is the same as inequality (12) from Theorem 3 of Section 2 for $a = 0$ and $b = \infty$. Therefore, from Theorem 3, we have characterizations of inequality (3) and an estimate of its least constant. Thus, the oscillatory properties of Equation (4) and the spectral properties of the operator L can be easily derived. When $r \equiv 1$ and the function $\upsilon^{-p'}$ is strongly singular at infinity, the oscillatory properties of the equation in form (7) are studied in [10], and the spectral properties of the operator in form (8) are investigated in [9,11] (Chapters 29 and 34), [12–14]. When the functions $\upsilon^{-p'}$ and r^{-1} are weakly singular at infinity, then there exists one of the limits $\lim\limits_{t\to\infty} f(t) = f(\infty)$ or $\lim\limits_{t\to\infty} D^1_r f(t) = D^1_r f(\infty)$ for all $f \in W^2_{p,\upsilon}(r)$. Suppose, for example, $D^1_r f(\infty)$ exists. Then, for $f \in \mathring{W}^2_{p,\upsilon}(r)$, we have $f(0) = D^1_r f(0) = D^1_r f(\infty) = 0$. Thus, differential inequality (3) is of second-order, but there exist three boundary conditions. This case is called the overdetermined case, which causes difficulties in establishing inequality (3), and such cases have not been studied well enough. The aim of the paper is to establish inequality (3) in the case when the functions $\upsilon^{-p'}$ and r^{-1} are weakly singular at infinity and regular at zero, so that there exists the values $f(0) = D^1_r f(0) = D^1_r f(\infty) = 0$, and in the symmetric case when the functions $\upsilon^{-p'}$ and r^{-1} are weakly singular at zero and regular at infinity, so that there exist the values $D^1_r f(0) = f(\infty) = D^1_r f(\infty) = 0$, then on the basis of the obtained results in terms of the coefficients to derive necessary and sufficient conditions for strong non-oscillation and oscillation of Equation (4), and to find conditions for boundedness from below and discreteness of the spectrum of the operator L. In addition, the paper aims to obtain two-sided estimates for the first eigenvalue of the operator L and criteria for its nuclearity.

The paper is organized as follows. Section 2 contains all the auxiliary statements and definitions necessary to prove the main results. In Section 3, we establish criteria for the validity of inequality (3) depending on the degree of singularity of the functions $\upsilon^{-p'}$ and r^{-1} at zero and infinity. In Section 4, the obtained results on inequality (3) are applied to study the oscillatory properties of differential Equation (4). Section 3 discusses the spectral properties of the operator L generated by differential expression (5).

2. Preliminaries

Suppose that $\chi_{(\alpha,\beta)}(\cdot)$ stands for the characteristic function of the interval $(\alpha, \beta) \subset I$.

Let $J = (a, b)$ and $-\infty \le a < b \le \infty$. Let ω be a non-negative function, ρ be a positive function locally integrable on the interval J. From the work [15], we have the following theorem.

Theorem 1. *Let $1 < p \le q < \infty$.*

(i) Inequality

$$\left(\int_a^b \left| \omega(t) \int_a^t f(s) ds \right|^q dt \right)^{\frac{1}{q}} \le C \left(\int_a^b |\rho(t) f(t)|^p dt \right)^{\frac{1}{p}} \tag{9}$$

holds if, and only if,

$$A^+ = \sup_{z\in J}\left(\int_z^b \omega^q(t)dt\right)^{\frac{1}{q}}\left(\int_a^z \rho^{-p'}(s)ds\right)^{\frac{1}{p'}} < \infty,$$

in addition,

$$A^+ \le C \le p^{\frac{1}{q}}(p')^{\frac{1}{p'}}A^+,$$

where C is the least constant in (9).

(ii) Inequality

$$\left(\int_a^b \left|\omega(t)\int_t^b f(s)ds\right|^q dt\right)^{\frac{1}{q}} \le C\left(\int_c^b |\rho(t)f(t)|^p dt\right)^{\frac{1}{p}} \tag{10}$$

holds if, and only if,

$$A^- = \sup_{z\in J}\left(\int_a^z \omega^q(t)dt\right)^{\frac{1}{q}}\left(\int_z^b \rho^{-p'}(s)ds\right)^{\frac{1}{p'}} < \infty,$$

In addition,

$$A^- \le C \le p^{\frac{1}{q}}(p')^{\frac{1}{p}}A^-,$$

where C is the least constant in (10).

Let

$$B_1^-(a,b) = \sup_{z\in J}\left(\int_z^b u^q(t)\left(\int_z^t r^{-1}(x)dx\right)^q dt\right)^{\frac{1}{q}}\left(\int_a^z v^{-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$B_2^-(a,b) = \sup_{z\in J}\left(\int_z^b u^q(t)dt\right)^{\frac{1}{q}}\left(\int_a^z\left(\int_s^z r^{-1}(x)dx\right)^{p'} v^{-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$B^-(a,b) = \max\{B_1^-(a,b), B_2^-(a,b)\}.$$

The following two statements follow from the results of the work [16].

Theorem 2. *Let* $1 < p \le q < \infty$. *The inequality*

$$\left(\int_a^b \left|u(t)\int_a^t\left(\int_s^t r^{-1}(x)dx\right)f(s)ds\right|^q dt\right)^{\frac{1}{q}} \le C\left(\int_a^b \left|v(t)f(t)\right|^p dt\right)^{\frac{1}{p}} \tag{11}$$

holds if, and only if, $B^-(a,b) < \infty$. *In addition,* $B^-(a,b) \le C \le 8p^{\frac{1}{p}}(p')^{\frac{1}{p'}}B^-(a,b)$, *where C is the least constant in* (11).

Let

$$B_1^+(a,b) = \sup_{z\in J}\left(\int_a^z\left(\int_t^z r^{-1}(x)dx\right)^q u^q(t)dt\right)^{\frac{1}{q}}\left(\int_z^b v^{-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$B_2^+(a,b) = \sup_{z\in J}\left(\int_a^z u^q(t)dt\right)^{\frac{1}{q}}\left(\int_z^b\left(\int_z^s r^{-1}(x)dx\right)^{p'} v^{-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$B^+(a,b) = \max\{B_1^+(a,b), B_2^+(a,b)\}.$$

Theorem 3. *Let $1 < p \le q < \infty$. Inequality*

$$\left(\int_a^b \left|u(t)\int_t^b\left(\int_t^s r^{-1}(x)dx\right)f(s)ds\right|^q dt\right)^{\frac{1}{q}} \le C\left(\int_a^b \left|v(t)f(t)\right|^p dt\right)^{\frac{1}{p}} \tag{12}$$

holds if, and only if, $B^+(a,b) < \infty$. In addition, $B^+(a,b) \le C \le 8p^{\frac{1}{q}}(p)^{\frac{1}{p'}}B^+(a,b)$, where C is the least constant in (12).

Depending on the degree of singularity of the functions $v^{-p'}$ and r^{-1} at zero and at infinity, the function $f \in W^2_{p,v}(I)$ has the finite limits $\lim\limits_{t\to 0^+} f(t) = f(0)$, $\lim\limits_{t\to 0^+} D^1_r f(t) = D^1_r f(0)$, $\lim\limits_{t\to\infty} f(t) = f(\infty)$ and $\lim\limits_{t\to\infty} D^1_r f(t) = D^1_r f(\infty)$ or does not have them.

Let $W^2_{p,v}(r, I_0)$ and $W^2_{p,v}(r, I_\infty)$ be the contraction sets of functions from $W^2_{p,v}(r, I)$ on $(0,1]$ and $[1,\infty)$, respectively. From the results of the work [17], we have the following statements.

Theorem 4. *Let $1 < p < \infty$.*
(i) If $v^{-1} \notin L_{p'}(I_\infty)$ and $r^{-1} \notin L_1(I_\infty)$ or $r^{-1} \in L_1(I_\infty)$ and

$$\int_1^\infty v^{-p'}(t)\left(\int_t^\infty r^{-1}(x)dx\right)^{p'} dx = \infty,$$

then $\mathring{W}^2_{p,v}(r, I_\infty) = W^2_{p,v}(r, I_\infty)$. (In this case, for all $f \in W^2_{p,v}(I)$, there do not exist $f(\infty)$ and $D^1_r f(\infty)$.)

(ii) If $v^{-1} \notin L_{p'}(I_\infty)$, $r^{-1} \in L_1(I_\infty)$ and $\int\limits_1^\infty v^{-p'}(t)\left(\int\limits_t^\infty r^{-1}(x)dx\right)^{p'} dt < \infty$, then

$$\mathring{W}^2_{p,v}(r, I_\infty) = \{f \in W^2_{p,v}(r, I_\infty) : f(\infty) = 0\}.$$

(In this case, for all $f \in W^2_{p,v}(I)$ there exists only $f(\infty)$.)

(iii) If $v^{-1} \in L_p(I_\infty)$, $r^{-1} \notin L_1(I_\infty)$ and $\int\limits_1^\infty v^{-p'}(t)\left(\int\limits_1^t r^{-1}(x)dx\right)^{p'} dt = \infty$, then

$$\mathring{W}^2_{p,v}(r, I_\infty) = \{f \in W^2_{p,v}(r, I_\infty) : D^1_r f(\infty) = 0\}.$$

(In this case, for all $f \in W^2_{p,v}(I)$, there exists only $D^1_r f(\infty)$.)
(iv) If $v^{-1} \in L_1(I_\infty)$ and $r^{-1} \in L_1(I_\infty)$, then

$$\mathring{W}^2_{p,v}(r, I_\infty) = \{f \in W^2_{p,v}(r, I_\infty) : f(\infty) = D^1_r f(\infty) = 0\}.$$

(In this case, for all $f \in W^2_{p,v}(I)$, there exist both $f(\infty)$ and $D^1_r f(\infty)$.)

We have one more similar theorem.

Theorem 5. *Let $1 < p < \infty$.*
(i) If $v^{-1} \notin L_{p'}(I_0)$ and $r^{-1} \notin L_1(I_0)$ or $r^{-1} \in L_1(I_0)$ and

$$\int_0^t v^{-p'}(t)\left(\int_0^t r^{-1}(x)dx\right)^{p'} dx = \infty,$$

then $\mathring{W}^2_{p,v}(r, I_0) = W^2_{p,v}(r, I_0)$.

(ii) If $v^{-1} \notin L_{p'}(I_0)$, $r^{-1} \in L_1(I_0)$ and $\int_0^1 v^{-p'}(t)\left(\int_0^t r^{-1}(x)dx\right)^{p'} dt < \infty$, then

$$\mathring{W}^2_{p,v}(r, I_0) = \{f \in W^2_{p,v}(r, I_0) : f(0) = 0\}.$$

(iii) If $v^{-1} \in L_{p'}(I_0)$, $r^{-1} \notin L_1(I_0)$ and $\int_0^1 v^{-p'}(t)\left(\int_t^1 r^{-1}(x)dx\right)^{p'} dt = \infty$, then

$$\mathring{W}^2_{p,v}(r, I_0) = \{f \in W^2_{p,v}(r, I_0) : D^1_r f(0) = 0\}.$$

(iv) If $v^{-1} \in L_1(I_0)$ and $r^{-1} \in L_1(I_0)$, then

$$\mathring{W}^2_{p,v}(r, I_0) = \{f \in W^2_{p,v}(r, I_0) : f(0) = D^1_r f(0) = 0\}.$$

3. Inequality (3)

For convenience, we accept the following notations: item (i) of Theorem 5 is denoted by $(i)^-$, item (i) of Theorem 4 is denoted by $(i)^+$, and so on. The following theorem lists all possible pairs of items of Theorems 4 and 5, for which the function $f \in \mathring{W}^2_{p,v}(I)$ has at most one boundary value at the endpoints of the interval I.

Theorem 6. *If the conditions of one of the following pairs of items of Theorems 4 and 5*

$$[(i)^+, (i)^-],\ [(ii)^+, (i)^-],\ [(iii)^+, (i)^-],\ [(i)^+, (ii)^-],\ [(i)^+, (iii)^-] \text{ and } [(iii)^+, (iii)^-]$$

hold, then inequality (3) does not hold.

The proof of Theorem 6 follows from the fact that it is possible to find a solution $f \in \mathring{W}^2_{p,v}(I)$ of the homogeneous equation $D^2_r f = 0$, such that the boundary conditions of these pairs are satisfied and the right-hand side of inequality (3) becomes zero, while its left-hand side differs from zero.

Under the conditions of the following pairs $[(i)^+, (iv)^-]$, $[(iv)^+, (i)^-]$, $[(iii)^+, (ii)^-]$, $[(ii)^+, (iii)^-]$ and $[(ii)^+, (ii)^-]$, the function $f \in \mathring{W}^2_{p,v}(I)$ has two boundary values at the endpoints of the interval I and inequality (3) is equivalent to the well-known integral inequalities (see [2]). In the cases $[(iv)^+, (ii)^-]$, $[(ii)^+, (iv)^-]$, $[(iv)^+, (iii)^-]$ and $[(iii)^+, (iv)^-]$, the function $f \in \mathring{W}^2_{p,v}(I)$ has three boundary conditions at the endpoints of the interval I; i.e., we get the overdetermined cases. In this paper, we investigate inequality (3) under the following pairs of conditions $[(iv)^+, (iii)^-]$ and $[(iii)^+, (iv)^-]$, then the obtained results that we apply to study the oscillatory and spectral properties of fourth-order differential operators. The rest of the cases $[(iv)^+, (ii)^-]$ and $[(ii)^+, (iv)^-]$ will be the subject of another paper.

Here, slightly changing the methods of investigation of the work [2], we obtain results that are convenient to apply to the above-mentioned problems of fourth-order differential operators.

Let $\tau \in I$. Assume that $B_1^+(\tau,\infty) \equiv B_1^+(\tau)$, $B_2^+(\tau,\infty) \equiv B_2^+(\tau)$,

$$B_3^+(\tau) = \left(\int_0^\tau u^q(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty \left(\int_\tau^s r^{-1}(x)dx\right)^{p'} v^{-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$F_1^-(\tau) = \sup_{0<z<\tau} \left(\int_0^z u^q(t)dt\right)^{\frac{1}{q}} \left(\int_z^\tau \left(\int_s^\tau r^{-1}(x)dx\right)^{p'} v^{-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$F_2^-(\tau) = \sup_{0<z<\tau} \left(\int_z^\tau \left(\int_t^\tau r^{-1}(x)dx\right)^{q} u^q(t)dt\right)^{\frac{1}{q}} \left(\int_0^z v^{-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$B^+(\tau) = \max\{B_1^+(\tau), B_2^+(\tau)\}, \;\; \mathcal{B}^+(\tau) = \max\{B^+(\tau), B_3^+(\tau)\},$$

$$F^-(\tau) = \max\{F_1^-(\tau), F_2^-(\tau)\}, \;\; \mathcal{B}^+F^- = \inf_{\tau\in I} \max\{\mathcal{B}^+(\tau), F^-(\tau)\}.$$

Let $v^{-1} \in L_{p'}(I)$. Then, for any $\tau \in I$, there exists, k_τ such that

$$\int_0^\tau v^{-p'}(t)dt = k_\tau \int_\tau^\infty v^{-p'}(t)dt. \tag{13}$$

In addition, k_τ increases in τ and $\lim\limits_{\tau\to 0^+} k_\tau = 0$, $\lim\limits_{\tau\to\infty} k_\tau = \infty$. Moreover, there exists $\tau_1 \in I$ such that $K_{\tau_1} = 1$ and $\int_0^{\tau_1} v^{-p'}(t)dt = \int_{\tau_1}^\infty v^{-p'}(t)dt$.

To prove the following theorem, we use the methods of the proof of Theorem 2.1 of the work [2].

Theorem 7. *Let $1 < p \le q < \infty$. Let $v^{-1} \in L_{p'}(I)$, $r^{-1} \in L_1(I_\infty)$, $r^{-1} \notin L_1(I_0)$ and*

$$\int_0^1 v^{-p'}(t) \left(\int_t^1 r^{-1}(x)dx\right)^{p'} dt = \infty.$$

Then for the least constant C in (3) *the estimates*

$$4^{-\frac{1}{p}} \mathcal{B}^+F^- \le C \le 11 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} \mathcal{B}^+F^- \tag{14}$$

and

$$\sup_{\tau\in I} (1 + k_\tau^{p-1})^{-\frac{1}{p}} F^-(\tau) \le C \le 11 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} F^-(\tau^-) \tag{15}$$

hold, where

$$\tau^- = \inf\{\tau > 0 : \mathcal{B}^+(\tau) \le F^-(\tau)\}. \tag{16}$$

Proof of Theorem 7. Sufficiency. From the conditions of Theorem 7, on the basis of (iv) of Theorem 4 and (iii) of Theorem 5, we get

$$\mathring{W}^2_{p,v}(I) = \{f \in W^2_{p,v}(I) : D_r^1 f(0) = f(\infty) = D_r^1 f(\infty) = 0\}. \tag{17}$$

For $\tau \in I$, we assume that $D_r^1 f(x) = \int\limits_0^x D_r^2 f(s)ds$ for $0 < x < \tau$, $D_r^1 f(x) = -\int\limits_x^\infty D_r^2 f(s)ds$ for $x > \tau$, and $f(t) = -\int\limits_t^\infty r^{-1}(x) D_r^1 f(x)dx$ for $t \in I$. Then, for $f \in \mathring{W}^2_{p,v}(I)$, we have

$$\begin{aligned} f(t) = \chi_{(0,\tau)}(t) \Bigg[\int\limits_\tau^\infty \left(\int\limits_\tau^s r^{-1}(x)dx \right) D_r^2 f(s)ds - \int\limits_t^\tau \left(\int\limits_s^\tau r^{-1}(x)dx \right) D_r^2 f(s)ds \\ - \int\limits_t^\tau r^{-1}(x)dx \int\limits_0^t D_r^2 f(s)ds \Bigg] + \chi_{(\tau,\infty)}(t) \int\limits_t^\infty \left(\int\limits_t^s r^{-1}(x)dx \right) D_r^2 f(s)ds. \end{aligned} \tag{18}$$

Replacing (18) into the left-hand side of (3) and using Minkowski's inequality for sums, then the Hölder's inequality, Theorems 1 and 3, we obtain

$$\begin{aligned} \left(\int\limits_0^\infty |u(t)f(t)|^q dt \right)^{\frac{1}{q}} &\leq p^{\frac{1}{q}} (p')^{\frac{1}{p'}} \left(F_1^-(\tau) + F_2^-(\tau) \right) \left(\int\limits_0^\tau |v(s) D_r^2 f(s)|^p ds \right)^{\frac{1}{p}} \\ &+ \left(8 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} B^+(\tau) + B_3^+(\tau) \right) \left(\int\limits_\tau^\infty |v(s) D_r^2 f(s)|^p ds \right)^{\frac{1}{p}} \\ \leq \left[(2 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} F^-(\tau))^{p'} + (9 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} \mathcal{B}^+(\tau))^{p'} \right]^{\frac{1}{p'}} &\left(\int\limits_0^\infty |v(s) D_r^2 f(s)|^p ds \right)^{\frac{1}{p}} \\ &\leq 11 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} \max\{\mathcal{B}^+(\tau), F^-(\tau)\} \left(\int\limits_0^\infty |v(s) D_r^2 f(s)|^p ds \right)^{\frac{1}{p}}. \end{aligned} \tag{19}$$

Since the left-hand side of (19) does not depend on $\tau \in I$, it follows that the right estimate in (14) holds.

The function $B^+(\tau)$ does not increase and the function $F^-(\tau)$ does not decrease. Let us show that for a sufficiently large τ, we have that $F^-(\tau) > \mathcal{B}^+(\tau)$. Let $\lim\limits_{\tau \to 0^+} F^-(\tau) = \infty$. Since $\lim\limits_{\tau\to\infty} B_3^+(\tau) < \infty$ follows from the finiteness of $B_2^+(\tau)$, it is obvious that $F^-(\tau) > \mathcal{B}^+(\tau)$ for a sufficiently large τ. If $\lim\limits_{\tau\to\infty} F^-(\tau) < \infty$, then from $\lim\limits_{\tau\to\infty} F_2^-(\tau) < \infty$, it follows that $\int\limits_1^\infty \left(\int\limits_t^\infty r^{-1}(x)dx \right)^q u^q(t)dt < \infty$. Then, from the estimates

$$B^+(\tau) \leq \left(\int\limits_\tau^\infty \left(\int\limits_t^\infty r^{-1}(x)dx \right)^q u^q(t)dt \right)^{\frac{1}{q}} \left(\int\limits_\tau^\infty v^{-p'}(s)ds \right)^{\frac{1}{p'}}$$

and

$$\begin{aligned} B_3^+(\tau) \leq \left(\int\limits_N^\infty \left(\int\limits_t^\infty r^{-1}(x)dx \right)^q u^q(t)dt \right)^{\frac{1}{q}} \left(\int\limits_\tau^\infty v^{-p'}(s)ds \right)^{\frac{1}{p'}} \\ + \left(\int\limits_0^N u^q(t)dt \right)^{\frac{1}{q}} \left(\int\limits_\tau^\infty \left(\int\limits_\tau^s r^{-1}(x)dx \right)^{p'} v^{-p'}(s)ds \right)^{\frac{1}{p'}} \end{aligned}$$

for some $N < \tau$, we have $\lim\limits_{\tau\to\infty} \mathcal{B}^+(\tau) = 0$. Therefore, for this case, we also have that $F^-(\tau) > \mathcal{B}^+(\tau)$ in some neighborhood of infinity. Hence, in (16), there exist $\tau^- > 0$ and $F^-(\tau^-) \geq \mathcal{B}^+(\tau^-)$. Thus, $F^-(\tau^-) \geq \mathcal{B}^+F^-$ and the right estimate in (15) holds.

Necessity. The idea of the proof of the necessary part is as follows. If, for $f \in \mathring{W}^2_{p,v}(I)$, we have that $D_r^2 f(s) \leq 0$ for $0 < s < \tau$ and $D_r^2 f(s) \geq 0$ for $s > \tau$, then in (18), all terms will be nonnegative, and when we substitute (18) into the left-hand side of (3), then each term on the left-hand side will be smaller than the right-hand side. This fact will prove to be the necessary part of Theorem 7. For this purpose, below we produce some function constructions. From the conditions of Theorem 7, we have $v^{-1} \in L_{p'}(I)$. Therefore, (13) holds. For $f \in \mathring{W}^2_{p,v}(I)$, we assume that $g = D_r^2 f$. Then, from (17), we obtain $g \in L_{p,v}(I)$ and $\int\limits_0^\infty g(t)dt = 0$. Let $\tilde{L}_{p,v}(I) = \{g \in L_{p,v} : \int\limits_0^\infty g(t)dt = 0\}$. Hence, the condition $f \in \mathring{W}^2_{p,v}(r, I)$ in (18) is equivalent to the condition $D_r^2 f = g \in \tilde{L}_{p,v}(I)$.

For $\tau \in I$, we consider two sets $\pounds_1 = \{g \in L_{p,v}(0,\tau) : g \leq 0\}$ and $\pounds_2 = \{g \in L_{p,v}(\tau,\infty) : g \geq 0\}$.

For each $g_1 \in \pounds_1$ and $g_2 \in \pounds_2$, we construct functions $g_2 \in \pounds_2$ and $g_1 \in \pounds_1$, such that $g(t) = g_1(t)$ for $0 < t \leq \tau$ and $g(t) = g_2(t)$ for $t > \tau$ belongs to the set $\tilde{L}_{p,v}(I)$.

We define a strictly decreasing function $\rho : (0,\tau) \to (\tau,\infty)$ from the relations

$$\int\limits_0^s v^{-p'}(t)dt = k_\tau \int\limits_{\rho(s)}^\infty v^{-p'}(t)dt,\ s \in (0,\tau),\quad \int\limits_0^{\rho^{-1}(s)} v^{-p'}(t)dt = k_\tau \int\limits_s^\infty v^{-p'}(t)dt,\ s \in (\tau,\infty), \tag{20}$$

where ρ^{-1} is the inverse function to the function ρ. From (20), it easily follows that the functions ρ and ρ^{-1} are locally absolutely continuous and $\rho(\tau) = \tau$, $\lim\limits_{s\to 0^+} \rho(s) = \infty$.

Differentiating the relations in (20), we have

$$\frac{1}{k_\tau} = \frac{v^{-p'}(\rho(s))}{v^{-p'}(s)}|\rho'(s)|,\ s \in (0,\tau),\ k_\tau = \frac{v^{-p'}(\rho^{-1}(s))}{v^{-p'}(s)}|(\rho^{-1}(s))'|,\ s \in (\tau,\infty). \tag{21}$$

For $g_1 \in \pounds_1$, we assume that

$$g_2(t) = -k_\tau g_1(\rho^{-1}(t))\frac{v^{-p'}(t)}{v^{-p'}(\rho^{-1}(t))},\ t > \tau. \tag{22}$$

Changing the variables $\rho^{-1}(t) = s$ and using the first relation in (21), the latter gives

$$\int\limits_\tau^\infty |v(t)g_2(t)|^p dt = k_\tau^{p-1}\int\limits_0^\tau |v(s)g_1(s)|^p ds < \infty, \tag{23}$$

i.e., $g_2 \in \pounds_2$. Similarly, for $g_2 \in \pounds_2$, we assume that

$$g_1(t) = -\frac{1}{k_\tau}g_2(\rho(t))\frac{v^{-p'}(t)}{v^{-p'}(\rho(t))},\ 0 < t < \tau, \tag{24}$$

and obtain that $g_1 \in \pounds_1$ and (23) holds.

In both cases, assuming that $g(t) = g_1(t)$ for $0 < t < \tau$ and $g(t) = g_2(t)$ for $t > \tau$, we have

$$\begin{aligned}\int_0^\infty |v(t)g(t)|^p dt &= (1 + k_\tau^{p-1}) \int_0^\tau |v(t)g_1(t)|^p dt \\ &= (1 + k_\tau^{1-p}) \int_\tau^\infty |v(t)g_2(t)|^p dt < \infty,\end{aligned} \tag{25}$$

i.e., $g \in L_{p,v}(I)$. For any $\tau \in I$, integrating both sides of (22) from τ to ∞ and (24) from 0 to τ, we obtain

$$\int_\tau^\infty g(t)dt = -\int_0^\tau g(t)dt,$$

i.e., $\int_0^\infty g(t)dt = 0$. Hence, $g \in \tilde{L}_{p,v}(I)$ is generated by the functions $g_1 \in \pounds_1$ and $g_2 \in \pounds_2$. Replacing the generated function $g \in \tilde{L}_{p,v}(I)$ in (3), and using (18), we obtain that inequality (3) has the form

$$\begin{aligned}&\left\{ \int_0^\tau \left[u(t) \int_\tau^\infty \left(\int_\tau^s r^{-1}(x)dx \right) g_2(s)ds + u(t) \int_t^\tau \left(\int_s^\tau r^{-1}(x)dx \right) |g_1(s)|ds \right.\right. \\ &\left.\left. + u(t) \int_t^\tau r^{-1}(x)dx \int_0^t |g_1(s)|ds \right]^q dt + \int_\tau^\infty \left(u(t) \int_t^\infty \left(\int_t^s r^{-1}(x)dx \right) g_2(s)ds \right)^q dt \right\}^{\frac{1}{q}} \\ &\qquad \le C \left(\int_0^\infty |v(t)g(t)|^p dt \right)^{\frac{1}{p}},\end{aligned} \tag{26}$$

where all terms in the left-hand side are non-negative.

Let the function $g \in \tilde{L}_{p,v}(I)$ be generated by $g_2 \in \pounds_2$. Then, from (25) and (26), we have

$$\left(\int_0^\tau u^q(t)dt \right)^{\frac{1}{q}} \int_\tau^\infty \left(\int_\tau^s r^{-1}(x)dx \right) g_2(s)ds \le C(1 + k_\tau^{1-p})^{\frac{1}{p}} \left(\int_\tau^\infty |v(t)g_2(t)|^p dt \right)^{\frac{1}{p}},$$

$$\left(\int_\tau^\infty \left(u(t) \int_t^\infty \left(\int_t^s r^{-1}(x)dx \right) g_2(s)ds \right)^q dt \right)^{\frac{1}{q}} \le C(1 + k_\tau^{1-p})^{\frac{1}{p}} \left(\int_\tau^\infty |v(t)g_2(t)|^p dt \right)^{\frac{1}{p}}.$$

Due to the arbitrariness of $g_2 \in \pounds_2$, on the basis of the reverse Hölder's inequality and Theorem 3, we obtain

$$B^+(\tau) \le C(1 + k_\tau^{1-p})^{\frac{1}{p}}, \; B_3^+(\tau) \le C(1 + k_\tau^{1-p})^{\frac{1}{p}},$$

i.e.,

$$\mathcal{B}^+(\tau) \le C(1 + k_\tau^{1-p})^{\frac{1}{p}}. \tag{27}$$

Similarly, for the function $g \in \tilde{L}_{p,v}(I)$ generated by $g_1 \in \pounds_1$, from (25) and (26), we have

$$F^-(\tau) \le C(1 + k_\tau^{p-1})^{\frac{1}{p}}. \tag{28}$$

From (27) and (28), we obtain

$$\mathcal{B}^{+}F^{-}=\inf_{\tau\in I}\max\{\mathcal{B}^{+}(\tau),F^{-}(\tau)\}\le C\inf_{\tau\in I}\Big[\max\{(1+k_{\tau}^{p-1})(1+k_{\tau}^{1-p})\}\Big]^{\frac{1}{p}}\le 4^{\frac{1}{p}}C,$$

which gives the left estimate in (14). Moreover, from (28), we get the left estimate in (15). The proof of Theorem 7 is complete. □

Let $0<\tau<\infty$. Let $B_1^{-}(0,\tau)\equiv B_1^{-}(\tau)$, $B_2^{-}(0,\tau)\equiv B_2^{-}(\tau)$,

$$B_3^{-}(\tau)=\left(\int_\tau^\infty u^q(t)dt\right)^{\frac{1}{q}}\left(\int_0^\tau\left(\int_s^\tau r^{-1}(x)dx\right)^{p'}v^{-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$F_1^{+}(\tau)=\sup_{z>\tau}\left(\int_z^\infty u^q(t)dt\right)^{\frac{1}{q}}\left(\int_\tau^z\left(\int_\tau^s r^{-1}(x)dx\right)^{p'}v^{-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$F_2^{+}(\tau)=\sup_{z>\tau}\left(\int_\tau^z\left(\int_\tau^t r^{-1}(x)dx\right)^{q}u^q(t)dt\right)^{\frac{1}{q}}\left(\int_z^\infty v^{-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$B^{-}(\tau)=\max\{B_1^{-}(\tau),B_2^{-}(\tau)\},\quad \mathcal{B}^{-}(\tau)=\max\{B^{-}(\tau),B_3^{-}(\tau)\},$$

$$F^{+}(\tau)=\max\{F_1^{+}(\tau),F_2^{+}(\tau)\},\quad \mathcal{B}^{-}F^{+}=\inf_{\tau\in I}\max\{\mathcal{B}^{-}(\tau),F^{+}(\tau)\}.$$

Theorem 8. *Let $1<p\le q<\infty$. Let $v^{-1}\in L_{p'}(I)$, $r^{-1}\in L_1(I_0)$, $r^{-1}\notin L_1(I_\infty)$ and*

$$\int_1^\infty v^{-p'}(t)\left(\int_1^t r^{-1}(x)dx\right)^{p'}dt=\infty.$$

Then, for the least constant C in (3), the estimates

$$4^{-\frac{1}{p}}\mathcal{B}^{-}F^{+}\le C\le 11p^{\frac{1}{q}}(p')^{\frac{1}{p'}}\mathcal{B}^{-}F^{+} \tag{29}$$

and

$$\sup_{\tau\in I}(1+k_{\tau}^{1-p})^{-\frac{1}{p}}F^{+}(\tau)\le C\le 11p^{\frac{1}{q}}(p')^{\frac{1}{p'}}F^{+}(\tau^{+}) \tag{30}$$

hold, where

$$\tau^{+}=\sup\{\tau>0:\mathcal{B}^{-}(\tau)\le F^{+}(\tau)\}. \tag{31}$$

Proof of Theorem 8. The conditions of Theorem 8 are symmetric to the conditions of Theorem 7. Therefore, the statement of Theorem 8 follows from the statement of Theorem 7. In inequality (3), under the conditions of Theorem 8, we change the variables $t=\frac{1}{x}$, then we obtain inequality (3) and the conditions of Theorem 7, where $u(x)$ is replaced by $\tilde{u}(x)=u(\frac{1}{x})x^{-\frac{2}{q}}$, $v(x)$ is replaced by $\tilde{v}(x)=v\left(\frac{1}{x}\right)x^{\frac{2}{p'}}$ and $r(x)$ is replaced by $\tilde{r}(x)=r\left(\frac{1}{x}\right)x^2$. Thus, the conditions of Theorem 8 turn to the conditions of Theorem 7 for the functions $\tilde{v}$ and $\tilde{r}$. Now, we use Theorem 7 and get the results with respect to the functions $\tilde{u}$, $\tilde{v}$ and $\tilde{r}$. Then, changing the variable to t, we obtain Theorem 8. The proof of Theorem 8 is complete. □

4. Oscillation Properties of Equation (4)

Two points t_1 and t_2, such that $t_1 \neq t_2$ of the interval I, are called conjugate with respect to Equation (4), if there exists a solution y of equation (4), such that $y(t_1) = y_2(t_2) = 0$ and $D_r^1 y(t_1) = D_r^1 y(t_2) = 0$. Equation (4) is called oscillatory at infinity (at zero), if for any $T \in I$, there exist conjugate points with respect to Equation (4) to the right (left) of T. Otherwise, Equation (4) is called non-oscillatory at infinity (at zero).

On the basis of Theorems 28 and 31 of [9], (see, e.g., Lemma 2.1 in [5]), we have the following variational lemmas.

Lemma 1. *Equation* (4) *is non-oscillatory at infinity if, and only if, there exists $T > 0$ and the inequality*

$$\int_T^\infty \left[v(t) \left| D_r^2 f(t) \right|^2 - u(t) \left| f(t) \right|^2 \right] dt \geq 0, f \in C_0^\infty(T, \infty), \tag{32}$$

holds.

Lemma 2. *Equation* (4) *is non-oscillatory at zero if, and only if, there exists $T > 0$ and the inequality*

$$\int_0^T \left[v(t) \left| D_r^2 f(t) \right|^2 - u(t) \left| f(t) \right|^2 \right] dt \geq 0, f \in C_0^\infty(0, T), \tag{33}$$

holds.

Equation (4) is the Euler–Lagrange equation of energy functional $\int_0^\infty \left[v(t) \left| D_r^2 f(t) \right|^2 - u(t) \left| f(t) \right|^2 \right] dt$.

Due to the compactness of supp$f$ for $f \in C_0^\infty(T, \infty)$, from (32), we have

$$\int_T^\infty u(t) \left| f(t) \right|^2 dt \leq \int_T^\infty v(t) \left| D_r^2 f(t) \right|^2 dt, f \in C_0^\infty(T, \infty). \tag{34}$$

Let $T \geq 0$. We consider the inequality

$$\int_T^\infty u(t) \left| f(t) \right|^2 dt \leq C_T \int_T^\infty v(t) \left| D_r^2 f(t) \right|^2 dt, f \in \mathring{W}_{p,v}^2(T, \infty). \tag{35}$$

From condition (34), we have the following lemma.

Lemma 3. *Let C_T be the least constant in* (35)*.*

(i) Equation (4) *is non-oscillatory at infinity if, and only if, there exists a constant $T > 0$, such that $0 < C_T \leq 1$ holds.*

(ii) Equation (4) *is oscillatory at infinity if, and only if, $C_T > 1$ for all $T \geq 0$.*

Proof of Lemma 3. Statements (i) and (ii) of Lemma 3 are equivalent. Let us prove statement (i). Let Equation (4) be non-oscillatory at infinity. Then, by Lemma 1, there exists $T > 0$, and (34) holds. This means that, for $T > 0$ inequality (35), holds with the least constant $C_T : 0 < C_T \leq 1$. Inversely, let $T > 0$ exist and inequality (35) hold with the least constant $C_T : 0 < C_T \leq 1$. Then, for $T > 0$, condition (34) is all the more correct. Therefore, by Lemma 1, Equation (35) is non-oscillatory at infinity. The proof of Lemma 3 is complete. □

We consider the inequality

$$\int_0^T u(t)\left|f(t)\right|^2 dt \leq C_T \int_0^T v(t)\left|D_r^2 f(t)\right|^2 dt, f \in \mathring{W}_{p,v}^2(0,T). \tag{36}$$

Similarly, we get one more lemma.

Lemma 4. *Let C_T be the least constant in (36).*

(i) Equation (4) is non-oscillatory at zero if, and only if, there exists a constant $T > 0$, such that $0 < C_T \leq 1$ holds.

(ii) Equation (4) is oscillatory at zero if, and only if, $C_T > 1$ for all $T \geq 0$.

On the basis of Lemmas 3, 4 and Theorems 7, 8, it is easy to establish different conditions of oscillation and non-oscillation of Equation (4) at zero and at infinity. Without dwelling on them, let us present problems, which are applied in the next Section 5.

We consider Equation (4) with the parameter $\lambda > 0$:

$$D_r^2\Big(v(t)D_r^2 y(t)\Big) - \lambda u(t)y(t) = 0, \quad t \in I. \tag{37}$$

Equation (37) is called strong oscillatory (non-oscillatory) at zero and at infinity, if it is oscillatory (non-oscillatory) for all $\lambda > 0$ at zero and at infinity, respectively.

From inequalities (35) and (36) for Equation (37), we respectively have

$$\lambda \int_T^\infty u(t)|f(t)|^2 dt \leq \lambda C_T \int_T^\infty v(t)|D_r^2 f(t)|^2 dt, \quad f \in \mathring{W}_{p,v}^2(T,\infty), \tag{38}$$

$$\lambda \int_0^T u(t)|f(t)|^2 dt \leq \lambda C_T \int_0^T v(t)|D_r^2 f(t)|^2 dt, \quad f \in \mathring{W}_{p,v}^2(0,T). \tag{39}$$

Lemma 5. *Let C_T be the least constant in (35) ((36)).*
(i) Equation (37) is strong non-oscillatory at infinity (at zero) if and only if $\lim\limits_{T\to\infty} C_T = 0$ ($\lim\limits_{T\to 0^+} C_T = 0$).
(ii) Equation (37) is strong oscillatory at infinity (at zero) if, and only if, $C_T = \infty$ ($C_T = \infty$) for any $T > 0$.

Proof of Lemma 5. Let us prove Lemma 5 at zero; the proof at infinity is similar.

(i) Let Equation (37) be non-oscillatory at zero. Then, by Lemma 4 for $\lambda > 0$, there exists T_1, such that $0 < \lambda C_{T_1} \leq 1$. Therefore, on the interval $(0, T_1)$, there do not exist conjugate points with respect to Equation (37). Then, for any $T:\ 0 < T < T_1$, on the interval $(0, T)$, there do not also exist conjugate points and $0 < \lambda C_T \leq 1$. Assume that $T_\lambda = \sup\{T > 0 : \lambda C_T \leq 1\}$. Then, $\lambda C_{T_\lambda} \leq 1$. Now, let equation be strong non-oscillatory at zero, then by Lemma 4 for any λ, there exists T_λ and $\lambda C_{T_\lambda} \leq 1$ or $C_{T_\lambda} \leq \frac{1}{\lambda}$. This gives that $\lim\limits_{\lambda\to\infty} C_{T_\lambda} = 0$. Let $0 < \lambda_1 < \lambda_2$ and $\lambda_2 C_{T_{\lambda_2}} \leq 1$. Then, $\lambda_1 C_{T_{\lambda_2}} \leq 1$. Hence, $T_{\lambda_1} \geq T_{\lambda_2}$ and T_λ do not increase in λ. Therefore, there exists $\lim\limits_{\lambda\to 0} T_\lambda = T_0$. If $T_0 > 0$, then $\lim\limits_{\lambda\to 0} C_{T_\lambda} = C_{T_0} = 0$. Then, from (36), it follows that $u(t) \equiv 0$ for $t \in (0, T_0)$. The obtained contradiction proves that $T_0 = 0$. Thus, $\lim\limits_{\lambda\to 0^+} C_{T_\lambda} = \lim\limits_{T\to 0^+} C_T = 0$.

Inversely, let $\lim\limits_{T\to 0^+} C_T = 0$. Then, for any λ, there exists $T(\lambda) > 0$ such that $\lambda C_{T(\lambda)} \leq 1$. Therefore, by Lemma 4, Equation (37) is non-oscillatory at zero for any $\lambda > 0$, which means that it is strong non-oscillatory at zero. The proof of Lemma 5 is complete. □

Now, on the basis of Lemma 5, we establish criteria of strong oscillation and non-oscillation of Equation (37) at zero and at infinity.

According to inequalities (35) and (36), in the expressions $B^-(\tau)$, $F^+(\tau)$, $B^+(\tau)$, and $F^-(\tau)$, we assume that $p = q = 2$; then, we replace u^2 by u and v^{-2} by v^{-1}. In addition, we assume that $\bar{B}^-(T,\tau) = (B^-(\tau))^2$, $\bar{F}^+(\tau) = (F^+(\tau))^2$, $\bar{B}^+(\tau) = (B^+(\tau))^2$, and $\bar{F}^-(\tau) = (F^-(\tau))^2$. Moreover, we take

$$\bar{B}_3^-(\tau) = \int\limits_\tau^\infty u(t)dt \int\limits_t^\tau \left(\int\limits_s^\tau r^{-1}(x)dx \right)^2 v^{-1}(s)ds$$

and

$$\bar{B}_3^+(\tau) = \int\limits_0^\tau u(t)dt \int\limits_\tau^\infty \left(\int\limits_\tau^s r^{-1}(x)dx \right)^2 v^{-1}(s)ds$$

instead of $B_3^-(\tau)$ and $B_3^+(\tau)$, respectively.

Theorem 9. *Let $v^{-1} \in L_1(I)$, $r^{-1} \notin L_1(I_0)$, $r^{-1} \in L_1(I_\infty)$ and*

$$\int\limits_0^1 v^{-1}(t) \left(\int\limits_t^1 r^{-1}(x)dx \right)^2 dt = \infty. \tag{40}$$

(i) Equation (37) is strong non-oscillatory at zero if, and only if,

$$\lim_{z\to 0^+} \int_0^z u(t)dt \int_z^\infty \left(\int_s^\infty r^{-1}(x)dx \right)^2 v^{-1}(s)ds = 0, \tag{41}$$

$$\lim_{z\to 0^+} \int_z^\infty \left(\int_t^\infty r^{-1}(x)dx \right)^2 u(t)dt \int_0^z v^{-1}(s)ds = 0. \tag{42}$$

(ii) Equation (37) is strong oscillatory at zero if, and only if,

$$\lim_{z\to 0^+} \int_0^z u(t)dt \int_z^\infty \left(\int_s^\infty r^{-1}(x)dx \right)^2 v^{-1}(s)ds = \infty \tag{43}$$

or

$$\lim_{z\to 0^+} \int_z^\infty \left(\int_t^\infty r^{-1}(x)dx \right)^2 u(t)dt \int_0^z v^{-1}(s)ds = \infty. \tag{44}$$

Proof of Theorem 9. (i) Suppose that Equation (37) is strong non-oscillatory at zero. Then, by Lemma 5, we have that $\lim\limits_{T\to 0^+} C_T = 0$ for the least constant C_T in inequality (36). From the left estimate in (15) for inequality (36), we have

$$\sup_{0<\tau<T} (1+k_\tau)^{-1} \bar{F}^-(\tau) \le C_T \tag{45}$$

for $T > 0$ and $\tau \in (0,T)$.

From the definition of k_τ on the interval $(0,T)$ it follows that $\lim\limits_{\tau\to 0^+} k_\tau = 0$. Therefore, $0 = \lim\limits_{T\to 0^+} C_T \ge \lim\limits_{\tau\to 0^+} (1+k_\tau)^{-1}\bar{F}^-(\tau) = \lim\limits_{\tau\to 0^+} \bar{F}^-(\tau)$. The latter gives that

$$\lim_{\tau\to 0^+} \bar{F}_i^-(\tau) = 0, \quad i = 1,2,\ldots \tag{46}$$

Denote the left-hand side of (42) by J. Then, there exists a sequence $\{z_n\}_{n=1}^\infty \subset I$, such that $\lim\limits_{n\to\infty} z_n = 0$ and

$$J = \lim_{n\to\infty} \int\limits_{z_n}^\infty \left(\int\limits_t^\infty r^{-1}(x)dx \right)^2 u(t)dt \int\limits_0^{z_n} v^{-1}(s)ds.$$

From the condition $r^{-1} \notin L_1(I_0)$ and (40) for any $\tau > 0$, we have

$$J = \lim_{n\to\infty} \sup_{0<z_n<\tau} \int_{z_n}^{\tau} \left(\int_t^{\tau} r^{-1}(x)dx \right)^2 u(t)dt \int_0^{z_n} v^{-1}(s)ds$$

$$\leq \lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_z^{\tau} \left(\int_t^{\tau} r^{-1}(x)dx \right)^2 u(t)dt \int_0^{z} v^{-1}(s)ds = \lim_{\tau\to 0^+} \bar{F}_2^-(\tau). \tag{47}$$

Similarly, we obtain

$$\lim_{z\to 0^+} \int_z^{\infty} \left(\int_s^{\infty} r^{-1}(x)dx \right)^2 v^{-1}(s)ds \int_0^z u(t)dt \leq \lim_{\tau\to 0^+} \bar{F_1}^-(\tau). \tag{48}$$

From (46)–(48), we have (41) and (42).
Inversely, let (41) and (42) hold. Then,

$$\lim_{z\to 0^+} \int_0^z u(t)dt \int_z^{\infty} \left(\int_s^{\infty} r^{-1}(x)dx \right)^2 v^{-1}(s)ds$$

$$= \lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_0^z u(t)dt \int_z^{\infty} \left(\int_s^{\infty} r^{-1}(x)dx \right)^2 v^{-1}(s)ds \geq \lim_{\tau\to 0^+} \bar{F_1}^-(\tau), \tag{49}$$

$$\lim_{z\to 0^+} \int_z^{\infty} \left(\int_t^{\infty} r^{-1}(x)dx \right)^2 u(t)dt \int_0^z v^{-1}(s)ds \geq \lim_{\tau\to 0^+} \bar{F_2}^-(\tau). \tag{50}$$

From (49) and (50), we obtain

$$\lim_{\tau\to 0^+} \bar{F}(\tau) = 0. \tag{51}$$

From the right side of (15) for inequality (36), we have

$$C_T \leq 22^2 \bar{F}^-(\tau^-), \tag{52}$$

where $\tau^- \in (0, T)$. Since $\lim\limits_{T\to 0^+} \tau^- = 0$, then from (51), we get

$$\lim_{T\to 0^+} C_T \leq 22^2 \lim_{T\to 0^+} \bar{F}^-(\tau^-) = 22^2 \lim_{\tau\to 0^+} \bar{F}^-(\tau) = 0.$$

Thus, $\lim\limits_{T\to 0^+} C_T = 0$ and by Lemma 5 Equation (37) is strong non-oscillatory at zero.

(ii) Let Equation (37) be strong oscillatory at zero. Then, by Lemma 5, for any $T > 0$, we have that $C_T = \infty$, where C_T is the least constant in (36). Therefore, from (52), we get $\bar{F}^-(\tau^-) = \infty$ for any $T > 0$. Since $\tau^- \in (0, T)$, then $\lim\limits_{\tau\to 0^+} \bar{F}^-(\tau) = \infty$. Hence, if $\lim\limits_{\tau\to 0^+} \bar{F_1}^-(\tau) = \infty$, then from (49), we get (43), and if $\lim\limits_{\tau\to 0^+} \bar{F_2}^-(\tau) = \infty$, then from (50) we get (44).

Inversely, let (43) hold. Then, from (48), it follows that $\lim\limits_{\tau\to 0^+} \bar{F_1}^-(\tau) = \infty$. Since $\bar{F}^-(\tau)$ does not decrease, then $\bar{F}^-(\tau) = \infty$ for any $\tau \in (0, T)$, and for any $T > 0$. Then, from (45), we get that $C_T = \infty$ for any $T > 0$. Hence, by Lemma 5, Equation (37) is strong oscillatory at zero. Similarly, if (44) holds, then from (47), we get that Equation (37) is strong oscillatory at zero. The proof of Theorem 9 is complete. □

Now, we assume that the function u together with the function v be positive, and sufficiently times continuously differentiable on the interval I. In the theory of oscillatory properties of differential equations, there is the reciprocity principle (see [18]), from which it follows that Equation (37) and its reciprocal equation

$$D_r^2\left(u^{-1}(t)D_r^2 y(t)\right) - \lambda v^{-1}(t)y(t) = 0, \ \ t \in I, \tag{53}$$

are simultaneously oscillatory or non-oscillatory.

On the basis of this reciprocity principle, from Theorem 9, we have the following statement.

Theorem 10. *Let $u \in L_1(I)$, $r^{-1} \in L_1(I_0)$, $r^{-1} \notin L_1(I_\infty)$ and*

$$\int\limits_1^\infty u(t)\left(\int\limits_1^t r^{-1}(x)dx\right)^2 dt = \infty.$$

(i) Equation (37) is strong non-oscillatory at zero if, and only if, (41) and (42).
(ii) Equation (37) is strong oscillatory at zero if, and only if, (43) and (44).

Proof of Theorem 10. The statement of Theorem 10 follows from the fact that the conditions of Theorem 10 are the conditions of Theorem 9 for Equation (53). Therefore, applying Theorem 9 to Equation (53), we obtain necessary and sufficient conditions for the non-oscillation and oscillation of Equation (53) at zero, while non-oscillation conditions (41) and (42) of Equation (37) are reduced to non-oscillation conditions (42) and (41) of Equation (53). Since, according to the reciprocity principle, the non-oscillation of Equation (53) at zero is equivalent to non-oscillation of Equation (37); i.e., we have that statement (i) of Theorem 10 is correct. Statement (ii) of Theorem 10 can be proven in the same way. The proof of Theorem 10 is complete. □

Similarly, on the basis of inequality (38), we have the following theorem.

Theorem 11. *Let $v^{-1} \in L_1(I)$, $r^{-1} \in L_1(I_0)$, $r^{-1} \notin L_1(I_\infty)$ and*

$$\int_1^\infty v^{-1}(t)\left(\int_1^t r^{-1}(x)dx\right)^2 dt = \infty.$$

(i) Equation (37) is strong non-oscillatory at infinity if, and only if,

$$\lim_{z\to\infty}\int_z^\infty u(t)dt\int_0^z\left(\int_0^s r^{-1}(x)dx\right)^2 v^{-1}(s)ds = 0, \tag{54}$$

$$\lim_{z\to\infty}\int_0^z\left(\int_0^t r^{-1}(x)dx\right)^2 u(t)dt\int_z^\infty v^{-1}(s)ds = 0. \tag{55}$$

(ii) Equation (37) is strong oscillatory at infinity if, and only if,

$$\lim_{z\to\infty}\int_z^\infty u(t)dt\int_0^z\left(\int_0^s r^{-1}(x)dx\right)^2 v^{-1}(s)ds = \infty \tag{56}$$

or

$$\lim_{z\to\infty}\int_0^z\left(\int_0^t r^{-1}(x)dx\right)^2 u(t)dt\int_z^\infty v^{-1}(s)ds = \infty. \tag{57}$$

Proof of Theorem 11. The conditions and the statement of Theorem 11 are symmetric to the conditions and the statement of Theorem 9, respectively. Therefore, arguing similarly as in Theorem 8, we obtain the validity of Theorem 11. The proof of Theorem 11 is complete. □

The next statement follows from the application of Theorem 11 to Equation (53), using the reciprocity principle, as in Theorem 10.

Theorem 12. *Let $u \in L_1(I)$, $r^{-1} \in L_1(I_0)$, $r^{-1} \notin L_1(I_\infty)$ and*

$$\int_1^\infty u(t)\left(\int_1^t r^{-1}(x)dx\right)^2 dt = \infty.$$

(i) Equation (37) is strong non-oscillatory at infinity if, and only if, (54) and (55) hold.
(ii) Equation (37) is strong oscillatory at infinity if, and only if, (56) and (57) hold.

5. Spectral Characteristics of Differential Operator L

The spectral properties of fourth and higher-order operators in form (8) have been studied in many works (see, e.g., [9,19] (Chapters 29 and 34), [12–14]), when the function v^{-1} is strong singular at zero and at infinity. Here, operator (5) is investigated in the case of weak singularity of the functions $v^{-p'}$ and r^{-1} at zero and at infinity.

Let the minimal differential operator $L_{\min}$ be generated by differential expression

$$ly(t) = \frac{1}{u(t)} D_r^2\left(v(t)D_r^2 y\right)$$

in the space $L_{2,u} \equiv L_2(u; I)$ with inner product $(f,g)_{2,u} = \int\limits_0^\infty f(t)g(t)u(t)dt$, i.e., $L_{\min}y = ly$ is an operator with the domain $D(L_{\min}) = C_0^\infty(I)$.

It is known that all self-adjoint extensions of the minimal differential operator L have the same spectrums (see [9]).

Let us consider the problem of boundedness from below, and the discreteness of the operator L.

One of the most important problems in the theory of singular differential operators is to find conditions which guarantee that any self-adjoint extension L of the operator $L_{\min}$ has a spectrum, which is discrete and bounded below; the so-called property BD [8]. Property BD means, roughly speaking, that the singular operator behaves like a regular one, since it is known that the spectrum of regular operators consists only of eigenvalues of finite multiplicities, with the only possible cluster point at infinity.

The relationship between the oscillatory properties of Equation (37) and spectral properties of the operator L is explained in the following statement.

Lemma 6 ([9]). *The operator L is bounded below and has a discrete spectrum if, and only if, Equation (37) is strong non-oscillatory.*

On the basis of Lemma 6, from Theorems 9–12 as corollaries, we obtain the following propositions.

Proposition 1. *Let the conditions of Theorem 9 or 10 hold. Then, the operator L is bounded below and has a discrete spectrum if, and only if, (41) and (42) hold.*

Proposition 2. *Let the conditions of Theorem 11 or 12 hold. Then, the operator L is bounded below and has a discrete spectrum if, and only if, (54) and (55) hold.*

The operator L_{min} is non-negative. Therefore, it has the Friedrich's extension L_F. By Propositions 1 and 2, the operator L_F has a discrete spectrum if, and only if, (41) and (42) hold under the conditions of Proposition 1, and (54) and (55) hold under the conditions of Proposition 2.

Since for $p = q = 2$, inequality (3) can be rewritten as $(f,f)_2 C^{-1} \le (L_F f, f)_{2,u}$, then from Theorems 7 and 8, we have the following propositions.

Proposition 3. *Let the conditions of Theorem 9 hold. Then, the operator L_F is positive-definite if, and only if, $\bar{\mathcal{B}}^+\bar{F}^- = \inf_{\tau\in I}\max\{\bar{\mathcal{B}}^+(\tau), \bar{F}^-(\tau)\} < \infty$. Moreover, there exist constants α, β : $0 < \alpha < \beta$ and the estimate $\alpha\bar{\mathcal{B}}^+\bar{F}^- \leq \lambda_1^{-1} \leq \beta\bar{\mathcal{B}}^+\bar{F}^-$ holds for the smallest eigenvalue λ_1 of the operator L_F.*

Proposition 4. *Let the conditions of Theorem 11 hold. Then, the operator L_F is positive-definite if, and only if, $\bar{\mathcal{B}}^-\bar{F}^+ = \inf_{\tau\in I}\max\{\bar{\mathcal{B}}^-(\tau), \bar{F}^+(\tau)\} < \infty$. Moreover, there exist constants α, β : $0 < \alpha < \beta$ and the estimate $\alpha\bar{\mathcal{B}}^-\bar{F}^+ \leq \lambda_1^{-1} \leq \beta\bar{\mathcal{B}}^-\bar{F}^+$ holds for the smallest eigenvalue λ_1 of the operator L_F.*

Let us note that for the operator L_F, from Theorem 7 under the conditions of Theorem 9, we have the following spectral problem

$$\begin{cases} D_r^2\Big(v(t)D_r^2 y(t)\Big) = \lambda u(t)y(t), \\ D_r' y(0) = D_r' y(\infty) = y(\infty) = 0, \end{cases}$$

while from Theorem 8 under the conditions of Theorem 11, we have the following spectral problem

$$\begin{cases} D_r^2\Big(v(t)D_r^2 y(t)\Big) = \lambda u(t)y(t), \\ y(0) = D_r' y(0) = D_r' y(\infty) = 0. \end{cases}$$

Since according to Rellih's lemma (see [20], p. 183), the operator L_F^{-1} has a discrete spectrum bounded below in $L_{2,u}$ if, and only if, the space with the norm $(L_F f, f)_{2,u}^{\frac{1}{2}}$ is compactly embedded into the space $L_{2,u}$, then from Propositions 1 and 2, we have one more statement.

Proposition 5. *Let the conditions of Theorem 9 (Theorem 11) hold. Then, the embedding $\mathring{W}^2_{2,v}(I) \hookrightarrow L_{2,u}$ is compact and the operator L_F^{-1} is completely continuous on $L_{2,u}$ if, and only if, (41) and (42) ((54) and (55)) hold.*

The following statement is from the work [7].

Lemma 7. *Let $H = H(I)$ be a certain Hilbert function space and $C[0,\infty)\bigcap H$ be dense in it. For any point $x_0 \in I$, we introduce the operator $E_{x_0}f = f(x_0)$ defined on $C[0,\infty)\bigcap H$, which acts in the space of complex numbers. Let us assume that E_{x_0} is a closure operator. Then, the norm of this operator is equal to the value $\left(\sum\limits_{n=1}^{\infty}|\varphi_n(x_0)|^2\right)^{\frac{1}{2}}$ (finite or infinite), where $\{\varphi_n(\cdot)\}_{n=1}^{\infty}$ is any complete orthonormal system of continuous functions in H.*

Lemma 8. *Let the conditions of Theorem 9 hold. Then, for $t \in I$*

$$\sup_{\tau\in I} D^+(t,\tau) \leq \sup_{f\in\mathring{W}^2_{2,v}} \frac{|f(t)|}{\|D_r^2 f\|_{2,v}} \leq \sqrt{2}\inf_{\tau\in I} D^+(t,\tau), \tag{58}$$

where

$$D^+(t,\tau) = \Big[\chi_{(0,\tau)}(t)\int_\tau^\infty\left(\int_\tau^s r^{-1}(x)dx\right)^2 v^{-1}(s)ds + \chi_{(0,\tau)}(t)\int_t^\tau\left(\int_s^\tau r^{-1}(x)dx\right)^2 v^{-1}(s)ds$$
$$+\chi_{(0,\tau)}(t)\left(\int_t^\tau r^{-1}(x)dx\right)^2\int_0^t v^{-1}(s)ds + \chi_{(\tau,\infty)}(t)\int_t^\infty\left(\int_t^s r^{-1}(x)dx\right)^2 v^{-1}(s)ds\Big]^{\frac{1}{2}}$$

and

$$\|D_r^2 f\|_{2,v} = \left(\int\limits_0^\infty v(t)|D_r^2 f(t)|^2 dt \right)^{\frac{1}{2}}.$$

Proof of Lemma 8. Let $\tau \in I$. In (18), for the function $f \in \mathring{W}^2_{2,v}$ we have

$$f(t) = \chi_{(0,\tau)}(t) \left[\int\limits_\tau^\infty \left(\int\limits_\tau^s r^{-1}(x)dx \right) D_r^2 f(s)ds \right.$$
$$\left. - \int\limits_t^\tau \left(\int\limits_s^\tau r^{-1}(x)dx \right) D_r^2 f(s)ds - \int\limits_t^\tau r^{-1}(x)dx \int\limits_0^t D_r^2 f(s)ds \right] \tag{59}$$
$$+ \chi_{(\tau,\infty)}(t) \int\limits_t^\infty \left(\int\limits_t^s r^{-1}(x)dx \right) D_r^2 f(s)ds.$$

In (59), we take the modulus in both parts and first applying the Hölder's inequality in the integrals of each term, then in the sum, we obtain

$$|f(t)| \le \sqrt{2} \inf_{\tau \in I} D^+(t,\tau) \|D_r^2 f\|_{2,v}, \quad \forall f \in \mathring{W}^2_{2,v}.$$

Therefore, the right estimate in (58) is valid.

Now, let us show the left estimate in (58). We fix $t \in I$ in (59) and select a function $D_r^2 f$ depending on t as follows

$$(D_r^2 f)_t(s) = \begin{cases} \chi_{(0,t)}(s) \left(\int_t^\tau r^{-1}(x)dx \right) v^{-1}(s) & \text{if } 0 < t < \tau, \\ -\chi_{(t,\tau)}(s) \left(\int_s^\tau r^{-1}(x)dx \right) v^{-1}(s) & \text{if } 0 < t < \tau, \\ -\chi_{(\tau,\infty)}(s) \int_\tau^s r^{-1}(x)dx v^{-1}(s) & \text{if } 0 < t < \tau, \\ \chi_{(t,\infty)}(s) \left(\int_t^s r^{-1}(x)dx \right) v^{-1}(s) & \text{if } t > \tau. \end{cases}$$

Replacing this function in (59), we get the value of the function $f(D_r^2 f)_t(z)$ at the point $z = t$:

$$f_t(t) = \chi_{(0,\tau)}(t) \int_\tau^\infty \left(\int_\tau^s r^{-1}(x)dx \right)^2 v^{-1}(s)ds + \chi_{(0,\tau)}(t) \int_t^\tau \left(\int_s^\tau r^{-1}(x)dx \right)^2 v^{-1}(s)ds$$
$$+ \chi_{(0,\tau)}(t) \left(\int_t^\tau r^{-1}(x)dx \right)^2 \int_0^t v^{-1}(s)ds$$
$$+ \chi_{(\tau,\infty)}(t) \int_t^\infty \left(\int_t^s r^{-1}(x)dx \right)^2 v^{-1}(s)ds = (D^+(t,\tau))^2. \tag{60}$$

Let us calculate the norm $L_{2,u}$ of the function $(D_r^2 f)_t$:

$$\left(\int_0^\infty v(s)|(D_r^2 f)_t(s)|^2 ds \right)^{\frac{1}{2}} = \left(\int_0^\tau v(s)|(D_r^2 f)_t(s)|^2 ds + \int_\tau^\infty v(s)|(D_r^2 f)_t(s)|^2 ds \right)^{\frac{1}{2}}$$
$$= \left[\chi_{(0,\tau)}(t) \int_\tau^\infty \left(\int_\tau^s r^{-1}(x)dx \right)^2 v^{-1}(s)ds + \chi_{(0,\tau)}(t) \int_t^\tau \left(\int_s^\tau r^{-1}(x)dx \right)^2 v^{-1}(s)ds \right.$$
$$+ \chi_{(0,\tau)}(t) \left(\int_t^\tau r^{-1}(x)dx \right)^2 \int_0^t v^{-1}(s)ds$$
$$\left. + \chi_{(\tau,\infty)}(t) \int_t^\infty \left(\int_t^s r^{-1}(x)dx \right)^2 v^{-1}(s)ds \right]^{\frac{1}{2}} = D^+(t,\tau). \tag{61}$$

From (60) and (61), we get

$$\sup_{f\in \mathring{W}^2_{2,v}} \frac{|f(t)|}{\|D^2_r f\|_{2,v}} \geq \frac{|f_t(t)|}{\|(D^2_r f)_t\|_{2,v}} = D^+(t,\tau)$$

for any $\tau \in I$. This relation proves the correctness of the left estimate in (58). The proof of Lemma 8 is complete. □

Let the operator L_F^{-1} be completely continuous on $L_{2,u}$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be eigenvalues and $\{\varphi_k\}_{k=1}^{\infty}$ be a corresponding complete orthonormal system of eigenfunctions of the operator L_F^{-1}.

Let

$$D^+(t) = \int_t^\infty \left(\int_s^\infty r^{-1}(x)dx\right)^2 v^{-1}(s)ds + \int_t^\infty r^{-1}(x)dx \int_0^t v^{-1}(s)ds, \quad t \in I.$$

Theorem 13. *Let the conditions of Theorem 9 hold. Let (41) and (42) hold. Then,*

$$(i) \qquad (D^+(t))^2 \leq \sum_{k=1}^{\infty} \frac{|\varphi_k(t)|^2}{\lambda_k} \leq 2(D^+(t))^2, t \in I. \tag{62}$$

(ii) The operator L_F^{-1} is nuclear if, and only if, $\int\limits_0^\infty u(t)\Big(D^+(t)\Big)^2 dt < \infty$ and for the nuclear norm $\|L_F^{-1}\|_{\sigma_1}$ of the operator L_F^{-1}, the relation

$$\int_0^\infty u(t)\Big(D^+(t)\Big)^2 dt \leq \|L_F^{-1}\|_{\sigma_1} = \sum_{k=1}^{\infty}\frac{1}{\lambda_k} \leq 2\int_0^\infty u(t)\Big(D^+(t)\Big)^2 dt \tag{63}$$

holds.

Proof of Theorem 13. By the condition of Theorem 13, we have that the operator L_F^{-1} is completely continuous on $L_{2,u}$ (see Proposition 5). In Lemma 7, we take $\mathring{W}^2_{2,v}(I)$ with the norm $\|D^2_r f\|_{2,v}$ as the space $H(I)$. Since the system of functions $\{\lambda_k^{-\frac{1}{2}}\varphi_k\}_{k=1}^{\infty}$ is a complete orthonormal system in the space $\mathring{W}^2_{2,v}(I)$, then by Lemma 7, we have

$$\|E_x\|^2 = \left(\sup_{f\in \mathring{W}^2_{2,v}(I)} \frac{|f(t)|}{\|D^2_r f\|_{2,v}}\right)^2 = \sum_{k=1}^{\infty}\frac{|\varphi_k(t)|^2}{\lambda_k},$$

where $E_t f = f(t)$. The latter and (58) give

$$\sup_{\tau\in I}\Big(D^+(t,\tau)\Big)^2 \leq \sum_{k=1}^{\infty}\frac{|\varphi_k(t)|^2}{\lambda_k} \leq 2\inf_{\tau\in I}\Big(D^+(t,\tau)\Big)^2. \tag{64}$$

Since

$$\inf_{\tau\in I} D^+(t,\tau) \leq \lim_{\tau\to\infty} D^+(t,\tau) = D^+(t), \;\; \sup_{\tau\in I} D^+(t,\tau) \geq \lim_{\tau\to\infty} D^+(t,\tau) = D^+(t),$$

then, from (64), we have (62). Multiplying both sides of (62) by u integrating them from zero to infinity, we get (63). The proof of Theorem 13 is complete. □

Let

$$D^-(t) = \int_0^t \left(\int_0^s r^{-1}(x)dx\right)^2 v^{-1}(s)ds + \int_0^t r^{-1}(x)dx \int_t^\infty v^{-1}(s)ds, \quad t \in I.$$

Similarly, we have the following statement.

Theorem 14. *Let the conditions of Theorem 11 hold. Let (54) and (55) hold. Then,*

$$(i) \qquad \left(D^-(t)\right)^2 \leq \sum_{k=1}^{\infty} \frac{|\varphi_k(t)|^2}{\lambda_k} \leq 2\left(D^-(t)\right)^2.$$

(ii) The operator L_F^{-1} is nuclear if, and only if, $\int\limits_0^\infty u(t)\left(D^-(t)\right)^2 dt < \infty$ and for the nuclear norm $\|L_F^{-1}\|_{\sigma_1}$ of the operator L_F^{-1} the relation

$$\int_0^\infty u(t)\left(D^-(t)\right)^2 dt \leq \|L_F^{-1}\|_{\sigma_1} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \leq 2\int_0^\infty u(t)\left(D^-(t)\right)^2 dt$$

holds.

6. Conclusions

In the paper, we establish inequality (3) and find two-sided estimate of its least constant, so that the finiteness of the values B_1^+, B_2^+, B_3^+, F_1^-, and F_2^- are necessary and sufficient for the validity of inequality (3). We extend the classical variational principle by proving Lemma 3, which gives the connection between inequality (3) and oscillatory properties of Equation (4). On the basis of the results on inequality (3) and Lemma 3, we obtain necessary and sufficient conditions for strong non-oscillation and oscillation of Equation (4). Let us note that, among the five values B_1^+, B_2^+, B_3^+, F_1^-, and F_2^- participating in the conditions for the validity of inequality (3), the non-oscillation and oscillation of Equation (4) depend only on the values F_1^- and F_2^-. On the basis of the connection between non-oscillation of Equation (4) and spectral properties of the operator L, we get its property BD, two-sided estimates for its first eigenvalue, and criteria for its nuclearity.

Author Contributions: All three authors have, on an equal level, discussed and posed the research questions in this paper. A.K. has helped to prove the main results and to type the manuscript. R.O. is the main author concerning the proofs of the main results. Y.S. has put the results into a more general frame and instructed how to write the manuscript in this final form. All authors have read and agreed to the published version of the manuscript.

Funding: The research was funded by the Ministry of Education and Science of the Republic of Kazakhstan, grant No. AP08856100 in the area "Research in the field of natural sciences".

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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