Optik - International Journal for Light and Electron Optics 261 (2022) 169120

Contents lists available at ScienceDirect

Optik - International Journal for Light and Electron Optics

journal homepage: www.elsevier.com/locate/ijleo

Original research article

Quadratic-phase wave packet transform

M. Younus Bhat [a], Aamir H. Dar [a], Didar Urynbassarova [b,c,*], Altyn Urynbassarova [d,e]

[a] Department of Mathematical Sciences, Islamic University of Science and Technology Awantipora, Pulwama, Jammu and Kashmir 192122, India
[b] School of Information and Electronics, Beijing Institute of Technology, Beijing 100081, China
[c] Beijing Key Laboratory of Fractional Signals and Systems, Beijing 100081, China
[d] Department of Technology and Ecology, School of Society, Technology and Ecology, Narxoz University, Almaty 050000, Kazakhstan
[e] Department of Information Security, Faculty of Information Technology, L.N. Gumilyov Eurasian National University, Nur-Sultan 010000, Kazakhstan

ARTICLE INFO

MSC:
42B10
42A05
42A38
44A20
42C40

Keywords:
Quadratic-phase Fourier transform
Quadratic-phase wave packet transform
Energy conservation
Uncertainty inequality

ABSTRACT

The quadratic-phase Fourier transform (QPFT) has gained much popularity in recent years because of its applications in image and signal processing. However, the QPFT is inadequate for localizing the quadratic-phase spectrum, which is required in some applications. In this paper, the quadratic-phase wave packet transform (QP-WPT) is proposed to address this problem, based on the wave packet transform (WPT) and QPFT. Firstly, we propose the definition of the QP-WPT and give its relation with windowed Fourier transform (WFT). Secondly, several notable inequalities and important properties of newly defined QP-WPT, such as boundedness, reconstruction formula, Moyal's formula, reproducing kernel are derived. Finally, we formulate several classes of uncertainty inequalities, such as Leib's uncertainty principle, logarithmic uncertainty inequality and the Heisenberg uncertainty inequality.

1. Introduction

The Fourier transform (FT) is an important tool in optical communication and signal processing [1]. However, owing to its global kernel the FT is incapable of obtaining information about local properties of the signal. But, the actual signals are often non-stationary or time-variable, so to overcome this problem, the short-time Fourier transform (STFT) is employed that uses a time window of fixed length applied at regular intervals so that we can obtain a portion of the signal considered to be stationary [2]. The resulting time-varying spectral depiction is critical for non-stationary signal analysis, but in this case it comes at fixed spectral and temporal resolution. The wavelet analysis [3,4] provides an attractive and pinch-hitting tool to the STFT by using an optical multichannel correlator with a bank of wavelet transform (WT) filters, which can provide a better illustration of the signal instead of the STFT. Nonetheless, in the high frequency region WT has poor frequency resolution. To solve this defect the wave packet transform (WPT) was proposed by combining the merits of STFT and WT [5,6]. WPT is a linear transform which uses the Weyl operator and the wave packages.

In recent years, researchers have successfully applied WPT in the fields of wireless communication, denoising, and image compression [7–14]. WPT is used widely in signal processing as it has some better morality than WT [15,16]. Moreover, it can realize multilevel decomposition and analyze the high frequency decomposition that is not achieved in traditional discrete WT. The frequency subbands of signal are selected via wave packet decomposition, that improves the time–frequency resolution capability

* Corresponding author at: School of Information and Electronics, Beijing Institute of Technology, Beijing 100081, China.
E-mail addresses: gyounusg@gmail.com (M.Y. Bhat), ahdkul740@gmail.com (A.H. Dar), didaruzh@mail.ru (D. Urynbassarova).

https://doi.org/10.1016/j.ijleo.2022.169120
Received 6 January 2022; Accepted 14 April 2022
Available online 20 April 2022

of the signal. However, the WPT is defined as the FT of the signal windowed with the wavelet, so the results obtained by WPT will not be optimal in dealing with chirp signals whose energy is not well concentrated in FT domain.

A superlative generalized version of the FT called quadratic-phase Fourier transform (QPFT) has been introduced by Castro et al. [17,18]. This novel transform has overthrown all the applicable signal processing tools as it provides a unified analysis of both transient and non-transient signals in an easy and insightful fashion. The QPFT is actually a generalization of several well known transforms like Fourier, fractional Fourier and linear canonical transforms whose kernel is in the exponential form. Due to its extra degrees of freedom, the QPFT has marked its importance in treatment of problems demanding several controllable parameters arising in diverse branches of science and engineering, including harmonic analysis, sampling, image processing, and so on [18–24].

Recently Prasad and Sharma [25] introduced the quadratic-phase Fourier wavelet transform (QPFWT), which is generalization of classical continuous wavelet transform (CWT) [26–32], continuous fractional wavelet transform [33–35], as well as generalization of linear canonical wavelet transform [35,36]. QPFWT intertwine the advantages of the quadratic-phase Fourier and wavelet transforms into a novel integral transform which assimilates their individual properties. However, the transform neither relies on the complete kernel of the QPFT nor exhibits any existing convolution structure in the QPFT. So Shah and Lone [19] introduced quadratic-phase wavelet in different approach which is completely reliant upon convolution associated with QPFT.

As one of the generalization of the classical WPT, the fractional wave packet transform (Fr-WPT) and linear canonical wave packet transform (LC-WPT) have been introduced to improve the performance in concentration [37–39]. They have attained a much more attention of the signal processing community and optics. But to the best of our knowledge theory about quadratic-phase wave packet transform (QP-WPT) have never been proposed up to date, therefore it is worthwhile to study the theory of QP-WPT based on the WPT and QPFT, which can be productive for signal processing theory and applications. Therefore, the cynosure of this paper is to rigorously study the QP-WPT.

The highlights of the paper are pointed out below:

- To introduce a novel integral transform coined as the quadratic-phase wave packet transform.
- To establish relationship between quadratic-phase wave packet transform with FT and windowed Fourier transform (WFT).
- To study several notable inequalities and important properties of newly defined QP-WPT, such as boundedness, reconstruction formula, Moyal's formula, reproducing kernel.
- To formulate several classes of uncertainty inequalities, such as Leib-type, the logarithmic uncertainty inequalities and the Heisenberg-type uncertainty inequalities associated with the QP-WPT.

The paper is organized as follows. In Section 2, we provide some preliminary results required in subsequent sections. In Section 3, we provide the definition of QP-WPT. Then, we investigated several basic properties of the QP-WPT which are important for signal representation in signal processing. In Section 4, we develop a series of uncertainty inequalities such as Leib's uncertainty principle, the logarithmic uncertainty inequality and the Heisenberg-type inequality associated with the QP-WPT. Finally, a conclusion is extracted in Section 5.

2. Preliminaries

In this section we recall some basic concepts and notations, which will be useful in our study on QP-WPT.

2.1. Fourier transform (FT)

We use the following definition of FT [1] on $L^1(\mathbb{R})$ space

$$\mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\xi} dt, \quad \forall \xi \in \mathbb{R}. \tag{2.1}$$

2.2. Continuous wavelet transform (CWT)

Wavelet transform presents an attractive alternative to the STFT by using a time–frequency window that changes with frequency, which can effectively provide resolution of varying granularity. The CWT of a signal $f(t) \in L^2(\mathbb{R})$ is defined as [28,29]

$$CWT_f(\beta, \alpha) = \frac{1}{\sqrt{\alpha}} \int_{\mathbb{R}} f(t) \psi^* \left(\frac{t-\beta}{\alpha} \right) dt, \tag{2.2}$$

where $*$ denotes the complex conjugate and t is time, β is the translation parameter, α is the scaling parameter and $\psi(t)$ is the transforming function, called mother wavelet. Here $\alpha > 0$ and ψ is normalized such that the $\|\psi\| = 1$ in $L^2(\mathbb{R})$ space.

2.3. Windowed fourier transform

The WFT of $f(t) \in L^2(\mathbb{R})$ with respect to the windowed function $\phi \in L^2(\mathbb{R})$ is defined as [40]

$$\mathcal{G}_\phi[f](w,\beta) = \int_{\mathbb{R}} f(t)\phi^*(t-\beta)e^{-i\xi t}dt \tag{2.3}$$

and the inverse of the function $f(t) \in L^2(\mathbb{R})$ is defined by [24]

$$f(t) = \frac{b}{2\pi\langle \phi, \psi\rangle}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{G}_\phi[f](\xi,\beta)e^{i\xi t}\psi(t-\beta)d\xi d\beta, \tag{2.4}$$

where, $\psi \in L^2(\mathbb{R})$.

2.4. Wave packet transform (WPT)

The WPT combines elements of STFT and CWT, and it can be viewed as [7,8]

$$WPT_f(\xi,\beta,\alpha) = \frac{1}{\sqrt{2\pi\alpha}}\int_{\mathbb{R}} f(t)\overline{\psi_\alpha(t-\beta)}e^{-i\xi t}dt \tag{2.5}$$

where $\psi_\alpha(t-\beta) = \psi\left(\frac{t-\beta}{\alpha}\right)$.

The WPT is the FT of a signal windowed with a wavelet that is dilated by α and translated by β.

Lemma 2.1 (*[38]*). *Let $\psi \in L^p(\mathbb{R}), \quad p \in [1,\infty)$. Then, $\|\psi_\alpha(t-\beta)\|_{L^p(\mathbb{R})} = \alpha^{(1/p-1/2)}\|\psi\|_{L^p(\mathbb{R})}$.*

2.5. Quadratic-phase fourier transform (QPFT)

In this subsection we introduce the QPFT, which is a neoteric addition to the classical integral transforms and we also gave its inversion formula and some other classical results which are already present in literature.

Definition 2.1. Given a parameter $\mu = (a,b,c,d,e)$, the QPFT of any signal f is defined by [25]

$$\mathcal{Q}_\mu[f](\xi) = \int f(t)K_\mu(t,\xi)dt, \tag{2.6}$$

where $K_\mu(t,\xi)$ is the quadratic-phase Fourier kernel, given by

$$K_\mu(t,\xi) = \sqrt{\frac{b}{2\pi i}}e^{i(at^2+bt\xi+c\xi+dt+e\xi)} \tag{2.7}$$

with $a,b,c,d,e \in \mathbb{R}, \quad b \neq 0$.

Theorem 2.1. *The inversion formula of the QPFT is given by [25]*

$$f(t) = \int \mathcal{Q}_\mu[f](\xi)\overline{K_\mu(t,\xi)}d\xi. \tag{2.8}$$

Using the inversion theorem, we can get the Parseval's relation given by [25]

$$\langle f,g\rangle = \langle \mathcal{Q}_\mu[f], \mathcal{Q}_\mu[g]\rangle \tag{2.9}$$

and Plancherel identity is given by

$$\int |\mathcal{Q}_\mu[f](\xi)|^2 d\xi = \int |f(t)|^2 dt. \tag{2.10}$$

Theorem 2.2 (*[19,20]*). *Let $f,g \in L^2(\mathbb{R})$ and $\alpha,\beta,\tau \in \mathbb{R}$, then*

- $\mathcal{Q}_\mu[\alpha f + \beta g](\xi) = \alpha\mathbb{Q}_\mu[f](w) + \beta\mathbb{Q}_\mu[g](w)$.
- $\mathcal{Q}_\mu[f(t-\tau)](\xi) = exp\{-i(a\tau^2 + b\tau\xi + d\tau)\}\mathcal{Q}_\mu[e^{-2ia\tau t}f(t)](\xi)$.
- $\mathcal{Q}_\mu[f(-t)](\xi) = \mathcal{Q}_{\mu'}[f(t)](-\xi), \quad \mu' = (a,b,c,-d,-e)$.
- $\mathcal{Q}_\mu[e^{i\alpha t}f(t)](\xi) = exp\{i(\alpha^2 + 2\alpha b\xi + \alpha eb)\frac{1}{b}\}\mathcal{Q}_\mu[f]\left(w + \frac{a}{b}\right)$.
- $\mathcal{Q}_\mu[\overline{f(t)}](w) = \overline{\mathcal{Q}_{-\mu}[f(t)](w)}$.

Theorem 2.3 (*Convolution [25]*). *If $f,g \in L^2(\mathbf{R})$, then*

$$\mathcal{Q}_\mu[f *_\mu g](\xi) = \sqrt{\frac{2\pi i}{b}}e^{-i(c\xi^2+e\xi)}\mathcal{Q}_\mu[f](\xi)\mathcal{Q}_\mu[e^{-ia(\cdot)^2-id(\cdot)}g](\xi), \tag{2.11}$$

where

$$(f *_\mu g)(t) = \int_{\mathbf{R}} f(x) g(t-x) e^{-ia(t^2-z^2)-id(t-z)} dz. \tag{2.12}$$

2.6. Quadratic-phase wavelet transform (QPWT)

The generalization of the classical CWT, continuous fractional wavelet transform, as well as generalization of linear canonical wavelet transform, is the quadratic-phase wavelet transform (QPWT).

For a signal $f(t) \in L^2(\mathbb{R})$, the continuous quadratic-phase wavelet transform of f with respect to an analyzing wavelet $\psi \in L^2(\mathbb{R})$ and the parameter set $\mu = (a, b, c, d, e)$ is defined by [25]

$$CQPWT_f(\beta, \alpha) = \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} f(t) \overline{\psi^\mu_{\beta,\alpha}(t)} dt, \tag{2.13}$$

where the family $\psi^\mu_{\beta,\alpha}(t)$ is called quadratic-phase wavelet (QPW) and is given by

$$\psi^\mu_{\beta,\alpha}(t) = \frac{1}{\sqrt{\alpha}} \psi\left(\frac{t-\beta}{\alpha}\right) e^{-ia(t^2-\beta^2)-id(t-\beta)}. \tag{2.14}$$

Lemma 2.2 (*[25]*). *If $\psi \in L^2(\mathbb{R})$, Then $\psi^\mu_{\beta,\alpha} \in L^2(\mathbb{R})$ with $\|\psi^\mu_{\beta,\alpha}\|^2 = \|\psi\|^2$.*

Now we are ready to introduce a novel integral transform the QP-WPT.

3. Quadratic-phase wavelet packet transform (QP-WPT)

In this section, we propose a definition of QP-WPT based on the idea of WPT by adding extra dimension to kernel and wavelet. We replace the wave packet transform kernel by the QPFT kernel and the wavelet by the quadratic-phase wavelet (QPW).

Definition 3.1 (*QP-WPT*). The QP-WPT transform of a function $f \in L^2(\mathbb{R})$ with respect to wavelet function ψ is defined as

$$\begin{aligned} W^\mu_f(\xi, \beta, \alpha) &= \int_{\mathbb{R}} f(t) \overline{\psi^\mu_{\beta,\alpha}(t)} K_\mu(t, \xi) dt \\ &= \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(at^2+bt\xi+c\xi^2+dt+e\xi)} f(t) \overline{\psi^\mu_{\beta,\alpha}(t)} dt, \end{aligned} \tag{3.1}$$

where $\psi^\mu_{\beta,\alpha}(t) = \psi_\alpha(t-\beta) e^{-ia(t^2-\beta^2)-id(t-\beta)}$ and $\psi_\alpha(t) = \frac{1}{\alpha}\psi\left(\frac{t}{\alpha}\right)$.

Remark 3.1. By varying the parameter $\mu = (a, b, c, d, e)$ Definition 3.1 embodies certain existing time–frequency transforms and also give birth to some novel time–frequency tools which are yet to be reported in the open literature which are listed below:

- For $\mu = (a/2b, -1/b, c/2b, 0, 0)$, Definition 3.1 boils down to the novel linear canonical wave packet transform

$$W^\mu_f(\xi, \beta, \alpha) = \int_{\mathbb{R}} f(t) K_\mu(t, \xi) \overline{\psi\left(\frac{t-\beta}{\alpha}\right) e^{i\frac{a}{2b}(t^2-\beta^2)}} dt.$$

- For $\mu = (\cot\theta, -\csc\theta, \cot\theta, 0, 0)$, $\theta \neq n\pi$ Definition 3.1 reduces to the novel fractional wave packet transform

$$W^\mu_f(\xi, \beta, \alpha) = \int_{\mathbb{R}} f(t) K_\mu(t, \xi) \overline{\psi\left(\frac{t-\beta}{\alpha}\right) e^{i\cot\theta(t^2-\beta^2)}} dt.$$

- For $\mu = (1, b, 0, 1, 0)$, $b \neq 0$, we can obtain the novel Fresnel wave packet transform

$$W^\mu_f(\xi, \beta, \alpha) = \int_{\mathbb{R}} f(t) K_\mu(t, \xi) \overline{\psi\left(\frac{t-\beta}{\alpha}\right) e^{i(t^2-\beta^2)+id(t-\beta)}} dt.$$

- For $\mu = (0, -1, 1, 0, 0)$, Definition 3.1 reduces to the classical wave packet transform

$$W^\mu_f(\xi, \beta, \alpha) = \int_{\mathbb{R}} f(t) K_\mu(t, \xi) \overline{\psi\left(\frac{t-\beta}{\alpha}\right)} dt.$$

Theorem 3.2. *Let $W^\mu_f(\xi, \beta, \alpha)$ and $Q_\mu[f]$ be the QP-WPT and QPFT of a function $f \in L^2(\mathbb{R})$, respectively and let $\psi^\mu_{\beta,\alpha}$ be the QPW, then we have*

$$\begin{aligned} W^\mu_f(\xi, \beta, \alpha) &= \sqrt{\alpha} \int_{\mathbb{R}} K_\mu(w, \beta) e^{-i[c(\alpha w)^2 + e(\alpha w) - 2cw\xi]} \\ &\quad \times Q_\mu[e^{ia(.)^2+id(.)} f(t)](w+\xi) Q_\mu[e^{-ia(.)^2-id(.)} \psi(.)](\alpha w) dw. \end{aligned} \tag{3.2}$$

Proof. Let us denote

$$f_{\xi,\mu} = \sqrt{\frac{b}{2\pi i}} e^{i(at^2+bt\xi+c\xi^2+dt+e\xi)} f(t).$$

On taking QPFT on both sides of above equation, we have

$$\begin{aligned}
\mathcal{Q}_\mu[f_{\xi,\mu}] &= \int_{\mathbb{R}} K(w,t) f_{\xi,\mu}(t)dt \\
&= \int_{\mathbb{R}} \sqrt{\frac{b}{2\pi i}} e^{i(at^2+btw+cw^2+dt+ew)} \sqrt{\frac{b}{2\pi i}} e^{i(at^2+bt\xi+c\xi^2+dt+e\xi)} f(t)dt \\
&= \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} \sqrt{\frac{b}{2\pi i}} e^{i[at^2+bt(w+\xi)+c(w+\xi)^2+dt+e(w+\xi)]} \\
&\qquad \times e^{i(at^2+dt)} f(t) e^{-i(2cw\xi)} dt \\
&= \sqrt{\frac{b}{2\pi i}} e^{-i2cw\xi} \mathcal{Q}_\mu[e^{i(at^2+dt)} f(t)](w+\xi).
\end{aligned}$$

From [25], we have

$$\mathcal{Q}_\mu[\psi^\mu_{\beta,\alpha}](w) = \sqrt{\alpha} e^{i(a\beta^2+b\beta w+cw^2+d\beta+ew)-ic(\alpha w)^2-ie(\alpha w)} \times \mathcal{Q}_\mu[e^{-ia(.)^2-id(.)}\psi(.)](\alpha w).$$

The QP-WPT is represented in terms of inner product of $f_{\xi,\mu}$ and $\psi^\mu_{\beta,\alpha}$ and by Parseval theorem of QPFT, we have

$$\begin{aligned}
W^\mu_f(\xi,\beta,\alpha) &= \langle f_{\xi,\mu}, \psi^\mu_{\beta,\alpha} \rangle \\
&= \left\langle \mathcal{Q}_\mu[f_{\xi,\mu}], \mathcal{Q}_\mu[\psi^\mu_{\beta,\alpha}] \right\rangle \\
&= \sqrt{\frac{\alpha b}{2\pi i}} \int_{\mathbb{R}} e^{i(a\beta^2+b\beta w+cw^2+d\beta+ew-c(\alpha w)^2-e(\alpha w)-2cw\xi)} \\
&\qquad \times \mathcal{Q}_\mu[e^{i(at^2+dt)} f(t)](w+\xi) \mathcal{Q}_\mu[e^{-ia(.)^2-id(.)}\psi(.)](\alpha w) dw.
\end{aligned}$$

Now using (2.7), we get the desired proof. □

Further, the definition of the QP-WPT in (3.1) can be rewritten as

$$\begin{aligned}
W^\mu_f(\xi,\beta,\alpha) &= \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(at^2+bt\xi+c\xi^2+dt+e\xi)+ia(t^2-\beta^2)+id(t-\beta)} \times f(t)\overline{\psi_\alpha(t-\beta)}dt \\
&= \int_{\mathbb{R}} f(t) \overline{\psi^\mu_{\xi,\beta,\alpha,}} dt,
\end{aligned} \tag{3.3}$$

where

$$\psi^\mu_{\xi,\beta,\alpha,}(t) = \overline{\left(\sqrt{\frac{b}{2\pi i}}\right)} e^{-i(at^2+bt\xi+c\xi^2+dt+e\xi)-ia(t^2-\beta^2)-id(t-\beta)} \psi_\alpha(t-\beta). \tag{3.4}$$

Proposition 3.1 (*Relation with WFT*).

$$\begin{aligned}
W^\mu_f(\xi,\beta,\alpha) &= \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(at^2+bt\xi+c\xi^2+dt+e\xi)+ia(t^2-\beta^2)+id(t-\beta)} f(t)\overline{\psi_\alpha(t-\beta)}dt \\
&= e^{i(c\xi^2+e\xi-a\beta^2-d\beta)} \int_{\mathbb{R}} \sqrt{\frac{b}{2\pi i}} e^{i(2at^2+bt\xi+2dt)} f(t)\overline{\psi_\alpha(t-\beta)}dt \\
&= e^{i(c\xi^2+e\xi-a\beta^2-d\beta)} \int_{\mathbb{R}} \sqrt{\frac{b}{2\pi i}} e^{i(2at^2+2dt)} f(t)\overline{\psi_\alpha(t-\beta)} e^{ibt\xi} dt \\
&= e^{i(c\xi^2+e\xi-a\beta^2-d\beta)} \mathcal{G}_{\psi^\mu}[h](b\xi,\beta),
\end{aligned} \tag{3.5}$$

where $h(t) = \sqrt{\frac{b}{2\pi i}} e^{i(2at^2+2dt)} f(t)$.

3.1. Basic properties of the QP-WPT

In this subsection we prove some notable inequalities associated with the QP-WPT. Moreover, we also investigate some basic properties of the QP-WPT which are important for signal representation in signal processing.

Lemma 3.1. *Let $\psi_\alpha \in Ł^p(\mathbb{R})$ and $f \in L^q(\mathbb{R})$ and $p,q \in [1,\infty)$ with $\frac{1}{p}+\frac{1}{q}$, then*

$$|W^\mu_f(\xi,\beta,\alpha)| \le \alpha^{(1/p-1/2)} \sqrt{\frac{b}{2\pi}} \|\psi\|_{L^p(\mathbb{R})} \|f\|_{L^q(\mathbb{R})}. \tag{3.6}$$

Proof. From (3.3), we have

$$\begin{aligned}|W_f^{\mu}(\xi,\beta,\alpha)| &= \left|\sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i(at^2+bt\xi+c\xi^2+dt+e\xi)+ia(t^2-\beta^2)+id(t-\beta)} f(t)\overline{\psi_\alpha(t-\beta)}dt\right| \\ &= \sqrt{\frac{b}{2\pi}}\left|\int_{\mathbb{R}} e^{i(at^2+bt\xi+c\xi^2+dt+e\xi)+ia(t^2-\beta^2)+id(t-\beta)} f(t)\overline{\psi_\alpha(t-\beta)}dt\right| \\ &\leq \sqrt{\frac{b}{2\pi}}\left|\int_{\mathbb{R}} f(t)\overline{\psi_\alpha(t-\beta)}dt\right|.\end{aligned}$$

By Lemma 2.1 and Holder's inequality, above yields

$$|W_f^{\mu}(\xi,\beta,\alpha)| \leq \alpha^{(1/p-1/2)}\sqrt{\frac{b}{2\pi}}\|\psi\|_{L^p(\mathbb{R})}\|f\|_{L^q(\mathbb{R})},$$

which completes the proof. □

Theorem 3.3 (*Boundedness*)**.** *For $\psi, f \in Ł^2(\mathbb{R})$, the QP-WPT is bounded on $Ł^2(\mathbb{R})$.*

Proof. By taking $p = q = 2$ in Lemma 3.1, we have:

$$|W_f^{\mu}(\xi,\beta,\alpha)| \leq \sqrt{\frac{b}{2\pi}}\|\psi\|_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R})},$$

which shows that the QP-WPT is bounded on $L^2(\mathbb{R})$. □

Theorem 3.4. *Let $\psi \in L^p(\mathbb{R})$ and $f \in L^1(\mathbb{R}) \cap L^1(\mathbb{R})$. Then we have*

$$\|W_f^{\mu}(\xi,\beta,\alpha)\|_{L^p(\mathbb{R})} \leq \alpha^{(1/p-1/2)}\sqrt{\frac{b}{2\pi}}\|\psi\|_{L^p(\mathbb{R})}\|f\|_{L^1(\mathbb{R})}. \tag{3.7}$$

Proof. By applying the Minkowski's inequality to (3.3), we obtain

$$\begin{aligned}\|W_f^{\mu}(\xi,\beta,\alpha)\|_{L^p(\mathbb{R})} &= \left(\int_{\mathbb{R}}\left|\sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i(at^2+bt\xi+c\xi^2+dt+e\xi)+ia(t^2-\beta^2)+id(t-\beta)} f(t)\overline{\psi_\alpha(t-\beta)}dt\right|^p d\beta\right)^{1/p} \\ &\leq \sqrt{\frac{b}{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|f(t)\overline{\psi_\alpha(t-\beta)}\right|^p d\beta\right)^{1/p} dt.\end{aligned}$$

Setting $t-\beta = y$, we have

$$\begin{aligned}\|W_f^{\mu}(\xi,\beta,\alpha)\|_{L^p(\mathbb{R})} &= \sqrt{\frac{b}{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|f(t)\overline{\psi_\alpha(y)}\right|^p dy\right)^{1/p} dt \\ &\leq \sqrt{\frac{b}{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|\overline{\psi_\alpha(y)}\right|^p dy\right)^{1/p} |f(t)|dt \\ &\leq \sqrt{\frac{b}{2\pi}}\|\psi_\alpha\|_{L^p(\mathbb{R})}\|f\|_{L^1(\mathbb{R})} \\ &\leq \alpha^{(1/p-1/2)}\sqrt{\frac{b}{2\pi}}\|\psi\|_{L^p(\mathbb{R})}\|f\|_{L^1(\mathbb{R})},\end{aligned}$$

which completes the proof. □

Theorem 3.5 (*Reconstruction theorem*)**.** *Every signal $f \in L^2(\mathbb{R})$, can be reconstructed from QP-WPT by the formula*

$$f(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} W_f^{\mu}(\xi,\beta,\alpha)\psi^{\mu}_{\xi,\beta,\alpha}(t)d\xi d\beta. \tag{3.8}$$

Proof. Let $h(t), \psi, \phi \in L^2(\mathbb{R})$. Assuming $\langle\phi,\psi\rangle \neq 0$ and ψ_α as a windowed function then by the inverse of the WFT (2.4), we have

$$h(t) = \frac{b}{2\pi\langle\phi,\psi\rangle}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{G}_{\psi^{\mu}}[h](b\xi,\beta)e^{-ibt\xi}\psi_\alpha(t-\beta)d\xi d\beta.$$

By virtue of (3.5), we have from above equation

$$\begin{aligned}\sqrt{\frac{b}{2\pi i}}e^{i(2at^2+2dt)}f(t) &= \frac{b}{2\pi\langle\phi,\psi\rangle}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i(c\xi^2+e\xi-a\beta^2-d\beta)}e^{-ibt\xi} \\ &\qquad\times\psi_\alpha(t-\beta)W_f^{\mu}(\xi,\beta,\alpha)d\xi d\beta, \\ f(t) &= \sqrt{\frac{bi}{2\pi}}\frac{1}{\langle\phi,\psi\rangle}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i(c\xi^2+e\xi-a\beta^2-d\beta+2dt+bt\xi+2at^2)}\end{aligned}$$

$$\times\psi_\alpha(t-\beta)W_f^\mu(\xi,\beta,\alpha)d\xi d\beta$$
$$=\frac{1}{\langle\phi,\psi\rangle}\int_{\mathbb{R}}\int_{\mathbb{R}}\psi^\mu_{\xi,\beta,\alpha}(t)W_f^\mu(\xi,\beta,\alpha)d\xi d\beta. \tag{3.9}$$

For perfect reconstruction take $\langle\phi,\psi\rangle=1$, above equation yields

$$f(t)=\int_{\mathbb{R}}\int_{\mathbb{R}}W_f^\mu(\xi,\beta,\alpha)\psi^\mu_{\xi,\beta,\alpha}(t)d\xi d\beta,$$

which completes the proof. □

Theorem 3.6 (*Moyal's Formula*). *Let $W_f^\mu(\xi,\beta,\alpha)$ and $W_g^\mu(\xi,\beta,\alpha)$ be the QP-WPT with respect to the wavelets ψ and ϕ respectively, then*

$$\langle W_f^\mu(\xi,\beta,\alpha),W_g^\mu(\xi,\beta,\alpha)\rangle_{L^2(\mathbb{R}^2)}=\overline{\langle\psi,\phi\rangle}_{L^2(\mathbb{R})}\langle f,g\rangle_{L^2(\mathbb{R})}. \tag{3.10}$$

Proof.

$$\begin{aligned}\langle W_f^\mu(\xi,\beta,\alpha),W_g^\mu(\xi,\beta,\alpha)\rangle &=\int_{\mathbb{R}^2}W_f^\mu(\xi,\beta,\alpha)\overline{W_g^\mu(\xi,\beta,\alpha)}d\xi d\beta\\ &=\int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}}f(t)\overline{\psi^\mu_{\beta,\alpha}(t)}K_\mu(t,\xi)dt\right.\\ &\qquad\left.\times\int_{\mathbb{R}}\overline{g(t')}\phi^\mu_{\beta,\alpha}(t')\overline{K_\mu(t',\xi)}dt'\right\}d\xi d\beta\\ &=\int_{\mathbb{R}^2}\int_{\mathbb{R}}f(t)\overline{g(t')}\phi^\mu_{\beta,\alpha}(t')\overline{\psi^\mu_{\beta,\alpha}(t)}\int_{\mathbb{R}}K_\mu(t,\xi)\overline{K_\mu(t',\xi)}d\xi dt dt' d\beta\\ &=\int_{\mathbb{R}^2}\int_{\mathbb{R}}f(t)\overline{g(t')}\phi_\alpha(t'-\beta)\overline{\psi_\alpha(t-\beta)}\frac{b}{2\pi}\int_{\mathbb{R}}e^{ib\xi(t-t')}d\xi dt dt' d\beta\\ &=\int_{\mathbb{R}^2}\int_{\mathbb{R}}f(t)\overline{g(t')}\phi_\alpha(t'-\beta)\overline{\psi_\alpha(t-\beta)}\delta(t-t')dt' dt d\beta\\ &=\int_{\mathbb{R}}\int_{\mathbb{R}}f(t)\overline{g(t)}\phi_\alpha(t-\beta)\overline{\psi_\alpha(t-\beta)}dt d\beta\\ &=\int_{\mathbb{R}}f(t)\overline{g(t)}dt\int_{\mathbb{R}}\frac{1}{\alpha}\phi\left(\frac{t-\beta}{\alpha}\right)\overline{\psi}\left(\frac{t-\beta}{\alpha}\right)d\beta\\ &=\overline{\langle\psi,\phi\rangle}_{L^2(\mathbb{R})}\langle f,g\rangle_{L^2(\mathbb{R})}.\end{aligned}$$

which completes the proof. □

Consequences of Theorem 3.6:

- If $\psi=\phi$, then

$$\langle W_f^\mu(\xi,\beta,\alpha),W_g^\mu(\xi,\beta,\alpha)\rangle_{L^2(\mathbb{R}^2)}=\|\psi\|^2_{L^2(\mathbb{R})}\langle f,g\rangle_{L^2(\mathbb{R})}. \tag{3.11}$$

- If $\psi=\phi$, and $f=g$, then

$$\langle W_f^\mu(\xi,\beta,\alpha),W_g^\mu(\xi,\beta,\alpha)\rangle_{L^2(\mathbb{R}^2)}=\|\psi\|^2_{L^2(\mathbb{R})}\|f\|^2_{L^2(\mathbb{R})}. \tag{3.12}$$

- If $\psi=\phi=1$, and $f=g$, then

$$\langle W_f^\mu(\xi,\beta,\alpha),W_g^\mu(\xi,\beta,\alpha)\rangle_{L^2(\mathbb{R}^2)}=\|f\|^2_{L^2(\mathbb{R})}. \tag{3.13}$$

Remark 3.7 (*Energy conservation*). Eq. (3.13) yields the conservation of energy for the QP-WPT

$$\int_{\mathbb{R}^2}\left|W_f^\mu(\xi,\beta,\alpha)\right|^2d\xi d\beta=\int_{\mathbb{R}}|f(t)|^2dt. \tag{3.14}$$

Theorem 3.8 (*Reproducing kernel*). *Let (ξ_0,β_0,α) be any point on the plane of (ξ,β,α), the necessary and sufficient condition that the function $W_f^\mu(\xi,\beta,\alpha)$ is the QP-WPT of some function is that $W_f^\mu(\xi,\beta,\alpha)$ must satisfy the following reproducing kernel formula*

$$W_f^\mu(\xi,\beta,\alpha)=\int_{\mathbb{R}}\int_{\mathbb{R}}W_f^\mu(\xi,\beta,\alpha)\mathbb{K}_{\psi^\mu}(\xi,\beta,\alpha:\xi_0,\beta_0,\alpha)d\xi d\beta, \tag{3.15}$$

where $W_f^\mu(\xi_0,\beta_0,\alpha)$ is value of function $W_f^\mu(\xi,\beta,\alpha)$ at (ξ_0,β_0,α), and $\mathbb{K}_{\psi^\mu}(\xi,\beta,\alpha:\xi_0,\beta_0,\alpha)$ is called the reproducing kernel given by

$$\mathbb{K}_{\psi^\mu}(\xi,\beta,\alpha:\xi_0,\beta_0,\alpha)=\langle\psi^\mu_{\xi,\beta,\alpha},\psi^\mu_{\xi_0,\beta_0,\alpha}\rangle. \tag{3.16}$$

Proof. From (3.3) and (3.8), we have

$$\begin{aligned} W_f^{\mu}(\xi,\beta,\alpha) &= \int_{\mathbb{R}} f(t)\overline{\psi^{\mu}_{\xi,\beta,\alpha}}(t)dt \\ &= \int_{\mathbb{R}}\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} W_f^{\mu}(\xi,\beta,\alpha)\psi^{\mu}_{\xi,\beta,\alpha}(t)d\xi d\beta\right\}\overline{\psi^{\mu}_{\xi,\beta,\alpha}}(t)dt. \end{aligned}$$

Setting $(\xi,\beta,\alpha)=(\xi_0,\beta_0,\alpha)$, we have

$$\begin{aligned} W_f^{\mu}(\xi_0,\beta_0,\alpha) &= \int_{\mathbb{R}}\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} W_f^{\mu}(\xi,\beta,\alpha)\psi^{\mu}_{\xi,\beta,\alpha}(t)d\xi d\beta\right\}\overline{\psi^{\mu}_{\xi_0,\beta_0,\alpha}}(t)dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} W_f^{\mu}(\xi,\beta,\alpha)\psi^{\mu}_{\xi,\beta,\alpha}(t)\overline{\psi^{\mu}_{\xi_0,\beta_0,\alpha}}(t)dtd\xi d\beta \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} W_f^{\mu}(\xi,\beta,\alpha)\left\{\int_{\mathbb{R}}\psi^{\mu}_{\xi,\beta,\alpha}(t)\overline{\psi^{\mu}_{\xi_0,\beta_0,\alpha}}(t)dt\right\}d\xi d\beta \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} W_f^{\mu}(\xi,\beta,\alpha)\mathbb{K}_{\psi^{\mu}}(\xi,\beta,\alpha:\xi_0,\beta_0,\alpha)d\xi d\beta, \end{aligned}$$

which completes the proof. □

4. Uncertainty principle's for the QP-WPT

Uncertainty principle has applications in two main areas: harmonic analysis and signal analysis. This principle in harmonic analysis stems from the uncertainty principle in quantum mechanics, which tells that a particle's velocity and position cannot be measured with infinite precision. In signal analysis, it tells that if one observes a signal only for a finite time, then the knowledge about the frequencies consisted by the signal is lost. In this section, we first prove QP-WPT Lieb's uncertainty principle by considering the relationship between the WFT and QP-WPT. Then we will obtain a logarithmic uncertainty principle associated with the QP-WPT by using the relation fundamental between FT and QP-WPT. Finally, we wil establish a generalization of the Heisenberg type uncertainty principle for the QP-WPT.

Theorem 4.1 (*Leib's uncertainty principle*). *For $\psi, f\in L^2(\mathbb{R})$ and $2\le p<\infty$, the following inequality holds:*

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|W_f^{\mu}(\xi,\beta,\alpha)\right|^p d\xi d\beta \le \frac{2}{p}(M_{\mu})^p\left(\|f\|_2\|\psi\|_2\right)^p, \tag{4.1}$$

where $(M_{\mu})=(2\pi)^{\frac{-1}{2}}|b|^{\frac{1}{2}-\frac{1}{p}}$.

Proof. The Lieb's uncertainty principle for the windowed Fourier transform [40,41] reads

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{G}_{\psi}[f](\xi,\beta)\right|^p d\xi d\beta \le \frac{2}{p}\left(\|f\|_2\|\psi\|_2\right)^p \tag{4.2}$$

for all $f,\psi\in L^2(\mathbb{R})$ and $2\le p<\infty$.

For $f\in L^2(\mathbb{R})$ we have function $h(t)=\sqrt{\frac{b}{2\pi i}}e^{i(2at^2+2dt)}f(t)\in L^2(\mathbb{R})$, therefore we can replace f in (4.2) by h as:

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{G}_{\psi^{\mu}_{\alpha}}[h](\xi,\beta)\right|^p d\xi d\beta &\le \frac{2}{p}\left(\|h\|_2\|\psi^{\mu}_{\alpha}\|_2\right)^p \\ &= \frac{2}{p}\left(\left(\int_{\mathbb{R}}\left|\sqrt{\frac{b}{2\pi i}}e^{i(2at^2+2dt)}f(t)\right|^2dt\right)^{\frac{1}{2}}\|\psi_{\alpha}\|_2\right)^p. \end{aligned} \tag{4.3}$$

Substituting $\xi=b\xi$ in (4.3), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}|b|\left|\mathcal{G}_{\psi^{\mu}_{\alpha}}[h](b\xi,\beta)\right|^p d\xi d\beta \le \frac{2}{p}\left(\frac{b}{2\pi}\right)^{\frac{p}{2}}\left(\left(\int_{\mathbb{R}}|f(t)|^2dt\right)^{\frac{1}{2}}\|\psi_{\alpha}\|_2\right)^p. \tag{4.4}$$

Using (3.5) in (4.4)

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|e^{-i(c\xi^2+e\xi-a\beta^2-d\beta)}W_f^{\mu}(\xi,\beta,\alpha)\right|^p d\xi d\beta \le \frac{2}{p|b|}\left(\frac{b}{2\pi}\right)^{\frac{p}{2}}\left(\left(\int_{\mathbb{R}}|f(t)|^2dt\right)^{\frac{1}{2}}\|\psi_{\alpha}\|_2\right)^p. \tag{4.5}$$

On further simplifying (4.5) and using Lemma 2.2, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|W_f^{\mu}(\xi,\beta,\alpha)\right|^p d\xi d\beta \le \frac{2}{p|b|}\left(\frac{b}{2\pi}\right)^{\frac{p}{2}}\left(\|f\|_2\|\psi\|_2\right)^p$$

$$= \frac{2}{p}\left(\frac{1}{|b|^{\frac{1}{p}}}\right)^p \left(\frac{|b|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\right)^p \left(\|f\|_2\|\psi\|_2\right)^p,$$

which completes the proof. □

Lemma 4.1 (*Relation between QP-WPT and FT*). *We have from* (3.3)

$$\begin{aligned}
W_f^\mu(\xi,\beta,\alpha) &= \sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i(at^2+bt\xi+c\xi^2+dt+e\xi)} f(t)\overline{\psi_{\beta,\alpha}^\mu}dt \\
&= \sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i(at^2+bt\xi+c\xi^2+dt+e\xi)+ia(t^2-\beta^2)+id(t-\beta)} f(t)\overline{\psi_\alpha(t-\beta)}dt \\
&= e^{i(c\xi^2+e\xi-a\beta^2-d\beta)}\sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i(2at^2+bt\xi+2dt)} f(t)\overline{\psi_\alpha(t-\beta)}dt \\
&= e^{i(c\xi^2+e\xi-a\beta^2-d\beta)}\sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i(2at^2+2dt)}e^{ib\xi t} f(t)\overline{\psi_\alpha(t-\beta)}dt \\
&= e^{i(c\xi^2+e\xi-a\beta^2-d\beta)}\sqrt{\frac{b}{i}}\mathcal{F}[g](b\xi),
\end{aligned} \tag{4.6}$$

where

$$g(t) = e^{i(2at^2+2dt)} f(t)\psi_\alpha(t-\beta). \tag{4.7}$$

Theorem 4.2 (*Logarithmic uncertainty principle*). *Let* $\psi \in L^2(\mathbb{R})$ *and* $W_f^\mu(\xi,\beta,\alpha)$ *be the QP-WPT of* $f \in \mathcal{S}(\mathbb{R})$ *[Schwartz space]. Then, the following logarithmic inequality holds:*

$$\begin{aligned}
&\|\psi\|^2\int_{\mathbb{R}}\ln|t||f(t)|^2dt + \int_{\mathbb{R}}\int_{\mathbb{R}}\ln|\xi|\left|W_f^\mu(\xi,\beta,\alpha)\right|^2 d\xi d\beta \\
&\geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|b|\right]\|f\|^2\|\psi\|^2.
\end{aligned}$$

Proof. For any $f \in \mathcal{S}(\mathbb{R})$ (Schwartz space in $L^2(\mathbb{R})$, the logarithmic uncertainty principle for the classical FT reads [20]

$$\int_{\mathbb{R}}\ln|t||f(t)|^2dt + \int_{\mathbb{R}}\ln|\xi||\mathcal{F}[f](\xi)|^2d\xi \geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi\right]\int_{\mathbb{R}}|f(t)|^2dt. \tag{4.8}$$

As $f \in \mathcal{S}(\mathbb{R})$, then it is evident that function g given in (4.7) belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$. Therefore we can replace f in (4.8) by g as

$$\int_{\mathbb{R}}\ln|t||g(t)|^2dt + \int_{\mathbb{R}}\ln|\xi||\mathcal{F}[g](\xi)|^2d\xi \geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi\right]\int_{\mathbb{R}}|g(t)|^2dt. \tag{4.9}$$

Changing ξ by $b\xi$, we obtain from (4.9)

$$\int_{\mathbb{R}}\ln|t||g(t)|^2dt + b\int_{\mathbb{R}}\ln|b\xi||\mathcal{F}[g](b\xi)|^2d\xi \geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi\right]\int_{\mathbb{R}}|g(t)|^2dt. \tag{4.10}$$

Applying Lemma 4.1 and (4.7) to (4.10), we obtain

$$\begin{aligned}
\int_{\mathbb{R}}\ln|t||f(t)\psi_\alpha(t-\beta)|^2dt + b\int_{\mathbb{R}}(\ln|b|+\ln|\xi|)\left|\sqrt{\frac{i}{b}}e^{-i(c\xi^2+e\xi-a\beta^2-d\beta)}W_f^\mu(\xi,\beta,\alpha)\right|^2 d\xi \\
\geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi\right]\int_{\mathbb{R}}|f(t)\psi_\alpha(t-\beta)|^2dt.
\end{aligned} \tag{4.11}$$

On further simplifying (4.11), we get

$$\begin{aligned}
\int_{\mathbb{R}}\ln|t||f(t)\psi_\alpha(t-\beta)|^2dt + \int_{\mathbb{R}}\ln|b|\left|W_f^\mu(\xi,\beta,\alpha)\right|^2 d\xi + \int_{\mathbb{R}}\ln|\xi|\left|W_f^\mu(\xi,\beta,\alpha)\right|^2 d\xi \\
\geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi\right]\int_{\mathbb{R}}|f(t)\psi_\alpha(t-\beta)|^2dt.
\end{aligned} \tag{4.12}$$

On integrating both sides of (4.12) with respect to β, we have

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}}\ln|t||f(t)\psi_\alpha(t-\beta)|^2dtd\beta + \ln|b|\int_{\mathbb{R}}\int_{\mathbb{R}}\left|W_f^\mu(\xi,\beta,\alpha)\right|^2 d\xi d\beta \\
+ \int_{\mathbb{R}}\int_{\mathbb{R}}\ln|\xi|\left|W_f^\mu(\xi,\beta,\alpha)\right|^2 d\xi d\beta \\
\geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi\right]\int_{\mathbb{R}}\int_{\mathbb{R}}|f(t)\psi_\alpha(t-\beta)|^2dtd\beta.
\end{aligned} \tag{4.13}$$

Now using (3.12) in (4.13), we get

$$\begin{aligned}\|\psi\|^2 \int_{\mathbb{R}} \ln|t||f(t)|^2 dt + \int_{\mathbb{R}}\int_{\mathbb{R}} \ln|\xi| \left|W_f^{\mu}(\xi,\beta,\alpha)\right|^2 d\xi d\beta \\ \geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi\right] \|f\|^2\|\psi\|^2 - \ln|b|\|f\|^2\|\psi\|^2 \\ = \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|b|\right] \|f\|^2\|\psi\|^2,\end{aligned}$$

which completes the proof. □

Theorem 4.3. *For $\psi, f \in L^2(\mathbb{R})$ and $W_f^{\mu}(\xi,\alpha,\beta)$ be the QP-WPT of the signal f, then the following inequality holds*

$$\int_{\mathbb{R}} t^2|f(t)|^2 dt \int_{\mathbb{R}}\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta \geq \left(\frac{1}{2|b|}\|f\|^2\|\psi\|\right)^2. \tag{4.14}$$

Proof.

The classical Heisenberg–Pauli–Weyl inequality in the QPFT domain (see [20] Theorem 3.2) is given by

$$\int_{\mathbb{R}} t^2|f(t)|^2 dt \int_{\mathbb{R}} \xi^2 |\mathcal{Q}_{\mu}[f](\xi)|^2 d\xi \geq \left(\frac{1}{2|b|}\int_{\mathbb{R}} |f(t)|^2 dt\right)^2. \tag{4.15}$$

Using the inverse transform for the QPFT into the left hand side and Plancherel identity for QPFT into the right hand side of the (4.15), we have

$$\int_{\mathbb{R}} t^2 |\mathcal{Q}_{\mu}^{-1}[\mathcal{Q}_{\mu}[f](\xi)]|^2(t) dt \int_{\mathbb{R}} \xi^2 |\mathcal{Q}_{\mu}[f](\xi)|^2 d\xi \geq \left(\frac{1}{2|b|}\int_{\mathbb{R}} |\mathcal{Q}_{\mu}[f](\xi)|^2 d\xi\right)^2. \tag{4.16}$$

For $f, \mathcal{Q}^{\mu}[f] \in L^2(\mathbb{R})$, we have $W_f^{\mu}(\xi,\beta,\alpha) \in L^2(\mathbb{R})$, so replacing $\mathcal{Q}^{\mu}[f]$ by $W_f^{\mu}(\xi,\beta,\alpha)$ in (4.16), we have

$$\int_{\mathbb{R}} t^2 |\mathcal{Q}_{\mu}^{-1}[W_f^{\mu}(\xi,\beta,\alpha)]|^2 dt \int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi \geq \left(\frac{1}{2|b|}\int_{\mathbb{R}} |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi\right)^2. \tag{4.17}$$

Which implies

$$\left(\int_{\mathbb{R}} t^2 |\mathcal{Q}_{\mu}^{-1}[W_f^{\mu}(\xi,\beta,\alpha)]|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi\right)^{1/2} \geq \frac{1}{2|b|}\int_{\mathbb{R}} |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi. \tag{4.18}$$

Now integrating (4.19) both sides by β, we have

$$\begin{aligned}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} t^2 |\mathcal{Q}_{\mu}^{-1}[W_f^{\mu}(\xi,\beta,\alpha)]|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi\right)^{1/2} d\beta \\ \geq \frac{1}{2|b|}\int_{\mathbb{R}}\int_{\mathbb{R}} |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta.\end{aligned} \tag{4.19}$$

Now applying Cauchy–Schwartz inequality, (4.19) yields

$$\begin{aligned}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} t^2 |\mathcal{Q}_{\mu}^{-1}[W_f^{\mu}(\xi,\beta,\alpha)]|^2 dt d\beta\right)^{1/2} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta\right)^{1/2} \\ \geq \frac{1}{2|b|}\int_{\mathbb{R}}\int_{\mathbb{R}} |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta.\end{aligned} \tag{4.20}$$

Now, using (3.12) in (4.20), we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} t^2 |f(t)\psi(t-\beta)|^2 dt d\beta\right)^{1/2} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta\right)^{1/2} \\ \geq \frac{1}{2|b|}\|f\|^2\|\psi\|^2.\end{aligned} \tag{4.21}$$

On further simplifying (4.21), we get

$$\begin{aligned}\left(\int_{\mathbb{R}} t^2 |f(t)|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta\right)^{1/2} \\ \geq \frac{1}{2|b|}\|f\|^2\|\psi\|.\end{aligned} \tag{4.22}$$

Which implies

$$\int_{\mathbb{R}} t^2|f(t)|^2 dt \int_{\mathbb{R}}\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta \geq \left(\frac{1}{2|b|}\|f\|^2\|\psi\|\right)^2,$$

which completes the proof. □

Remark 4.4. By varying the parameter $\mu = (a, b, c, d, e)$ the Heisenberg-type inequality (4.14), embodies certain existing Heisenberg-type inequalities and also give birth to some novel Heisenberg-type inequalities which are yet to be reported in the open literature which are listed below:

- For $\mu = (a/2b, -1/b, c/2b, 0, 0)$, the Heisenberg-type inequality (4.14) boils down to the novel Heisenberg inequality for linear canonical wave packet transform (see Theorem 6.2 [38])

$$\int_{\mathbb{R}} t^2|f(t)|^2 dt \int_{\mathbb{R}}\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta \geq \left(\frac{|b|}{2}\|f\|^2\|\psi\|\right)^2.$$

- For $\mu = (\cot\theta, -\csc\theta, \cot\theta, 0, 0)$, $\theta \neq n\pi$, we can obtain the novel Heisenberg inequality for the fractional wave packet transform

$$\int_{\mathbb{R}} t^2|f(t)|^2 dt \int_{\mathbb{R}}\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta \geq \left(\frac{\sin\theta}{2}\|f\|^2\|\psi\|\right)^2.$$

- For $\mu = (0, -1, 1, 0, 0)$, we can obtain the novel Heisenberg inequality for the classical wave packet transform

$$\int_{\mathbb{R}} t^2|f(t)|^2 dt \int_{\mathbb{R}}\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta \geq \left(\frac{1}{2}\|f\|^2\|\psi\|\right)^2.$$

5. Conclusion

Based on quadratic-phase Fourier transform (QPFT) and the classical wave packet transform (WPT) theory, we in this paper propose a novel integral transform coined as quadratic-phase wave packet transform (QP-WPT), which rectifies the limitations of the WPT and QPFT. Overall, it not only combines the advantages of QPFT and WPT, but also preserves the properties of its conventional counterpart, and has better mathematical properties. Besides studying some notable inequalities and the fundamental properties including the Moyal's formula, inversion formula and a reproducing kernel, we also formulated several classes of uncertainty inequalities, such as Leib's uncertainty principle, the logarithmic uncertainty inequality and the Heisenberg inequality.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

This work is supported by the Research Grant (No. JKST&IC/SRE/J/357-60) provided by JKST&IC, UT of J&K, India.

References

[1] A.V. Oppenheim, R.W. Schafer, Discrete-Time Signal Processing, Englewood Cliffs, Prentice-Hall, New Jersey, 1989.
[2] L. Cohen, Time-Frequency Analysis, Englewood Cliffs, Prentice Hall, New Jersey, 1995.
[3] B. Torresani, Wavelets associated with representations of the affine Weyl–Heisenberg group, J. Math. Phys. 32 (1991) 1273–1279.
[4] E. Freysz, B. Pouligny, F. Argoul, A. Arneodo, Optical wavelet transform of fractal aggregates, Phys. Rev. Lett. 64 (1990) 745–748.
[5] B. Torresani, Time-frequency representations: wavelet packets and optimal decomposition, Ann. Inst. Henri Poincare 56 (1992) 215–234.
[6] T.E. Posch, The Wave Packet Transform as Applied to Signal Processing, Victoria, BC, Canada, 1992.
[7] Y. Luo, G.T. Schuster, Wave packet transform and data compression, SEG Tech. Prog. Expanded Abstracts 11 (4) (1992) 1187–1190.
[8] T.E. Posch, The wave packet transform as applied to signal processing, conference record of the twenty-sixth asilomar conference on signals, Syst. Comput. (1992) 484–487.
[9] H. Yang, Synchrosqueezed wave packet transforms and diffeomorphismbased spectral analysis for 1D general mode decompositions, Appl. Comput. Harmon. Anal. 39 (1) (2015) 33–66.
[10] C.J. Zarowski, Wavelet packet transform initialization for piecewise polynomial and bandlimited inputs, IEEE Trans. Signal Process. 47 (1999) 224–226.
[11] A. Pommer, A. Uhl, Selective encryption of wavelet packet subband structures for secure transmission of visual data, IEEE Benelux Signal Process. Symp. (2002) 25–28.
[12] X. Sui, K. Wan, Y. Zhang, Pattern recognition of semg based on wavelet packet transform and improved svm, Optik 176 (2019) 228–235.
[13] Q. Yi, H. Wang, R. Guo, S. Li, Y. Jiang, Laser ultrasonic quantitative recognition based on wavelet packet fusion algorithm and svm, Optik 149 (2017) 206–219.
[14] H. Zhang, J. Zhang, K. Qiu, Performance comparison of wavelet packet transform based and conventional coherent optical OFDM transmission system, Optik 125 (2014) 2647–2651.
[15] J.R. Partington, Ünalmış B, On the windowed fourier transform and wavelet transform of almost periodic functions, Appl. Comput. Harmon. Anal. 10 (1) (2001) 45–60.
[16] J. Barros, R.I. Diego, Analysis of harmonics in power systems using the wavelet-packet transform, IEEE Trans. Instrum. Meas. 57 (1) (2008) 63–69.
[17] L.P. Castro, M.R. Haque, M.M. Murshed, S. Saitoh, N.M. Tuan, Quadratic Fourier transforms, Ann. Funct. Anal. 5 (1) (2014) 10–23, http://dx.doi.org/10.15352/afa/1391614564.
[18] L.P. Castro, L.T. Minh, N.M. Tuan, New convolutions for quadratic-phase Fourier integral operators and their applications, Mediterr. J. Math. 15 (13) (2018) 1–17, http://dx.doi.org/10.1007/s00009-017-1063-y.
[19] F.A. Shah, K.S. Nisar, W.Z. Lone, A.Y. Tantary, Uncertainty principles for the quadratic-phase Fourier transforms, Math. Methods Appl. Sci. (2021) http://dx.doi.org/10.1002/mma.7417.
[20] F.A. Shah, W.Z. Lone, A.Y. Tantary, Short-time quadratic-phase Fourier transform, Optik 245 (2021) http://dx.doi.org/10.1016/j.ijleo.2021.167689.

[21] S. Saitoh, Theory of reproducing kernels: Applications to approximate solutions of bounded linear operator functions on Hilbert spaces, Amer. Math. Soc. Trans. Ser. 230 (2) (2010) 107–134.
[22] M.Y. Bhat, A.H. Dar, Wigner-ville distribution and ambiguity function associated with the quadratic-phase Fourier transform, 2022, in press.
[23] M.Y. Bhat, A.H. Dar, The algebra of 1-D quaternion quadratic-phase Fourier transform, 2022, in press.
[24] M.Y. Bhat, A.H. Dar, The 2-D hyper-complex gabor quadratic-phase Fourier transform and uncertainty principles, 2022, in press.
[25] A. Prasad, P.B. Sharma, The quadratic-phase Fourier wavelet transform, Math. Methods Appl. Sci. (2019) 1–17.
[26] C.K. Chui, An Introduction to Wavelets, Academic Press, New York, 1992, http://dx.doi.org/10.2307/2153134.
[27] I. Daubechies, Ten lecture on wavelets, in: CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM Publ., Philadelphia, PA, 2006.
[28] L. Debnath, Wavelet Transforms and their Applications, Birkhauser, Boston, MA, 2002.
[29] R.S. Pathak, The Wavelet Transform, 4 Amsterdam, Atlantis Press/Word Scientific, 2009.
[30] A. Rieder, The wavelet transform on Sobolev spaces and its approximation properties, Numer. Math. 58 (8) (1991) 875–894.
[31] V. Perrier, C. Basdevant, Besov norms in terms of the continuous wavelet transform application to structure functions, Math. Models Methods Appl. Sci. 6 (5) (1996) 649–664.
[32] A. Prasad, S. Manna, A. Mahato, V.K. Singh, The generalized continuous wavelet transform associated with the fractional Fourier transform, J. Comput. Appl. Math. 259 (2014) 183–194.
[33] D. Mendlovic, Z. Zalevsky, D. Mas, J. Garcia, C. Ferreira, Fractional wavelet transform, Appl. Opt. 36 (20) (1997) 4801–4806.
[34] J. Shi, N. Zhang, X. Liu, A novel fractional wavelet transform and its applications, Sci. China Inf. Sci. 55 (6) (2012) 1270–1279.
[35] D. Wei, Y.M. Li, Generalized wavelet transform based on the convolution operator in the linear canonical transform domain, Optik 125 (16) (2014) 4491–4496.
[36] Y. Guo, B.Z. Li, The linear canonical wavelet transform on some function space, Int. J. Wavelets Multiresolut. Inf. Process. 16 (1) (2018) 1850010.
[37] D. Wei, Y.J. Zhang, A new fractional wave packet transform, Optik 231 (2021) http://dx.doi.org/10.1016/j.ijleo.2021.166357.
[38] A. Prasad, M. Kundu, Linear canonical wave packet transform, Integral Transforms Spec. Funct. (2020) http://dx.doi.org/10.1080/10652469.2020.1867128.
[39] Y. Li, D. Wei, The wave packet transform associated with the linear canonical transform, Optik (2015) http://dx.doi.org/10.1016/j.ijleo.2015.07.103.
[40] M. Bahri, R. Ashino, Some properties of windowed linear canonical transform and its logarithmic uncertainty principle, Int. J. Wavelets Multiresolut. Inf. Process. 14 (3) (2016) 1650015.
[41] K. Grochenig, Foundation of Time-Frequency Analysis, Birkhauser, Boston, 2001.