Results in Physics 55 (2023) 107173

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.elsevier.com/locate/rinp

Phase portraits and new exact traveling wave solutions of the (2+1)-dimensional Hirota system

Gaukhar Shaikhova [a,*], Bayan Kutum [b], Arailym Syzdykova [a]

[a] L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
[b] Karaganda Buketov University, Karaganda, Kazakhstan

ARTICLE INFO

Keywords:
(2+1)-dimensional Hirota system
Periodic wave
Soliton
Traveling wave
Phase portraits

ABSTRACT

In this paper, we investigate the (2+1)-dimensional Hirota system of equations, which is essential for modeling physical events since it contains the NLS equation and the cmKdV equation. We apply the Jacobi elliptic function method to obtain new results. By using some ansatz in terms of finite Jacobi elliptic functions, we have obtained various exact solutions, such as dark solitons, bright solitons, and periodic solutions. Moreover, for some reductions as $\alpha = 0, \beta = 1$ and $\alpha = 1, \beta = 0$, we obtain the exact solutions for the (2+1)-dimensional NLS equation and the (2+1)-dimensional cmKdV equation. The dynamics of the obtained solutions are presented in the figures. We also analyze the phase portraits of the (2+1)-dimensional Hirota system.

Introduction

The study of nonlinear equations is developing intensively among engineers and scientists because these equations model various difficult processes that occur in real life and can be applied in many scientific disciplines [1–6]. Due to the complexity of their structure, there is no single method for dealing with nonlinear equations. Various researchers have proposed methods and techniques to construct exact solutions of different nonlinear equations, such as the first integral method [7,8], the Hirota method [9,10], the Kudryashov method [11,12], the sine–cosine method [13–15], the Darboux transformation method [16,17], the tanh method [18,19], the generalized unified method [20,21] and the Lie symmetry method [22–24].

In this paper, we study the (2+1)-dimensional Hirota system of equations given by [25]

$$\begin{aligned} & iq_t + \alpha q_{xy} + i\beta q_{xxy} - qv + i(qw)_x = 0, \\ & v_x + 2\alpha\delta(|q|^2)_y - 2i\beta\delta(q^*_{xy}q - q^*q_{xy}) = 0, \\ & w_x - 2\beta\delta(|q|^2)_y = 0, \end{aligned} \tag{1}$$

where $q(x,y,t)$, $v(x,y,t)$, $w(x,y,t)$ are wave functions, α, β are real constants, $\delta = \pm 1$. The symbol $*$ stands for the complex conjugate. The system (1) is a generalization of the Hirota equation in the (2+1)-dimension and has great importance for applied magnetism [25]. Namely, in [25] Eqs. (1) are proposed for the first time, and the researchers presented the gauge/geometric equivalence with the spin system. The Eqs. (1) are integrable, nonlinear, and dispersive and admits soliton solutions. As is well-known, the soliton has applications in fiber optics, biology, magnets. In some articles, Eqs. (1) are studied by the Darboux transformation (DT) [26,27], the extended tanh method [28].

The Hirota system (HS) (1) are the typical model of mathematical physics comprising the cmKdV equation and the NLS equation:

If $\alpha = 0, \beta = 1$, Eqs. (1) reduce to the (2+1)-dimensional cmKdV equations [29–31]

$$\begin{aligned} & q_t + q_{xxy} + iqv + (qw)_x = 0, \\ & v_x - 2i\delta(q^*_{xy}q - q^*q_{xy}) = 0, \\ & w_x - 2\delta(|q|^2)_y = 0. \end{aligned} \tag{2}$$

If $\alpha = 1, \beta = 0$, Eqs. (1) reduce to the (2+1)-dimensional NLS equations

$$\begin{aligned} & iq_t + q_{xy} - qv = 0, \\ & v_x + 2\delta(|q|^2)_y = 0. \end{aligned} \tag{3}$$

In this paper, we investigate the (2+1)-dimensional HS (1) using the Jacobi elliptic function method and the bifurcation theory of planar dynamical systems. The Jacobi elliptic function method has been extensively applied to nonlinear equations to construct various solutions. For instance, this method has been applied to the perturbed nonlinear Schrodinger equation [32,33], the (2 + 1)-dimensional Nizhnik–Novikov–Veselov equation [34], the fractional nonlinear Schrodinger–Hirota equation [35], the sine–Gordon equation [36], the AB system [37], the generalized variable-coefficient Gardner equation [38], the generalized (3+1)-dimensional nonlinear Schrodinger equation

* Corresponding author.
E-mail address: g.shaikhova@gmail.com (G. Shaikhova).

https://doi.org/10.1016/j.rinp.2023.107173
Received 3 October 2023; Received in revised form 27 October 2023; Accepted 6 November 2023
Available online 11 November 2023

[39], the nonlinear Schrödinger equation in combined harmonic-lattice potentials [40] and others. The researchers obtained solutions in the form of solitary waves and periodic solutions. As for the bifurcation theory of planar dynamical systems, it was applied to the conformable Fokas–Lenells model [41], the generalized Schrodinger equation [42], the generalized q-deformed Sinh-Gordon equation [43], and the discrete space–time logistic model [44].

The paper consists of several sections. In Section "The mathematical analysis", we present the mathematical analysis of the governing equations. Section "Description of the Jacobi elliptic function method" describes the main steps of the Jacobi elliptic function method. In Section "Application", we apply this technique to obtain the new exact wave solutions. Section "Physical interpretation of the solutions" presents some obtained solutions using 2-D, 3-D, and contour plots. Phase portraits are presented in Section "Phase portraits". In Section "Conclusions", we give the conclusion of our research work.

The mathematical analysis

To obtain the exact analytical solutions of Eqs. (1), we introduce the following transformation

$$q(x,y,t)=e^{i(ax+by+dt)}Q(x,y,t), \tag{4}$$

where a,b,d are real constants and $Q(x,y,t)$ is the real valued function, Eqs. (1) are reduced to the following system

$$(-d-\alpha ab+\beta ba^2)Q+(\alpha-2a\beta)Q_{xy}-\beta bQ_{xx}-vQ-awQ+ \tag{5}$$
$$+i(Q_t+(\alpha a-a^2\beta)Q_y+(\alpha b-2ab\beta)Q_x+\beta Q_{xxy}+w_xQ+wQ_x)=0,$$
$$v_x+2\alpha\delta(Q^2)_y-4\beta\delta(aQ_yQ+bQ_xQ)=0, \tag{6}$$
$$w_x-2\beta\delta(Q^2)_y=0. \tag{7}$$

Substituting the wave transformation:

$$Q(x,y,t)=Q(\xi)=Q(kx+ly+ct), \tag{8}$$
$$v(x,y,t)=v(\xi)=v(kx+ly+ct), \tag{9}$$
$$w(x,y,t)=w(\xi)=w(kx+ly+ct), \tag{10}$$

into Eqs. (5)–(7), we obtain that

$$(-d-\alpha ab+\beta ba^2)Q+k(\alpha l-2a\beta l-\beta bk)Q''-vQ-awQ+ \tag{11}$$
$$+i((c+\alpha al+\alpha bk-2ab\beta k-a^2\beta l)Q'+\beta k^2lQ'''+kw'Q+kwQ')=0,$$
$$kv'+2\alpha\delta l(Q^2)'-2\beta\delta(alQ^2+bkQ^2)'=0, \tag{12}$$
$$kw'-2\beta\delta l(Q^2)'=0. \tag{13}$$

Integrating Eqs. (12)–(13) once, we obtain

$$v=\frac{1}{k}(2\delta(\beta(al+bk)-\alpha l)Q^2+R_1),\quad w=\frac{1}{k}(2\beta\delta lQ^2+S_1), \tag{14}$$

where R_1,S_1 are constants of integration. Substituting Eqs. (14) into Eq. (11), and separating real and imaginary parts, we get the ordinary differential equations:

$$(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})Q+k(\alpha l-2a\beta l-\beta bk)Q''+$$
$$+\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)Q^3=0, \tag{15}$$
$$(c+\alpha al+\alpha bk-2ab\beta k-a^2\beta l+S_1)Q'+\beta k^2lQ'''+2\beta\delta l(Q^3)'=0. \tag{16}$$

Integrating Eq. (16) once, with respect to ξ, gives

$$(c+\alpha al+\alpha bk-2ab\beta k-a^2\beta l+S_1)Q+\beta k^2lQ''+2\beta\delta lQ^3=L, \tag{17}$$

where L is a constant of integration. As the same function $Q(\xi)$ satisfies both Eqs. (15) and (17), we have the next constraint condition:

$$k^2l\beta(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})=$$
$$=k(\alpha l-2a\beta l-\beta bk)(c+\alpha al+\alpha bk-2ab\beta k-a^2\beta l+S_1),\quad L=0. \tag{18}$$

By using the condition (18), we have

$$c=\frac{kl\beta(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{\alpha l-2a\beta l-\beta b}-\alpha(al+bk)+\beta(2abk+a^2l)-S_1. \tag{19}$$

Next we solve Eq. (15)

$$(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})Q+k(\alpha l-2a\beta l-\beta bk)Q''+$$
$$+\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)Q^3=0, \tag{20}$$

Description of the Jacobi elliptic function method

In the following, we would like to outline the main steps of the method:

Step 1. Nonlinear evolution equation

$$F(u,u_t,u_x,u_y,u_{xx},u_{yy},u_{xy},\dots)=0, \tag{21}$$

by using the wave transformation $u(x,y,t)=u(\xi)$, $\xi=x+y+ct$, where c is constant, is reduced to a nonlinear ordinary differential equation:

$$E(u,u',u'',u''',\dots)=0. \tag{22}$$

Step 2. We introduce some ansatz in terms of the finite Jacobi elliptic function expansion in the following forms:

1. $sn(\xi)$ expansion

$$u(\xi)=a_0+\sum_{j=1}^{n}a_jsn^j(\xi), \tag{23}$$

2. $cn(\xi)$ expansion

$$u(\xi)=a_0+\sum_{j=1}^{n}a_jcn^j(\xi), \tag{24}$$

2. $dn(\xi)$ expansion

$$u(\xi)=a_0+\sum_{j=1}^{n}a_jdn^j(\xi), \tag{25}$$

where $sn(\xi),cn(\xi),dn(\xi)$ are the Jacobi elliptic functions.

Properties of the triangular function

$$cn^2(\xi)+sn^2(\xi)=dn^2(\xi)+m^2sn^2(\xi)=1, \tag{26}$$
$$ns^2(\xi)=1+cs^2(\xi),\quad ns^2(\xi)=m^2+ds^2(\xi), \tag{27}$$
$$sc^2(\xi)+1=nc^2(\xi),\quad m^2sd^2(\xi)+1=nd^2(\xi). \tag{28}$$

Derivatives of the Jacobi elliptic functions

$$sn'(\xi)=cn(\xi)dn(\xi),\quad cn'(\xi)=-sn(\xi)dn(\xi),$$
$$dn'(\xi)=-m^2sn(\xi)cn(\xi), \tag{29}$$
$$ns'(\xi)=-ds(\xi)cs(\xi),\quad ds'(\xi)=-cs(\xi)ns(\xi),$$
$$cs'(\xi)=-ns(\xi)ds(\xi), \tag{30}$$
$$sc'(\xi)=nc(\xi)dc(\xi),\quad nc'(\xi)=sc(\xi)dc(\xi),$$
$$cd'(\xi)=cs(\xi)nd(\xi),\quad nd'(\xi)=m^2sd(\xi)cd(\xi), \tag{31}$$

where m is a modulus.

Step 3. The parameter n can be found by balancing the nonlinear term and the highest order derivative term on Eq. (22).

Step 4. Respectively substitute Eqs. (23)–(25) into Eq. (22) along with Eqs. (26)–(28) and (29)–(31) and then respectively set all coefficients of $sn^j(\xi),cn^j(\xi),dn^j(\xi)$ (j=0,1,2,3, …) to be zero to get an over-determined system of nonlinear algebraic equations with respect to a_j (j=0,1,2,3, …). By solving the obtained system, we find the coefficients a_j (j=0,1,2,3, …). This way, we can get the solutions with the Jacobi elliptic function.

Since

$$\lim_{m\longrightarrow 1}sn(\xi)=tanh(\xi),\quad \lim_{m\longrightarrow 1}cn(\xi)=sech(\xi),\quad \lim_{m\longrightarrow 1}dn(\xi)=sech(\xi), \tag{32}$$
$$\lim_{m\longrightarrow 1}ns(\xi)=coth(\xi),\quad \lim_{m\longrightarrow 1}cs(\xi)=csch(\xi),\quad \lim_{m\longrightarrow 1}ds(\xi)=csch(\xi), \tag{33}$$

$$\lim_{m\longrightarrow 0} sn(\xi)=sin(\xi),\quad \lim_{m\longrightarrow 0} cn(\xi)=cos(\xi),\quad \lim_{m\longrightarrow 0} dn(\xi)=1, \tag{34}$$

$$\lim_{m\longrightarrow 0} sn(\xi)=csc(\xi),\quad \lim_{m\longrightarrow 0} cs(\xi)=cot(\xi),\quad \lim_{m\longrightarrow 0} ds(\xi)=csc(\xi). \tag{35}$$

Application

The sn solutions

Following the method, the solution of Eq. (20) can be obtained by transformation

$$Q(\xi)=a_0+a_1 sn(\xi). \tag{36}$$

To find the sn solution we use Eq. (36) and its second order derivative

$$Q''(\xi)=-(1+m^2)a_1 sn(\xi)+2m^2 a_1 sn^3(\xi), \tag{37}$$

and

$$Q^3(\xi)=a_0^3+3a_0^2 a_1 sn(\xi)+3a_0 a_1^2 sn^2(\xi)+a_1^3 sn^3(\xi). \tag{38}$$

Substitute (36)–(38) into (20) we get

$$\begin{aligned}&(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})(a_0+a_1 sn(\xi))+\\&+k(\alpha l-2a\beta l-\beta bk)(-(1+m^2)a_1 sn(\xi)+2m^2 a_1 sn^3(\xi))+\\&+\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)(a_0^3+3a_0^2 a_1 sn(\xi)+3a_0 a_1^2 sn^2(\xi)+a_1^3 sn^3(\xi))=0.\end{aligned} \tag{39}$$

We equate the coefficients of each pair of the $sn(\xi)$ functions and get a system of algebraic equations

$$sn^0(\xi)\ :\ (-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+ +\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)a_0^2)a_0=0, \tag{40}$$

$$sn^1(\xi)\ :\ (-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}- -k(\alpha l-2a\beta l-\beta bk)(1+m^2)+ +3a_0^2\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk))a_1=0, \tag{41}$$

$$sn^2(\xi)\ :\ 3a_0 a_1^2\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)=0, \tag{42}$$

$$sn^3(\xi)\ :\ (\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)a_1^2+2m^2k(\alpha l-2a\beta l-\beta bk))a_1=0. \tag{43}$$

By solving the system (40)–(43), we obtain:

$$a_0=0,\quad a_1=\pm m\sqrt{\frac{-k^2}{\delta}},\quad d=-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}-k(\alpha l-2a\beta l-\beta bk)(1+m^2). \tag{44}$$

Substituting Eq. (44) into Eq. (36) and then obtained result in Eqs. (4) and (14) we derive the sn solutions for *the (2+1)-dimensional HS* (1)

$$q_{11}(x,y,t)=\pm e^{i(ax+by+dt)}m\sqrt{\frac{-k^2}{\delta}}sn(kx+ly+ct),\quad \delta<0, \tag{45}$$

$$v_{11}(x,y,t)=-2(\beta(bk+al)-\alpha l)m^2ksn^2(kx+ly+ct)+\frac{R_1}{k}, \tag{46}$$

$$w_{11}(x,y,t)=-2\beta m^2lksn^2(kx+ly+ct)+\frac{S_1}{k}, \tag{47}$$

where $c=\frac{kl\beta(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{\alpha l-2a\beta l-\beta b}-\alpha(al+bk)+\beta(2abk+a^2l)-S_1$, $d=-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}-k(\alpha l-2a\beta l-\beta bk)(1+m^2)$.

We notice that the Jacobi elliptic functions degenerate to the following function

$$sn(\xi)\to tanh(\xi),\quad as\quad m\to 1. \tag{48}$$

Taking $m=1$, we have the hyperbolic function solutions

$$q_{12}(x,y,t)=\pm e^{i(ax+by+dt)}\sqrt{\frac{-k^2}{\delta}}tanh(kx+ly+ct),\quad \delta<0, \tag{49}$$

$$v_{12}(x,y,t)=-2(\beta(bk+al)-\alpha l)ktanh^2(kx+ly+ct)+\frac{R_1}{k}, \tag{50}$$

$$w_{12}(x,y,t)=-2\beta lktanh^2(kx+ly+ct)+\frac{S_1}{k}, \tag{51}$$

where $c=\frac{kl\beta(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{\alpha l-2a\beta l-\beta b}-\alpha(al+bk)+\beta(2abk+a^2l)-S_1$, $d=-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}-2k(\alpha l-2a\beta l-\beta bk)$.

Reductions

In case $\alpha=0,\beta=1$ in Eqs. (45)–(47), we obtain the sn solutions for *the (2+1)-dimensional cmKdV equations*

$$q_{13}(x,y,t)=\pm e^{i(ax+by+dt)}m\sqrt{\frac{-k^2}{\delta}}sn(kx+ly+ct),\quad \delta<0, \tag{52}$$

$$v_{13}(x,y,t)=-2(bk+al)m^2ksn^2(kx+ly+ct)+\frac{R_1}{k}, \tag{53}$$

$$w_{13}(x,y,t)=-2m^2lksn^2(kx+ly+ct)+\frac{S_1}{k}, \tag{54}$$

where $c=-\frac{kl(-d+ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{(2al+b)}+(2abk+a^2l)-S_1$, $d=ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+k(2al+bk)(1+m^2)$.

Or with $m=1$ we get

$$q_{14}(x,y,t)=\pm e^{i(ax+by+dt)}\sqrt{\frac{-k^2}{\delta}}tanh(kx+ly+ct),\quad \delta<0, \tag{55}$$

$$v_{14}(x,y,t)=-2(bk+al)ktanh^2(kx+ly+ct)+\frac{R_1}{k}, \tag{56}$$

$$w_{14}(x,y,t)=-2lktanh^2(kx+ly+ct)+\frac{S_1}{k}, \tag{57}$$

where $c=-\frac{kl(-d+ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{(2al+b)}+(2abk+a^2l)-S_1$, $d=ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+2k(2al+bk)$.

In case $\alpha=1,\beta=0$ and $S_1=0$ in Eqs. (45)–(47), we obtain the sn solutions for *the (2+1)-dimensional NLS equations*

$$q_{15}(x,y,t)=\pm e^{i(ax+by+dt)}m\sqrt{\frac{-k^2}{\delta}}sn(kx+ly+ct),\quad \delta<0, \tag{58}$$

$$v_{15}(x,y,t)=2m^2lksn^2(kx+ly+ct)+\frac{R_1}{k}, \tag{59}$$

where $c=-(al+bk)$, $d=-ab-\frac{R_1}{k}-kl(1+m^2)$.

Or with $m=1$ we get

$$q_{16}(x,y,t)=\pm e^{i(ax+by+dt)}\sqrt{\frac{-k^2}{\delta}}tanh(kx+ly+ct),\quad \delta<0, \tag{60}$$

$$v_{16}(x,y,t)=2lktanh^2(kx+ly+ct)+\frac{R_1}{k}, \tag{61}$$

where $c=-(al+bk)$, $d=-ab-\frac{R_1}{k}-2kl$.

The cn solutions

According to the method, the solution of Eq. (20) can be found by transformation

$$Q(\xi)=a_0+a_1 cn(\xi). \tag{62}$$

To find the cn solution we use Eq. (62) and its second order derivative

$$Q''(\xi)=(2m^2-1)a_1 cn(\xi)-2m^2 a_1 cn^3(\xi), \tag{63}$$

and

$$Q^3(\xi)=a_0^3+3a_0^2 a_1 cn(\xi)+3a_0 a_1^2 cn^2(\xi)+a_1^3 cn^3(\xi). \tag{64}$$

Substitute (62)–(64) into (20) we get

$$\begin{aligned}&(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})(a_0+a_1 cn(\xi))+\\&+k(\alpha l-2a\beta l-\beta bk)((2m^2-1)a_1 cn(\xi)-2m^2 a_1 cn^3(\xi))+\\&+\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)(a_0^3+3a_0^2 a_1 cn(\xi)+3a_0 a_1^2 cn^2(\xi)+a_1^3 cn^3(\xi))=0.\end{aligned} \tag{65}$$

We equate coefficients of each pair of the $cn(\xi)$ functions and get a system of algebraic equations

$$cn^0(\xi):(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+ +\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)a_0^2)a_0=0, \tag{66}$$

$$cn^1(\xi):(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+ +k(\alpha l-2a\beta l-\beta bk)(2m^2-1)+ +3a_0^2\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk))a_1=0, \tag{67}$$

$$cn^2(\xi):3a_0a_1^2\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)=0, \tag{68}$$

$$cn^3(\xi):(\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)a_1^2-2m^2k(\alpha l-2a\beta l-\beta bk))a_1=0. \tag{69}$$

By solving the system (66)–(69), we obtain:

$$a_0=0,\quad a_1=\pm m\sqrt{\frac{k^2}{\delta}},$$
$$d=-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+k(\alpha l-2a\beta l-\beta bk)(2m^2-1). \tag{70}$$

By substituting Eq. (70) into Eq. (62) and then obtained result in Eqs. (4) and (14) we derive the cn solutions for *the (2+1)-dimensional HS* (1)

$$q_{21}(x,y,t)=\pm e^{i(ax+by+dt)}m\sqrt{\frac{k^2}{\delta}}cn(kx+ly+ct),\quad \delta>0, \tag{71}$$

$$v_{21}(x,y,t)=2(\beta(bk+al)-\alpha l)m^2kcn^2(kx+ly+ct)+\frac{R_1}{k}, \tag{72}$$

$$w_{21}(x,y,t)=2\beta m^2lkcn^2(kx+ly+ct)+\frac{S_1}{k}, \tag{73}$$

where $c=\frac{kl\beta(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{\alpha l-2a\beta l-\beta b}-\alpha(al+bk)+\beta(2abk+a^2l)-S_1$, $d=-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+k(\alpha l-2a\beta l-\beta bk)(2m^2-1)$.

We notice that the Jacobi elliptic functions degenerate into the following function

$$cn(\xi)\to sech(\xi),\quad as\quad m\to 1. \tag{74}$$

Taking $m=1$, we have the hyperbolic function solutions for Eqs. (1)

$$q_{22}(x,y,t)=\pm e^{i(ax+by+dt)}\sqrt{\frac{k^2}{\delta}}sech(kx+ly+ct),\quad \delta>0, \tag{75}$$

$$v_{22}(x,y,t)=2(\beta(bk+al)-\alpha l)ksech^2(kx+ly+ct)+\frac{R_1}{k}, \tag{76}$$

$$w_{22}(x,y,t)=2\beta lksech^2(kx+ly+ct)+\frac{S_1}{k}, \tag{77}$$

where $c=\frac{kl\beta(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{\alpha l-2a\beta l-\beta b}-\alpha(al+bk)+\beta(2abk+a^2l)-S_1$, $d=-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+k(\alpha l-2a\beta l-\beta bk)$.

Reductions

In case $\alpha=0,\beta=1$ for Eqs. (71)–(73), we obtain the cn solutions for *the (2+1)-dimensional cmKdV equations*

$$q_{23}(x,y,t)=\pm e^{i(ax+by+dt)}m\sqrt{\frac{k^2}{\delta}}cn(kx+ly+ct),\quad \delta>0, \tag{78}$$

$$v_{23}(x,y,t)=2(bk+al)m^2kcn^2(kx+ly+ct)+\frac{R_1}{2}, \tag{79}$$

$$w_{23}(x,y,t)=2m^2lkcn^2(kx+ly+ct)+\frac{S_1}{k}, \tag{80}$$

where $c=-\frac{kl(-d+ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{(2al+b)}+(2abk+a^2l)-S_1$, $d=ba^2-\frac{R_1}{k}-\frac{aS_1}{k}-k(2al+bk)(2m^2-1)$.

Or with $m=1$ we get

$$q_{24}(x,y,t)=\pm e^{i(ax+by+dt)}\sqrt{\frac{k^2}{\delta}}sech(kx+ly+ct),\quad \delta>0, \tag{81}$$

$$v_{24}(x,y,t)=2(bk+al)ksech^2(kx+ly+ct)+\frac{R_1}{k}, \tag{82}$$

$$w_{24}(x,y,t)=2lksech^2(kx+ly+ct)+\frac{S_1}{k}, \tag{83}$$

where $c=-\frac{kl(-d+ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{(2al+b)}+(2abk+a^2l)-S_1$, $d=ba^2-\frac{R_1}{k}-\frac{aS_1}{k}-k(2al+bk)$.

In case $\alpha=1,\beta=0$ and $S_1=0$ for Eqs. (71)–(73), we obtain the cn solutions for *the (2+1)-dimensional NLS equations*

$$q_{25}(x,y,t)=\pm e^{i(ax+by+dt)}m\sqrt{\frac{k^2}{\delta}}cn(kx+ly+ct),\quad \delta>0, \tag{84}$$

$$v_{25}(x,y,t)=-2m^2lkcn^2(kx+ly+ct)+\frac{R_1}{k}, \tag{85}$$

where $c=-(al+bk)$, $d=-ab-\frac{R_1}{2}+kl(2m^2-1)$.

Or with $m=1$ we get

$$q_{26}(x,y,t)=\pm e^{i(ax+by+dt)}\sqrt{\frac{k^2}{\delta}}sech(kx+ly+ct),\quad \delta>0, \tag{86}$$

$$v_{26}(x,y,t)=-2lksech^2(kx+ly+ct)+\frac{R_1}{k}, \tag{87}$$

where $c=-(al+bk)$, $d=-ab-\frac{R_1}{2}+kl$.

The dn solutions

The dn solution of Eq. (20) are obtained by transformation

$$Q(\xi)=a_0+a_1dn(\xi). \tag{88}$$

and its second-order derivative

$$Q''(\xi)=(2-m^2)a_1dn(\xi)-2a_1dn^3(\xi), \tag{89}$$

and

$$Q^3(\xi)=a_0^3+3a_0^2a_1dn(\xi)+3a_0a_1^2dn^2(\xi)+a_1^3dn^3(\xi). \tag{90}$$

Substitute (88)–(90) into (20) we get

$$(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})(a_0+a_1dn(\xi))+ +k(\alpha l-2a\beta l-\beta bk)((2-m^2)a_1dn(\xi)-2a_1dn^3(\xi))+ +\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)(a_0^3+3a_0^2a_1dn(\xi)+3a_0a_1^2dn^2(\xi)+a_1^3dn^3(\xi))=0. \tag{91}$$

We equate the coefficients of each pair of the $dn(\xi)$ functions and have a system of algebraic equations

$$dn^0(\xi):(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+ +\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)a_0^2)a_0=0, \tag{92}$$

$$dn^1(\xi):(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+ +k(\alpha l-2a\beta l-\beta bk)(2-m^2)+ +3a_0^2\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk))a_1=0, \tag{93}$$

$$dn^2(\xi):3a_0a_1^2(\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk))=0, \tag{94}$$

$$dn^3(\xi):(\frac{2\delta}{k}(\alpha l-2a\beta l-\beta bk)a_1^2-2k(\alpha l-2a\beta l-\beta bk))a_1=0. \tag{95}$$

By solving the system (92)–(95), we obtain:

$$a_0=0,\quad a_1=\pm\sqrt{\frac{k^2}{\delta}},$$
$$d=-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+k(\alpha l-2a\beta l-\beta bk)(2-m^2). \tag{96}$$

Substituting Eq. (96) into Eq. (88) and then obtained result in Eqs. (4) and (14) we derive the dn solutions for *the (2+1)-dimensional HS* (1)

$$q_{31}(x,y,t)=\pm e^{i(ax+by+dt)}\sqrt{\frac{k^2}{\delta}}dn(kx+ly+ct),\quad \delta>0, \tag{97}$$

$$v_{31}(x,y,t)=2(\beta(bk+al)-\alpha l)kdn^2(kx+ly+ct)+\frac{R_1}{k}, \tag{98}$$

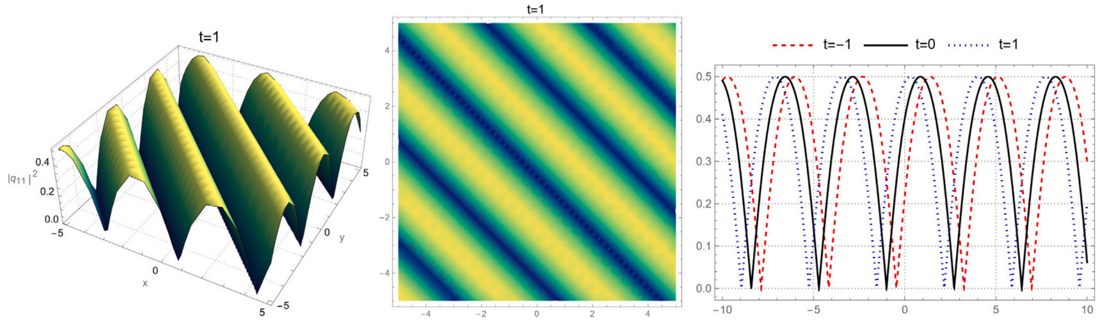

Fig. 1. The 3D, Contour, and 2D plots for q_{11} with the parameters $a=1, b=1, k=1, l=1, \delta=-1, \alpha=1, \beta=1, R_1=0, S_1=0, m=0.5, d=5, c=4.25$.

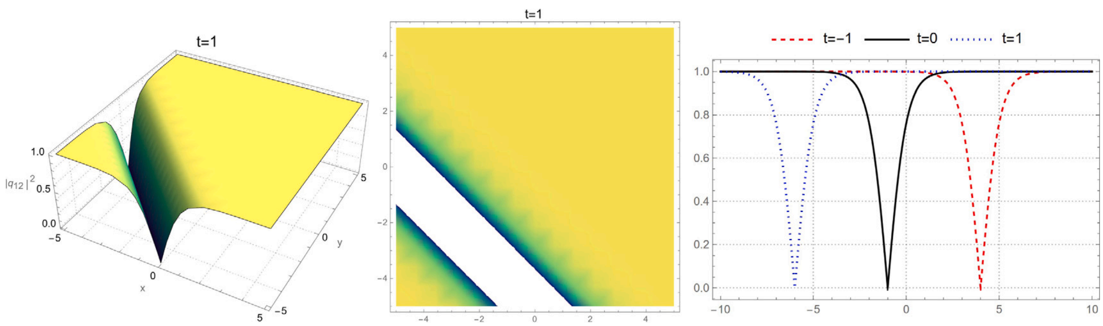

Fig. 2. The 3D, Contour, and 2D plots for q_{12} with the parameters $a=1, b=1, k=1, l=1, \delta=-1, \alpha=1, \beta=1, R_1=0, S_1=0, m=1, d=5, c=8$.

$$w_{31}(x,y,t) = 2\beta lkdn^2(kx+ly+ct)+\frac{S_1}{k}, \tag{99}$$

where $c = \frac{kl\beta(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{\alpha l-2a\beta l-\beta b} - \alpha(al+bk)+\beta(2abk+a^2l)-S_1$, $d=-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+k(\alpha l-2a\beta l-\beta bk)(2-m^2)$.

We notice that the Jacobi elliptic functions degenerate to the following function

$$dn(\xi)\to sech(\xi), \quad as \quad m\to 1. \tag{100}$$

Taking $m=1$, we have the hyperbolic function solutions

$$q_{32}(x,y,t) = \pm e^{i(ax+by+dt)}\sqrt{\frac{k^2}{\delta}}sech(kx+ly+ct), \quad \delta>0, \tag{101}$$

$$v_{32}(x,y,t) = 2(\beta(bk+al)-\alpha l)ksech^2(kx+ly+ct)+\frac{R_1}{k}, \tag{102}$$

$$w_{32}(x,y,t) = 2\beta lksech^2(kx+ly+ct)+\frac{S_1}{k}, \tag{103}$$

where $c = \frac{kl\beta(-d-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{\alpha l-2a\beta l-\beta b} - \alpha(al+bk)+\beta(2abk+a^2l)-S_1$, $d=-\alpha ab+\beta ba^2-\frac{R_1}{k}-\frac{aS_1}{k}+k(\alpha l-2a\beta l-\beta bk)$.

Reductions

In case $\alpha=0, \beta=1$ in Eqs. (97)–(99), we obtain the dn solutions for *the (2+1)-dimensional cmKdV equations*

$$q_{33}(x,y,t) = \pm e^{i(ax+by+dt)}\sqrt{\frac{k^2}{\delta}}dn(kx+ly+ct), \quad \delta>0, \tag{104}$$

$$v_{33}(x,y,t) = 2(bk+al)kdn^2(kx+ly+ct)+\frac{R_1}{k}, \tag{105}$$

$$w_{33}(x,y,t) = 2lkdn^2(kx+ly+ct)+\frac{S_1}{k}, \tag{106}$$

where $c=-\frac{kl(-d+ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{(2al+b)}+(2abk+a^2l)-S_1$, $d=ba^2-\frac{R_1}{k}-\frac{aS_1}{k}-k(2al+bk)(2-m^2)$.

Or with $m=1$ we get

$$q_{34}(x,y,t) = \pm e^{i(ax+by+dt)}\sqrt{\frac{k^2}{\delta}}sech(kx+ly+ct), \quad \delta>0, \tag{107}$$

$$v_{34}(x,y,t) = 2(bk+al)ksech^2(kx+ly+ct)+\frac{R_1}{k}, \tag{108}$$

$$w_{34}(x,y,t) = 2lksech^2(kx+ly+ct)+\frac{S_1}{k}, \tag{109}$$

where $c=-\frac{kl(-d+ba^2-\frac{R_1}{k}-\frac{aS_1}{k})}{(2al+b)}+(2abk+a^2l)-S_1$, $d=ba^2-\frac{R_1}{k}-\frac{aS_1}{k}-k(2al+bk)$.

In case $\alpha=1, \beta=0$ and $S_1=0$ in Eqs. (97)–(99), we obtain the dn solutions for *the (2+1)-dimensional NLS equations*

$$q_{35}(x,y,t) = \pm e^{i(ax+by+dt)}\sqrt{\frac{k^2}{\delta}}dn(kx+ly+ct), \quad \delta>0, \tag{110}$$

$$v_{35}(x,y,t) = -2lkdn^2(kx+ly+ct)+\frac{R_1}{k}, \tag{111}$$

where $c=-(al+bk)$, $d=-ab-\frac{R_1}{k}+lk(2-m^2)$.

Or with $m=1$ we get

$$q_{36}(x,y,t) = \pm e^{i(ax+by+dt)}\sqrt{\frac{k^2}{\delta}}sech(kx+ly+ct), \quad \delta>0, \tag{112}$$

$$v_{36}(x,y,t) = -2lksech^2(kx+ly+ct)+\frac{R_1}{2}, \tag{113}$$

where $c=-(al+bk)$, $d=-ab-\frac{R_1}{k}+lk$.

Physical interpretation of the solutions

In this section, we present the graphical representation of obtained solutions.

Fig. 1 demonstrates the 3D, contour plot and 2D surfaces of q_{11}, by taking the values $a=1, b=1, k=1, l=1, \delta=-1, \alpha=1, \beta=1, R_1=0, S_1=0, m=0.5, d=5, c=4.25$, respectively, which has a periodic solution. In the 2D representation, the evolutions of the solutions are given as $t=-1, t=0, t=1$.

Fig. 2 shows the 3D, contour plot and 2D dark soliton solution surfaces of q_{12}, for the values $a=1, b=1, k=1, l=1, \delta=-1, \alpha=1, \beta=1, R_1=0, S_1=0, m=1, d=5, c=8$, respectively. For the 2D surface dynamics, the solutions are presented with different time parameters $t=-1, t=0, t=1$ as a dark soliton. We see that the dark soliton keeps directions from the right to left.

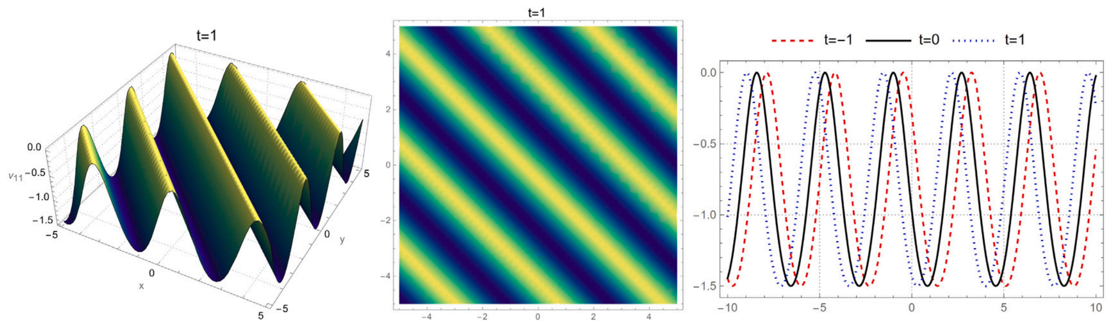

Fig. 3. The 3D, Contour, and 2D plots for v_{11} with the parameters $a = 1, b = 1, k = 1, l = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, m = 0.5, d = 5, c = 4.25$.

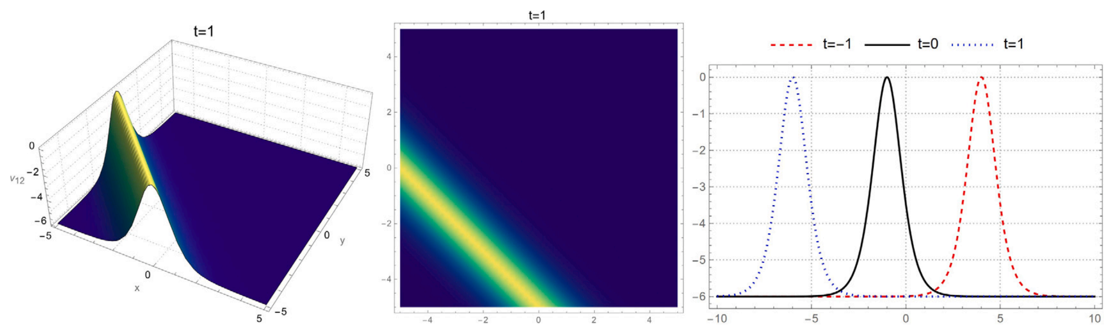

Fig. 4. The 3D, Contour, and 2D plots for v_{12} with the parameters $a = 1, b = 1, k = 1, l = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, m = 1, d = 5, c = 8$.

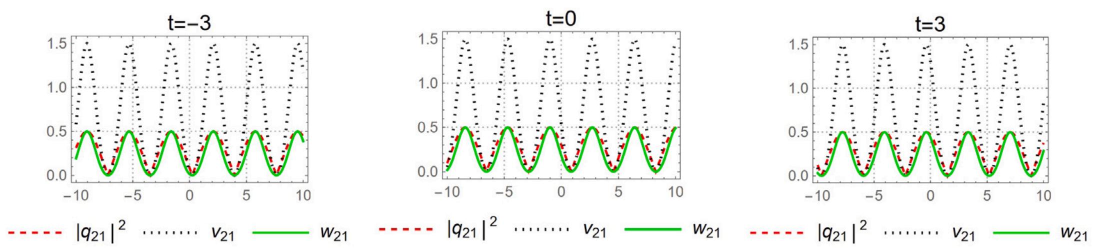

Fig. 5. The dynamics of the solutions q_{21}, v_{21}, w_{21} with the parameters $a = 1, b = 1, k = 1, l = 1, \delta = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, m = 1, d = 2, c = 3.5$.

Fig. 3 depict the 3D, contour plot and 2D surfaces for v_{11} with parameters $a = 1, b = 1, k = 1, l = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, m = 0.5, d = 5, c = 4.25$, respectively. It is the periodic solution for $t = -1, t = 0, t = 1$ in the 2D plot.

Fig. 4 contains the solution v_{12} with parameters $a = 1, b = 1, k = 1, l = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, m = 1, d = 5, c = 8$. From this figure it can be seen that the solution v_{12} is a bell-shaped soliton solution that evolves with $t = -1, t = 0, t = 1$ in the 2D representation.

In Fig. 5 we see the periodic shape of the solutions q_{21}, v_{21}, w_{21} with values $a = 1, b = 1, k = 1, l = 1, \delta = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, m = 1, d = 2, c = 3.5$. The dynamics of the solutions are shown with different time parameters $y = 1, t = -3, t = 0, t = 3$.

Fig. 6 shows the propagation of the bright soliton solutions q_{22}, v_{22}, w_{22} in 2D plot at $y = 1, t = -3, t = 0, t = 3$ with parameters $a = 1, b = 1, k = 1, l = 1, \delta = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, m = 1, d = 2, c = -4$. It is noted that bell-shaped bright solitons keep their widths, directions, and amplitudes invariant during propagation.

The considered figures show that the different choices of the parameters a, b, d, c, m yield a few waveforms such as periodic solutions, bright soliton, and dark soliton. Thus, the Jacobi elliptic functions method can yield various solutions in applying nonlinear wave equations.

Phase portraits

In this section, we present the phase portraits of Eqs. (1). Firstly, we can rewrite Eq. (20) as follows

$$Q'' - AQ + BQ^3 = 0, \tag{114}$$

where $A = \frac{(d+\alpha ab-\beta ba^2+\frac{R_1}{k}+\frac{aS_1}{k})}{k(\alpha l-2a\beta l-\beta bk)}$, $B = \frac{2\delta}{k^2}$.

Let $Q' = \varphi$, we obtain the following planar dynamical system

$$\begin{cases} \frac{dQ}{d\xi} = \varphi, \\ \frac{d\varphi}{d\xi} = AQ - BQ^3, \end{cases} \tag{115}$$

where $A = \frac{(d+\alpha ab-\beta ba^2+\frac{R_1}{k}+\frac{aS_1}{k})}{k(\alpha l-2a\beta l-\beta bk)}$, $B = \frac{2\delta}{k^2}$. The above system (115) is a Hamiltonian system with a Hamiltonian function

$$H(Q, \varphi) = \frac{\varphi^2}{2} - \frac{AQ^2}{2} + \frac{BQ^4}{4} = h, \tag{116}$$

where h is the Hamiltonian constant. To derive the equilibrium points of system (115), the following system

$$\begin{cases} \varphi = 0, \\ AQ - BQ^3 = 0, \end{cases} \tag{117}$$

is solved. For non-zero parameters A and B, Eq. (117) has three equilibrium points as follows:

$$M_1 = (0, 0), \quad M_2 = (\sqrt{\frac{A}{B}}, 0), \quad M_3 = (-\sqrt{\frac{A}{B}}, 0). \tag{118}$$

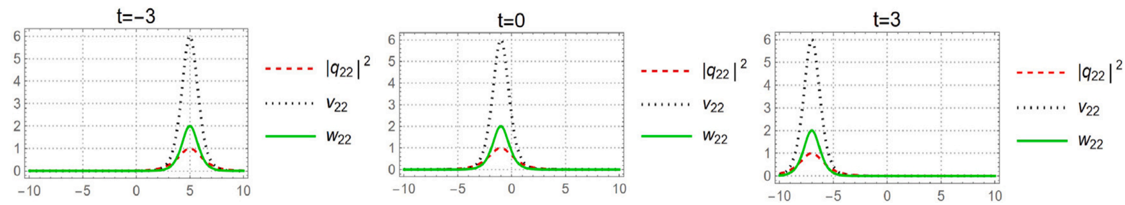

Fig. 6. The dynamics of the solutions q_{22}, v_{22}, w_{22} with the parameters $a = 1, b = 1, k = 1, l = 1, \delta = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, m = 1, d = 2, c = -4$.

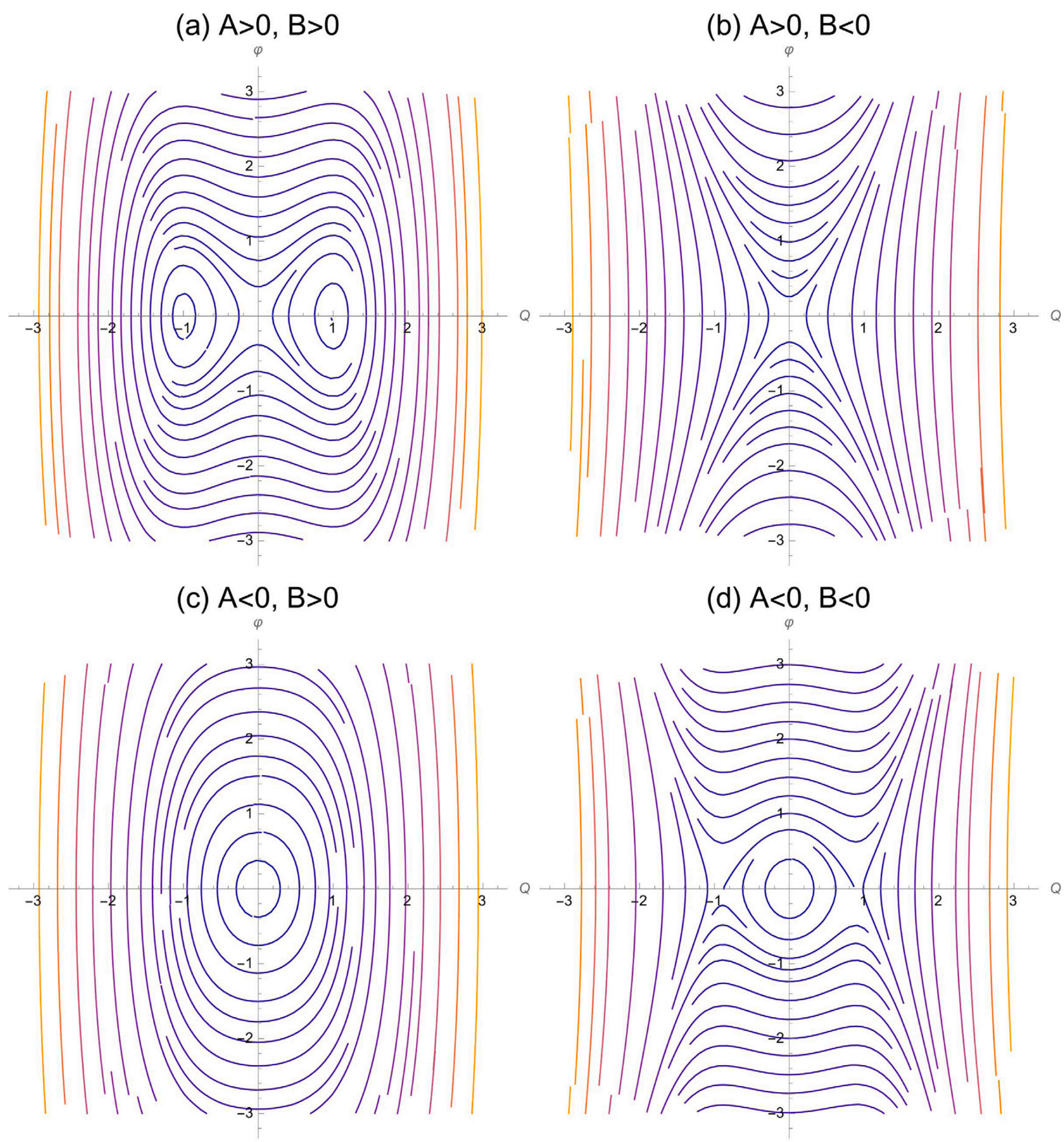

Fig. 7. The phase portraits of system (115).

The determinant of the Jacobian matrix of the system (115) is

$$D(Q,\varphi) = \begin{vmatrix} 0 & 1 \\ A - 3BQ^2 & 0 \end{vmatrix} = 3BQ^2 - A. \tag{119}$$

We know that

I. If $D(Q,\varphi) < 0$, then (Q,φ) is a saddle point;
II. If $D(Q,\varphi) > 0$, then (Q,φ) is a center;
III. If $D(Q,\varphi) = 0$, then (Q,φ) is a cuspidal point.

Here are possible outcomes resulting from varying the parameters involved

Case 1: $A > 0,\ B > 0$

By choosing a parameter regime such as $d = 0.36, a = 0.3, b = 0.2, k = 1, l = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, \delta = 1$, we find three equilibrium points $M_1 = (0,0)$, $M_2 = (1.0025, 0)$ and $M_3 = (-1.0025, 0)$ as shown in Fig. 7 (a). It can be seen that M_2 and M_3 represent a center, while M_1 is a saddle point.

Case 2: $A > 0,\ B < 0$

By choosing a parameter regime such as $d = 0.36, a = 0.3, b = 0.2, k = 1, l = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, \delta = -1$, we find that the only real point is $M_1 = (0, 0)$ (saddle point) as presented in Fig. 7 (b).

Case 3: $A < 0,\ B > 0$

By choosing a parameter regime such as $d = -0.36, a = 0.3, b = 0.2, k = 1, l = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, \delta = 1$, we find that the only real point is $M_1 = (0, 0)$ (center point) as presented in Fig. 7(c).

Case 4: $A < 0,\ B < 0$

By choosing a parameter regime such as $d = -0.36, a = 0.3, b = 0.2, k = 1, l = 1, \alpha = 1, \beta = 1, R_1 = 0, S_1 = 0, \delta = -1$, we find three equilibrium points: $M_1 = (0, 0)$, $M_2 = (0.8916, 0)$ and $M_3 = (-0.8916, 0)$ as shown in Fig. 7 (d). It can be seen that M_2 and M_3 represent a saddle point, while M_1 is a center.

Conclusions

In this paper, our attention has been focused on the (2+1)-dimensional Hirota system of equations. This model of mathematical physics encompasses the (2+1)-dimensional NLS equation and the (2+1)-dimensional cmKdV equation. It is also a (2+1)-dimensional integrable spin model that is put to use in applied magnetism. The Jacobi elliptic function method was applied to obtain new solutions. As a result, various types of exact solutions, such as dark solitons, bright solitons, and periodic wave solutions, were obtained. Also, in some reductions as $\alpha = 0, \beta = 1$ and $\alpha = 1, \beta = 0$, we received the exact solutions for the (2+1)-dimensional NLS equation and the (2+1)-dimensional cmKdV equation. To illustrate the obtained results, we plot 3D, 2D, and contour profiles by setting suitable values of the involved parameters. In addition, we obtain the phase portraits according to the bifurcation theory of planar dynamic systems.

The first advantage of the Jacobi elliptic function method is that, unlike existing methods such as Hirota's bilinear method or the inverse scattering method, tedious algebra and guesswork can be avoided. Secondly, the Jacobi elliptic function solutions degenerate into hyperbolic or trigonometric function solutions when the modulus $m = 1$ or $m = 0$. Indeed, different waveforms can be produced by choosing the parameters, such as the bell shape, the anti-bell shape, and other solutions. The obtained results are new because the used methods have not been applied for Eqs. (1) before. Moreover, this work extends the work on the (2+1)-dimensional Hirota system of equations [(25)–(28)] by deriving a variety of exact solutions. We presented detailed calculations and believed that used research methods could be useful for readers. And other researchers can apply these methods to other nonlinear equations. Moreover, it will also be interesting to study the stability and geometry properties of Eqs. (1). Related work is underway, and results will be reported separately.

CRediT authorship contribution statement

Gaukhar Shaikhova: Conceptualization, Investigation, Writing – original draft, Writing – review & editing, Supervision. **Bayan Kutum:** Conceptualization, Investigation, Writing – original draft, Writing – review & editing. **Arailym Syzdykova:** Investigation, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: The authors report financial support was provided by Committee of Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

Data availability

No data was used for the research described in the article.

Acknowledgments

The research work is prepared with the financial support of the Committee of Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan, IRN project AP09057947.

References

[1] Wazwaz A. Partial differential equations and solitary waves theory. Berlin: Springer; 2009.
[2] Manikandan K, Serikbayev N, Manigandan M, Sabareeshwaran M. Dynamical evolutions of optical smooth positons in variable coefficient nonlinear Schrödinger equation with external potentials. Optik 2023;288:171203.
[3] Verhulst F. Nonlinear differential equations and dynamical systems. Berlin: Springer; 1996.
[4] Liu Y-H, Hao H-Q, Zhang J-W. Rogue wave solutions on different periodic backgrounds for the (2+1)-dimensional Heisenberg ferromagnetic spin chain equation. J Math Anal Appl 2023;527(1):127421.
[5] Cinar M, Cakicioglu H, Secer A, Ozisik M, Bayram M. Optical solitons of improved perturbed nonlinear Schrödinger equation with cubic-quintic-septic and triple-power laws in optical metamaterials. Phys Scr 2023;98:7.
[6] Lü D, Dai S. A remark on Chern–Simons-Schrödinger equations with Hartree type nonlinearity. J Nonlinear Var Anal 2023;7:409–20.
[7] Dehghan M, Hamidi A, Shakourifar M. The solution of coupled Burgers, equations using Adomian–Pade technique. Appl Math Comput 2007;189(2):1034–47.
[8] Hosseini K, Ansari R, Gholamin P. Exact solutions of some nonlinear systems of partial differential equations by using the first integral method. J Math Anal Appl 2012;387(2):807–14.
[9] Liu JG, Zhu WH. Multiple rogue wave, breather wave and interaction solutions of a generalized (3 + 1)-dimensional variable-coefficient nonlinear wave equation. Nonlinear Dynam 2021;103:1841–50.
[10] Tian Y, Liu JG. Study on dynamical behavior of multiple lump solutions and interaction between solitons and lump wave. Nonlinear Dynam 2021;104:1507–17.
[11] Hosseini K, Mayeli P, Kumar D. New exact solutions of the coupled sine-Gordon equations in nonlinear optics using the modified Kudryashov method. J Modern Opt 2018;65(3):361–4.
[12] Kudryashov NA. On types of nonlinear nonintegrable equations with exact solutions. Phys Lett A 1991;155:269–75.
[13] Yao Sh-W, Behera S, Inc M, Rezazadeh H, Virdi JPS, Mahmoud W, et al. Analytical solutions of conformable Drinfel'd–Sokolov–Wilson and Boiti Leon Pempinelli equations via sine–cosine method. Results Phys 2022;42:105990.
[14] Javadvahidi J, Zekavatmanda SM, Rezazadeh H, Mehmet M, Akinlar A, Chugh YCh. New solitary wave solutions to the coupled Maccari's system. Results Phys 2021;21:103801.
[15] Ala V, Rakhimzhanov B. Exact solutions of beta fractional Fokas-Lenells equation via sine-cosine method. Bull South Ural State Univ: Math Model, Progr Comput Softw 2023;16(2):5–13.
[16] Chen H, Zheng Sh. Darboux transformation for nonlinear Schrödinger type hierarchies. Physica D 2023;454:133863.
[17] Matveev V, Salle MA. Darboux transformations and solitons. Berlin, Germany: Springer-Verlag; 1991.
[18] Wazwaz AM. The Tanh and the Sine-Cosine methods for the complex modified KdV and the generalized KdV equations. Comput Math Appl 2005;49:1101–12.
[19] Burdik C, Shaikhova G, Rakhimzhanov B. Soliton solutions and travelling wave solutions for the two-dimensional generalized nonlinear Schrödinger equations. Eur Phys J Plus 2021;136:1095, (1–17).
[20] Osman MS, Machado JAT. New nonautonomous combined multi-wave solutions for (2 + 1)-dimensional variable coefficients kdv equation. Nonlinear Dynam 2018;93(2):733–40.
[21] Javid A, Raza N, Osman MS. Multi-solitons of thermophoretic motion equation depicting the wrinkle propagation in substrate-supported graphene sheets. Commun Theor Phys 2019;71(4):362–6.
[22] Cherniha R, Muzyka L. Lie symmetries of the Shigesada–Kawasaki–Teramoto system. Commun Nonlinear Sci Numer Simul 2017;45:81–92.
[23] Serikbayev N, Saparbekova A. Symmetry and conservation laws of the (2+1)-dimensional nonlinear Schrödinger-type equation. Int J Geom Methods Mod Phys 2023;2350172.
[24] Ghanbari B, Kumar S, Niwas M, Baleanu D. The Lie symmetry analysis and exact Jacobi elliptic solutions for the Kawahara–KdV type equations. Results Phys 2021;23:104006.
[25] Myrzakulov R, Mamyrbekova GK, Nugmanova GN, Lakshmanan M. Integrable (2+ 1)-dimensional spin models with self-consistent potentials. Symmetry 2015;7(3):1352–75.
[26] Yesmakhanova K, Shaikhova G, Bekova G. Soliton solutions of the Hirota's system. AIP Conf Proc 2016;1759:020147.
[27] Liu Y-H, Zhang J-W. Localized wave solutions and their superposition and conversion mechanism for the (2+1)-dimensional Hirota's system. Optik 2023;277:170717.

[28] Bekova G, Shaikhova G, Yesmakhanova K. Travelling wave solutions for the two-dimensional Hirota system of equations. AIP Conf Proc 2018;1997:020039.
[29] Shaikhova GN, Serikbayev N, Yesmakhanova K, Myrzakulov R. Nonlocal complex modified Korteweg–de Vries equations: reductions and exact solutions. In: Proceedings of the twenty-first international conference on geometry, integrability and quantization. 2020, p. 265–71.
[30] Yuan F, Zhu X, Wang Y. Deformed solitons of a typical set of (2+1)-dimensional complex modified korteweg-de vries equations. Int J Appl Math Comput Sci 2020;30(2):337–50.
[31] Shaikhova G, Kutum B, Myrzakulov R. Periodic traveling wave, bright and dark soliton solutions of the (2+1)-dimensional complex modified Korteweg–de Vries system of equations by using three different methods. AIMS Math 2022;7(10):18948–70.
[32] Muniyappan A, Sharmil M, Priya EK, Sumithra S, Biswas A, Yıldırı Y, et al. W-shaped chirp free and chirped bright, dark solitons for perturbed nonlinear Schrödinger equation in nonlinear optical fibers. Proc Estonian Acad Sci 2023;72(2):128–44.
[33] Biswasa A, Ekicid M, Sonmezoglud A, Belice MR. Highly dispersive optical solitons with cubic–quintic–septic law by extended Jacobi's elliptic function expansion. Optik - Int J Light Electron Opt 2019;183:571–8.
[34] Ren Y-J, Zhang H-Q. A generalized F-expansion method to find abundant families of Jacobi Elliptic Function solutions of the (2 + 1)-dimensional Nizhnik–Novikov–Veselov equation. Chaos Solitons Fractals 2006;27:959–79.
[35] Sharif A. Jacobielliptic function approach to a conformable fractional nonlinear Schrödinger–Hirota equation. Partial Differ Equ Appl Math 2023;8:100541.
[36] Fan E, Zhang J. Applications of the Jacobi elliptic function method to special-type nonlinear equations. Phys Lett A 2002;305:383–92.
[37] Zhang J, Hao H-Q. Soliton solutions of the AB system via the Jacobi elliptic function expansion method. Optik-Int J Light Electron Opt 2021;244:167541.
[38] Lu D. New exact solutions for the generalized variable-coefficient Gardner equation with forcing term. Appl Math Comput 2012;219:2732–8.
[39] Belic M, Petrovic N, Zhong W-P, Xie R-H, Chen G. Analytical light bullet solutions to the generalized (3+1)-dimensional nonlinear Schrödinger equation. Phys Rev Lett 2008;101:123904, (1–4).
[40] Deng W, He J-R, Xue L. Snakelike similaritons in combined harmonic-lattice potentials with a varying source. Nonlinear Dynam 2020;100:1599–609.
[41] Zh Li, Ch Huang. Bifurcation, phase portrait, chaotic pattern and optical soliton solutions of the conformable Fokas–Lenells model in optical fibers. Chaos Solitons Fractals 2023;169:113237.
[42] Hosseini K, Hincal E, Ilie M. Bifurcation analysis, chaotic behaviors, sensitivity analysis, and soliton solutions of a generalized Schrödinger equation. Nonlinear Dyn 2023;111:17455–62.
[43] Kazmi SS, Jhangeer A, Raza N, Alrebdi HI, Abdel-Aty A-H, Eleuch H. The analysis of bifurcation, quasi-periodic and solitons patterns to the new form of the generalized q-deformed Sinh-Gordon equation. Symmetry 2023;15:1324.
[44] Xu L. Bifurcation analysis for a discrete space–time model of logistic type. J Nonlinear Funct Anal 2022;2022(29):1–9.