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ORIGINAL ARTICLE

Cosmic string in gravity's rainbow

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Abstract In this paper, we study the various cylindrical solutions (cosmic strings) in gravity's rainbow scenario. In particular, we calculate the gravitational field equations corresponding to energy-dependent background. Further, we discuss the possible Kasner, quasi-Kasner and non-Kasner exact solutions of the field equations. In this framework, we find that quasi-Kasner solutions cannot be realized in gravity's rainbow. Assuming only time-dependent metric functions, we also analyse the time-dependent vacuum cosmic strings in gravity's rainbow, which are completely different than the other GR solutions.

Keywords Cosmic string · Gravity's rainbow

1 Overview and motivation

A strong notion of an observer independent minimum length scale has been found in all theories of quantum gravity, for instance, in string theory (Amati et al. 1989), noncommutative geometry (Girelli et al. 2005), loop quantum gravity (Rovelli 1998; Carlip 2001) and Lorentzian dynamical triangulations (Will 2014; Ambjorn and Loll 1998; Ambjorn et al. 2000, 2001). Here we point out that the nascent GW astronomy (Abbott et al. 2017) could help in discriminating among general relativity or alternative theories (Corda 2009). There is no harm to assume this minimum measurable length scale as the Planck scale. The mathematical ground of general theory of relativity is based on a smooth manifold which breaks down when energies of probe reaches the order of Planck energy (Maggiore 1993; Park 2008). Keeping this point in mind, one may expect a radically new picture of spacetime, which includes departure from the standard relativistic dispersion relation. A departure from the standard dispersion relation indicates that the system incorporates a breaking of Lorentz invariance. Indeed, Lorentz symmetry is one of the most remarkable symmetries in nature which along with the Poincaré symmetry fix the standard form of energy (E)-momentum ($\vec{p}$) dispersion relation, i.e., $E^2 - |\vec{p}|^2 = m^2$. A modification in the standard energy-momentum dispersion relation occurs in the ultraviolet limit of most of the quantum gravity theories (t'Hooft 1996; Kostelecky and Samuel 1989; Amelino-Camelia et al. 1998; Gambini and Pullin 1999; Carroll et al. 2001). In fact, a modification in the energy-momentum dispersion relation is studied in Horava-Lifshitz gravity in the ultraviolet region (Horava 2009a, 2009b). Although the broken Lorentz invariance is considered in ultraviolet limit, the velocity of light c and the Planck energy E_p should not be modified. The study of such modified energy-momentum dispersion relation (MDR) is known as double special relativity (DSR) (or non-linear special relativity) (Magueijo and Smolin 2002a, 2002b).

An extension of DSR into a general relativity framework which has at its foundation the proposal that the geometry of a spacetime runs with the energy scale at which the geometry is being probed is known as gravity's rainbow (or might

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be called "doubly general relativity") (Magueijo and Smolin 2004). In this regard, they found that the cosmological distances, in an expanding universe, become energy dependent. In fact, by considering the energy dependent time, they addressed the horizon problem without inflation or a varying speed of light (Magueijo and Smolin 2004). Earlier, it has been found that the gravity's rainbow produces a deformation to the spacetime metric which becomes significant at the Planck scale of the particle's energy/momentum. Moreover, it is realized that quantum corrections can become relevant not only for particles approaching the Planck energy but, due to the one loop contribution, even for low-energy particles as far as Planckian length scales are considered (Garattini and Mandanici 2012). The gravity's rainbow illustrates a new mass-temperature relation and define a minimum mass and maximum temperature for rainbow black hole predicting the existence of black hole remnant (Li et al. 2009; Ling et al. 2006). It has also been found that the gravity's rainbow prevents black holes from evaporating completely (Ling et al. 2007; Ali 2014), just like the standard uncertainty principle prevents the hydrogen atom from collapsing (Adler et al. 2001; Cavaglia et al. 2003). The quantum corrections due to rainbow functions of the metric are studied in thermodynamics of the massive BTZ black holes (Hendi et al. 2017). It has been found there that in semi-classical/quantum regime, thermodynamics of the black holes would be modified into a level which differs from classical case. More precisely, the different orders of the rainbow functions affect the high energy and asymptotical behaviors of the solutions and their leading terms (Hendi et al. 2017).

Moreover, the gravity's rainbow has also been studied at various occasions in recent past. For instance, the critical behavior of the black holes in Gauss-Bonnet gravity's rainbow was discussed and found that the generalization to a charged case puts an energy dependent restriction on different parameters (Hendi et al. 2016d). By considering rainbow functions in terms of power-law of the Hubble parameter, the Starobinsky model of inflation, from the perspectives of gravity's rainbow, was investigated (Chatrabhuti et al. 2016). Very recently, the modifications on Hawking-Page phase transition (Feng et al. 2016; Kim et al. 2016) and wave function of the universe (Khodadi et al. 2016) are also discussed. In gravity's rainbow framework, the Hawking, Unruh, freefall and fiducial temperatures of the black hole have also been investigated (Yadav et al. 2016; Gim and Kim 2016). In addition, remnants of black objects (Ali et al. 2014), asymptotic flatness (Hackett 2006), nonsingular universes in Einstein and Gauss-Bonnet gravities (Awad et al. 2013; Hendi et al. 2016c) have been studied in the gravity's rainbow background. The zero point energy in a spherically symmetric background combining the high energy distortion of gravity's rainbow with the modification induced by a $f(R)$ theory has been interpreted in Garattini (2013), Hendi et al. (2016b). Within the context of gravity's rainbow modified geometry, motivated from quantum gravity corrections at the Planck energy scale, it is shown that the distortion of the metric leads to a Wheeler-DeWitt equation whose solution admits outgoing plane waves and consequently, a period of cosmological inflation may arise without the need of introducing an inflation field (Garattini and Sakellariadou 2014). In cosmology of early Universe, we investigate the generally accepted doctrine that the universe is affecting to what we termed as "topological defects" through exhaustion of all sources of matter, and suggest that by virtue of a cosmic string mechanism which maintains its available energy is self-gravitating. Energy is being "degraded" in objects which are in the cosmos, but "elevated" or raised to a higher level in strings (Vilenkin and Shellard 2000; Hindmarsh and Kibble 1995). The modified Friedmann-Robertson-Walker(FRW) equations are also derived in the contexts of gravity's rainbow (Ling 2007). From the astrophysical perspective, it was shown that the existence of energy dependent spacetime can modify the hydrostatic equilibrium equation (or modified TOV) of stars (Hendi et al. 2016a). One main motivation for us to study exact cylindrical solutions in gravitational theories is to describe such topological defects by Riemannian geometry. A simple description of the above topological defects is to find the cylindrical solution by solving highly non linear field equations. In general relativity (GR), the simplest cylindrical model described by the class of exact cylindrical solutions were found by Kasner and later on studied by several authors (Kasner 1921, 1925; Linet 1985, 1986; Tian 1986).

On the other hand, the spontaneous symmetry breaking mechanism and cosmological phase transitions together urged us to confront the possibility of topological defects playing a significant role in cosmology (Vilenkin and Shellard 2000). The cosmic strings, in fact, provide a viable fluctuation spectrum for galaxy formation (Zel'dovich 1980). The study of topological defects and cosmological implications of strings are subjects of sustained interest (Hindmarsh and Kibble 1995). The cylindrical solutions play a major role in the study gravitational systems. For instance, cylindrical solutions describe the gravitational waves by an effective energy tensor, in terms of a gravitational potential generalizing the Newtonian potential (Hayward 2000). Also, cylindrical symmetry gets relevance in order to study the exact solutions in general relativity (not only due to the theoretical reasons but also for the physical realization in objects such as cosmic strings). The cylindrical solutions are discussed from the viewpoint of exact solutions in $f(R)$ gravity theories (Azadi et al. 2008). Although the substantial progress has been made in this area but the cosmic string solutions in gravity's rainbow framework are still unexplored. This provides an opportunity to us to bridge this gap. One of the oldest branch in GR is to find exact solutions of certain types of gravitational theories as Riemannian metrics $g_{\mu\nu}$ satisfy some types of field equations. Recently, exact solution finds different interesting applications in other complex problems of physics (Kramer et al. 1980). A celebrated application is found when exact forms for a type of gravity can be used to probe the quantum theory on the associated spatial boundary. Generally speaking to proceed with exact solutions, we need to have two basic choices: first option is to fixing symmetry of the background metric $g_{\mu\nu}$; second is to give the symmetry to the matter contents $T_{\mu\nu}$. Although, in gravitational theory, there is no simple and direct relation between the symmetry of source of the gravitational field and the symmetry of the metric because of non-linearity of field equations and breaking of linear approximations, but still we can probe symmetry very carefully from matter. As we know, topology is an independent parameter and can be imposed on geometry after we fix the general form of metric. It also becomes important when some types of the topological effects are needed suddenly by assigning an independent metric. In the context of modified theories of gravity, cosmic strings are investigated in $f(R)$ gravity (Azadi et al. 2008; Momeni and Gholizade 2009), teleparallel theories (Baker 1990; Maluf and Goya 2001; Houndjo et al. 2012), brane worlds (Dvali et al. 2000), Kaluza-Klein models (Furtado et al. 1999), Lovelock Lagrangians (Simon 1990), Gauss-Bonnet (Cheng and Liu 2008; Rodrigues et al. 2014; Houndjo et al. 2014), Born-Infeld (Gibbons and Herdeiro 2001; Ferraro and Fiorini 2011), bimetric theories (Reddy et al. 2006), non-relativistic models of gravity (Momeni 2011), scalar-tensor theories (Gundlach and Ortiz 1990; Bezerra et al. 2003; Ferreira et al. 2000; Barros and Romero 1995; Emília and Guimarães 1997), Brans-Dicke theory (Delice 2006a, 2006b; Baykal et al. 2010; Baykal and Delice 2005; Kirezli et al. 2013; Çiftci and Delice 2015), dilation gravity (Tseytlin and Vafa 1992; Gregory and Santos 1997), non-minimally coupled models of gravity (Harko and Lake 2015a), Mimetic gravity (Momeni et al. 2016) and, recently, in the Bose-Einstein condensate strings (Harko and Lake 2015b). However, the cosmic string is still unexplored in gravity's rainbow setting. Here, we try to bridge this gap.

In this paper, we first briefly review the basics of energy-dependent Einstein field equations described by a specific rainbow functions. Following the basic properties of cosmic strings, we write the metric of the cosmic string in gravity's rainbow. Implementation the metric of the cosmic string in gravity's rainbow to the Einstein field equations leads to a set of differential equations in terms of rainbow function. In order to realize the exact solutions of these differential equations, we consider the two parametric metric solution (so-called Kasner solution, which is an unique exact solution for the Einstein equations with cylindrical symmetry). In addition to that we also discuss the possibility of the quasi-Kasner and non-Kasner solutions. In this regard, we find that the quasi-Kasner solutions cannot be realized in gravity's rainbow. In this setting, we further compute the Ricci and Kretschmann scalars and observe that the gravity's rainbow cosmic strings have same singularities as in the standard GR theory. We discuss the time-dependent cosmic strings also in gravity's rainbow. The time-dependent vacuum solutions are based on the assumption that all metric functions depend on time only not on space.

We organize this work as follows. In Sect. 2, we discuss the basic set-up gravity's rainbow. The equations of motion for cosmic string in gravity's rainbow is computed in Sect. 3. The realization of Kasner's solution in gravity's rainbow is given in Sect. 4. We try to emphasize the spherically symmetric solution for gravity's rainbow by considering energy as a function of radial coordinate only in Sect. 5. The time dependent cosmic string solutions are discussed in Sect. 6. The behavior of cosmic string in gravity's rainbow is discussed by considering energy as a function of time only in Sect. 7. We summarize results in the last section.

2 Basic set-up of gravity's rainbow

Gravity's rainbow (doubly general relativity) is an extension of DSR into a general relativity framework, which justifies a modified dispersion relation given by (Magueijo and Smolin 2002a, 2002b)

$$E^2 f^2(E/E_p) - |\vec{p}|^2 g^2(E/E_p) = m^2c^4, \tag{1}$$

where E_p refers the Planck energy. Here, functions $f(E/E_p)$ and $g(E/E_p)$ are known as the rainbow functions. This modification in the energy-momentum relation due to rainbow functions becomes significant in the ultraviolet limit. However, the following constrained are required to reproduce the standard dispersion relation in the infrared limit:

$$\lim_{E/E_p\to 0} f(E/E_p) = 1; \qquad \lim_{E/E_p\to 0} g(E/E_p) = 1. \tag{2}$$

In order to express the energy-dependent metrics in a one-parameter family, we write (Magueijo and Smolin 2004)

$$g(E) = \eta^{ab} e_a(E/E_p) \otimes e_b(E/E_p) \tag{3}$$

where the energy-dependent set of orthonormal frame fields $e_a = (e_0, e_i)$ are

$$\begin{aligned} e_0(E/E_p) &= \frac{1}{f(E/E_p)}\tilde{e}_0, \\ e_i(E/E_p) &= \frac{1}{g(E/E_p)}\tilde{e}_i. \end{aligned} \tag{4}$$

Here, the quantities $(\tilde{e}_0, \tilde{e}_i)$ are the energy independent frame fields. Here, we also note that, in the limit $E/E_p \to 0$, this corresponds to usual general relativity. Eventually, these gravity's rainbow functions modify the black hole metrics. Motivated from loop quantum gravity considerations (Alfaro et al. 2002; Sahlmann and Thiemann 2006; Smolin 2006), our analysis is based on the following specific rainbow functions (Amelino-Camelia et al. 1998):

$$f(E/E_p) = 1; \qquad g(E/E_p) = \sqrt{1 - \eta\left(\frac{E}{E_p}\right)^n} \tag{5}$$

where η refers to the rainbow parameter. We follow the natural units $c = \hbar = k_B = 1$ throughout the paper.

3 Metric and equations of motion

In this section, we compute the equations of motion for cosmic string in gravity's rainbow. This type of cylindrical solution has treated popular in the literature for some time with the name of cosmic string as a model to describe topological defects of early cosmology and closed time like curves.

According to standard definition, the general cosmic string metric has the following basic properties:

$$g_{\mu\nu}(t, r, \varphi, z) = \begin{cases} g_{ta} = 0, & a = \{r, \varphi, z\} \\ \partial_t g_{\mu\nu} = 0, & \mu, \nu = \{t, r, \varphi, z\} \\ \partial_z, \partial\varphi, & \text{symmetries} \\ \mathcal{R}^3 \times \mathcal{S}^1, & \text{topology.} \end{cases} \tag{6}$$

Inevitably, the cosmic string with cylindrical symmetry describes an exact solution in general relativity with a metric which presents an interior solution when radius tends to zero.

The cosmic string metric $g_{\mu\nu}$, $\mu = \{t, x^a\}$, $a = 1, 2, 3$, is supposed to be static (i.e. with vanishing off diagonal components $g_{ta} = 0, a = \{r, \varphi, z\}$ and time independent $\partial_t g_{\mu\nu} = 0$) and cylindrical symmetric. It means we suppose that the distribution of matter fields in such space time forms a static stress-energy tensor. By symmetry, we will think about existence of a pair of commuting killing vectors $\partial_z, \partial\varphi$. From kinematical point of view, the classical trajectories (orbits) are closed around the z axis. The unique coordinate to be used to describe the metric $g_{\mu\nu}$ is the (semi) radial coordinate r (because it doesn't mean distance in general). This coordinate r is initiating from the z axis when $r = 0$ and it is supposed to be extended smoothly up to the spatial infinity $r = \infty$. By definition, the topology of the space time is uniquely well defined by the $\mathcal{R}^3 \times \mathcal{S}^1$, here $\mathcal{R}$ denotes the domain of real numbers and $\mathcal{S}^1$ defines a unit circle.

Let us start by writing the metric of the time-independent cosmic string in gravity's rainbow scenario as follows,

$$ds^2 = \frac{A(r)}{f^2(E/E_p)}dt^2 - \frac{1}{g^2(E/E_p)}dr^2 - \frac{B(r)}{g^2(E/E_p)}d\varphi^2 - \frac{C(r)}{g^2(E/E_p)}dz^2, \tag{7}$$

where $A(r)$, $B(r)$, $C(r)$ are some functions depend on cylindrical coordinate r.

The Einstein field equation obtained by varying the usual Einstein-Hilbert action with respect to the rainbow's metric $g_{\mu\nu}(E/E_p)$ is given as follows:

$$R(E/E_p)_{\mu\nu} - \frac{1}{2}g(E/E_p)_{\mu\nu}R = 8\pi G(E/E_p)T_{\mu\nu}, \tag{8}$$

whereas the energy dependent Newton's universal gravitational constant $G(E/E_p)$ becomes the conventional Newton's universal gravitational constant $G = G(0)$ in the limit $E/E_p \to 0$. We assume that the matter fields fill the space-time with energy-momentum tensor $T_\mu^\nu = \text{diag}(\rho, -p_r, -p_\varphi, -p_z)$, where ρ is energy density and p_i corresponds to different components of pressure field. By substituting (7) in field equation (8), we obtain the following set of ordinary differential equations:

$$\frac{B'^2}{B^2} - \frac{B'C'}{BC} + \frac{C'^2}{C^2} - \frac{B''}{B} - \frac{C''}{C} = 32\pi G(E/E_p)\frac{\rho}{g^2}, \tag{9}$$

$$\frac{A'B'}{AB} + \frac{A'C'}{AC} + \frac{B'C'}{BC} = 32\pi G(E/E_p)\frac{p_r}{g^2}, \tag{10}$$

$$\frac{A'^2}{A^2} - \frac{A'C'}{AC} + \frac{C'^2}{C^2} - 2\frac{A''}{A} + 2\frac{C''}{C} = -32\pi G(E/E_p)\frac{p_\varphi}{g^2}, \tag{11}$$

$$\frac{A'^2}{A^2} - \frac{A'B'}{AB} + \frac{B'^2}{B^2} - 2\frac{A''}{A} - 2\frac{B''}{B} = -32\pi G(E/E_p)\frac{p_z}{g^2}. \tag{12}$$

In order to discuss the behavior of these equations, we need to them. We emphasize this in forthcoming sections.

4 Realization of Kasner's solution in gravity's rainbow

Here, to find the exact solution, we focus, in particular, Kasner's solution in gravity's rainbow. The two parametric metric, so-called Kasner solution, is an unique exact solution for the Einstein equations with cylindrical symmetry in GR (Kasner 1921, 1925; Kramer et al. 1980). This is given by following line element:

$$ds^2 = (kr)^{2a}dt^2 - dr^2 - \beta^2(kr)^{2(b-1)}r^2d\varphi^2 - (kr)^{2c}dz^2, \tag{13}$$

here k defines an appropriate length scale and β is a constant and it is related directly to the deficit angle of the conical space-time. By solving Einstein equations (8), we are considering not only the form of metric functions for which

$R_{\mu\nu} = 0$ but also every possible value of the parameters $\{a, b, c\}$ satisfying,

$$a + b + c = a^2 + b^2 + c^2 = 1. \tag{14}$$

In Kasner metric, Ricci scalar vanishes (i.e. $R = 0$), however, for quasi-Kasner solution one can have non-vanishing Ricci scalar (i.e. $R \neq 0$). We will show that the Kasner metric is a trivial solution in gravity's rainbow. A possible question will be, "does the quasi-Kasner solution with $R \neq 0$ solves our gravity's rainbow system described by the equations Eqs. (9)–(12) or not?". We address this problem in the following situations:

- *Quasi-Kasner solutions in Rainbow scenario*: $A = (kr)^{2a}$, $B = \beta^2 r^2 (kr)^{2(b-1)}$, $C = (kr)^{2c}$:

The condition $T_{\mu\nu} \neq 0$ in (9)–(12) would not harm us to find the quasi-Kasner's solutions. It will be interesting enough to find something similar to GR solutions. Substituting these values of solutions in Eqs. (9)–(12), we observe that for particular values of parameters (a, b, c) the quasi-Kasner is a solution for field equations in gravity's rainbow. These are

$$ds^2 = \frac{dt^2}{f^2(E/E_p)} - \frac{dr^2}{g^2(E/E_p)} - \frac{\beta^2 k^2 d\varphi^2}{g^2(E/E_p)} - \frac{(kr)^2 dz^2}{g^2(E/E_p)}, \tag{15}$$

$$ds^2 = \frac{dt^2}{f^2(E/E_p)} - \frac{dr^2}{g^2(E/E_p)} - \frac{\beta^2 r^2 d\varphi^2}{g^2(E/E_p)} - \frac{dz^2}{g^2(E/E_p)}, \tag{16}$$

$$ds^2 = \frac{(kr)^2 dt^2}{f^2(E/E_p)} - \frac{dr^2}{g^2(E/E_p)} - \frac{\beta^2 k^2 d\varphi^2}{g^2(E/E_p)} - \frac{dz^2}{g^2(E/E_p)}. \tag{17}$$

From the above expressions, it is obvious that if we put the values of parameters (a, b, c) into the metric (13), we obtain a class of Kasner metrics with $R = 0$. This implies that the quasi-Kasner solutions cannot be realized in gravity's rainbow.

- *Non-Kasner family of the exact solutions*:

In order to study the non-Kesner type of solution, we first eliminate A, C and B respectively from Eqs. (9)–(12) and, as a result, we obtain:

$$B''' = -\frac{B'^2 B'' - 2B''^2 B}{B' B} \tag{18}$$

$$C'' = -\frac{2B'' BB'C - B'^3 C + B'' B^2 C'}{B' B^2} \tag{19}$$

$$A' = -\frac{B' C' AB - 2B'^2 AC + 4B'' B AC}{-BC'^2 + B'BC}, \tag{20}$$

along with the following constraint:

$$-B'^2 BC'C - B'^3 C^2 + B' B^2 C'^2 + 2B'' B^2 BC' + 2B'' B' BC^2 = 0. \tag{21}$$

Now, it is possible to solve these three differential equations given in (18)–(20). In this way, we found the following exact solutions for system of equations:

$$A(r) = l_3 \exp\left[-l_1 \int dr \frac{l_4(l_1 + \frac{n}{3} - 2)(r - r_0)^{\frac{n}{2}} + 3l_5(l_1 - \frac{n}{3} - 2)(r - r_0)^{-\frac{n}{2}}}{l_4(l_1 - \frac{n}{3} - \frac{2}{3})(r - r_0)^{1+\frac{n}{2}} + l_5(lC_1 + \frac{n}{3} - \frac{2}{3})(r - r_0)^{1-\frac{n}{2}}}\right], \tag{22}$$

$$B(r) = \tilde{l}_3 (r - r_0)^{l_1}, \tag{23}$$

$$C(r) = l_4 (r - r_0)^{-\frac{1}{2}l_1 + 1 + \frac{1}{2}n} + l_5 (r - r_0)^{-\frac{1}{2}l_1 + 1 - \frac{1}{2}n}, \tag{24}$$

where $\tilde{l}_3 = l_3 e^{i\pi l_1}$, $n = \sqrt{-3l_1^2 + 4l_1 + 4}$, $l_3 > 0$, $l_1 \in [-\frac{2}{3}, 2]$, $n \in \mathcal{R}$ and $r \geq l_2$.

Here we note that for $n = 0$ the metric converts to the following form:

$$ds^2 = \frac{\rho^2 d\tilde{t}^2}{f^2(E/E_p)} - \frac{d\rho^2}{g^2(E/E_p)} - \frac{\mu d\varphi^2}{g^2(E/E_p)} - \frac{d\tilde{z}^2}{g^2(E/E_p)}, \tag{25}$$

where we defined $\mu = l_4 + l_5$, $\rho = r - l_2$, $\tilde{t} = t\sqrt{\tilde{l}_3}$, $\tilde{z} = z\sqrt{l_3}$. In fact, this metric corresponds to the vacuum Levi-Civita metric and it coincides with the cosmic string. Here, we also mention that the azimuthal angle $\varphi \neq\in (0, 2\pi]$, but the geometry still remains close to flat space. The deficit angle, in this case, is $1 - 4\eta(E/E_p) = \mu(E/E_p)$, where $\eta(E/E_p)$ is the gravitational mass per unit length of the spacetime.

We compute Ricci scalar and get

$$R=g^2(E/E_p)\big[e^{i\pi(1+l_1)}n^2 l_1 l_4 l_5\big]\rho^{-1+2l_1}. \tag{26}$$

Here we notice that the Ricci scalar $R\neq 0$. The Kretschmann scalar $\mathcal{K}=R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ corresponding to this metric is a non-singular function for $\rho\in\mathcal{R}$, however, the Kretschmann scalar has singularity at $\rho=0$, $\rho=(\pm i\sqrt{\frac{l_4(-2+3l_1-n)}{l_5(3l_1+n-2)}})^{-2/n}$ (with $l_5\neq 0$), $\rho=\exp(-\frac{1}{n}\log\frac{l_4}{l_5})$ (with $\frac{l_4}{l_5}>0$). Thus, we can conclude that gravity's rainbow cosmic strings have same singularities as in the standard GR theory.

5 When $E=E(r)$

In this section, we consider the general $f(R)$ case, where modified Einstein equations become (Azadi et al. 2008),

$$FR_{\mu\nu}-\nabla_\mu\nabla_\nu F=\frac{1}{4}g_{\mu\nu}(FR-\Box F), \tag{27}$$

where $F(R)=\frac{df(R)}{dR}$. From above, we can write

$$\frac{FR_{\mu\mu}-\nabla_\mu\nabla_\mu F}{g_{\mu\mu}}=\frac{1}{4}(FR-\Box F)=\frac{1}{4}g_{\mu\nu}(FR-\Box F) := A_\mu. \tag{28}$$

This means that the combination A_μ is independent of the index μ and therefore $A_\mu=A_\nu$ for all μ,ν. Now, in case $E=E(r)$, the metric (7) becomes

$$ds^2=\frac{A(r)}{f^2(E(r)/E_p)}dt^2-\frac{1}{g^2(E(r)/E_p)}dr^2-\frac{B(r)}{g^2(E(r)/E_p)}d\varphi^2-\frac{C(r)}{g^2(E(r)/E_p)}dz^2. \tag{29}$$

The nonzero components of the metric tensor have the following expressions:

$$g_{00}=\frac{A(r)}{f^2(E(r)/E_p)},\qquad g_{11}=-\frac{1}{g^2(E(r)/E_p)},$$
$$g_{22}=-\frac{B(r)}{g^2(E(r)/E_p)},\qquad g_{33}=-\frac{C(r)}{g^2(E(r)/E_p)}.$$

It is easy to find the inverse of the above components as

$$g^{00}=\frac{f^2(E(r)/E_p)}{A(r)},\qquad g^{11}=-g^2(E(r)/E_p),$$
$$g^{22}=-\frac{g^2(E(r)/E_p)}{B(r)},\qquad g^{33}=-\frac{g^2(E(r)/E_p)}{C(r)}.$$

With these metric components, it is straightforward to calculate the Christoffel symbols $\Gamma^\mu_{\nu\delta}$ with following definition:

$$\Gamma^\mu_{\nu\delta}=\frac{1}{2}g^{\mu\theta}\left(\frac{\partial g_{\theta\nu}}{\partial x^\delta}+\frac{\partial g_{\theta\delta}}{\partial x^\nu}-\frac{\partial g_{\nu\delta}}{\partial x^\theta}\right). \tag{30}$$

The calculation leads to the expressions of nonzero components as

$$\begin{aligned}
&\Gamma^0_{10}=\frac{A'}{2A}-\frac{f_E E'}{fE_p},\qquad \Gamma^1_{00}=\frac{g^2A'}{2f^2}-\frac{Ag^2f_EE'}{f^3E_p},\\
&\Gamma^1_{11}=-\frac{g_EE'}{gE_p},\qquad \Gamma^1_{22}=-\frac{1}{2}B'+\frac{Bg_EE'}{hE_p},\\
&\Gamma^1_{33}=-\frac{1}{2}C'+\frac{Cg_EE'}{gE_p},\qquad \Gamma^2_{21}=\frac{1}{2}\frac{B'}{B}-\frac{g_EE'}{gE_p},\\
&\Gamma^3_{31}=\frac{1}{2}\frac{C'}{C}-\frac{g_EE'}{gE_p}.
\end{aligned} \tag{31}$$

The Ricci tensor is defined by

$$R_{\mu\nu}=\frac{\partial\Gamma^\delta_{\mu\nu}}{\partial x^\delta}-\frac{\partial\Gamma^\delta_{\mu\delta}}{\partial x^\nu}+\Gamma^\delta_{\mu\nu}\Gamma^\theta_{\delta\theta}-\Gamma^\theta_{\mu\delta}\Gamma^\delta_{\nu\theta}. \tag{32}$$

With the help of above definition, the different components are calculated by

$$R_{00}=\frac{1}{2}\frac{g^2}{f^2}A''+\frac{g^2A'B'}{4Bf^2}+\frac{g^2A'C'}{4Cf^2}-\frac{g^2A'^2}{4Af^2}-\frac{gA'g_EE'}{2f^2E_p}-\frac{g^2A'f_EE'}{f^3E_p}-\frac{Ag^2B'f_EE'}{2Bf^3E_p}$$
$$-\frac{Ag^2C'f_EE'}{2Cf^3E_p}+\frac{2Ag^2f_E^2E'^2}{f^4E_p^2}-\frac{Ag^2f_{EE}E'^2}{f^3E_p^2}-\frac{Ag^2f_EE''}{f^3E_p}+\frac{Agf_Eg_EE'^2}{f^3E_p^2}, \tag{33}$$

$$R_{11}=-\frac{A''}{2A}+\frac{A'f_EE'}{AfE_p}-\frac{A'g_EE'}{2AgE_p}+\frac{A'^2}{4A^2}-\frac{B''}{2B}+\frac{B'g_EE'}{2BhE_p}+\frac{B'^2}{4B^2}-\frac{C''}{2C}+\frac{C'g_EE'}{2CgE_p}$$
$$+\frac{C'^2}{4C^2}+\frac{f_{EE}E'^2}{fE_p^2}+\frac{f_Eg_EE'^2}{fgE_p^2}+\frac{f_EE''}{fE_p}-\frac{2f_E^2E'^2}{f^2E_p^2}+\frac{2g_{EE}E'^2}{gE_p^2}+\frac{2g_EE''}{gE_p}-\frac{2g_E^2E'^2}{g^2E_p^2}, \tag{34}$$

$$R_{22}=-\frac{A'B'}{4A}+\frac{BA'g_EE'}{2AgE_p}-\frac{B''}{2}-\frac{B'C'}{4C}+\frac{B'f_EE'}{2fE_p}+\frac{B'g_EE'}{gE_p}+\frac{B'^2}{4B}+\frac{BC'g_EE'}{2CgE_p}$$
$$-\frac{Bf_Eg_EE'^2}{fgE_p^2}+\frac{Bg_{EE}E'^2}{gE_p^2}+\frac{Bg_EE''}{gE_p}-\frac{2Bg_E^2E'^2}{g^2E_p^2}, \tag{35}$$

$$R_{33}=-\frac{A'C'}{4A}+\frac{CA'g_E E'}{2AgE_p}-\frac{B'C'}{4B}+\frac{CB'g_E E'}{2BgE_p}-\frac{C''}{2}+\frac{C'f_E E'}{2fE_p}+\frac{C'g_E E'}{hE_p}+\frac{C'^2}{4C}-\frac{Cf_E g_E E'^2}{fgE_p^2}+\frac{Cg_{EE}E'^2}{gE_p^2}+\frac{Cg_E E''}{gE_p}-\frac{2Cg_E^2 E'^2}{g^2E_p^2}. \tag{36}$$

The mixed Ricci tensor is computed as

$$R_0^0=\frac{g^2A''}{2A}+\frac{g^2A'B'}{4AB}+\frac{g^2A'C'}{4AC}-\frac{g^2A'f_E E'}{AfE_p}-\frac{gA'g_E E'}{2AE_p}-\frac{g^2A'^2}{4A^2}-\frac{g^2B'f_E E'}{2BfE_p}-\frac{g^2C'f_E E'}{2CfE_p}-\frac{g^2f_{EE}E'^2}{fE_p^2}-\frac{g^2f_E E''}{fE_p}+\frac{gf_E g_E E'^2}{fE_p^2}+\frac{2g^2f_E^2E'^2}{f^2E_p^2}, \tag{37}$$

$$R_1^1=\frac{g^2A''}{2A}-\frac{g^2A'f_E E'}{AfE_p}+\frac{gA'g_E E'}{2AE_p}-\frac{g^2A'^2}{4A^2}+\frac{g^2B''}{2B}-\frac{gB'g_E E'}{2BE_p}-\frac{g^2B'^2}{4B^2}+\frac{g^2C''}{2C}-\frac{gC'g_E E'}{2CE_p}-\frac{g^2C'^2}{4C^2}-\frac{g^2f_{EE}E'^2}{fE_p^2}-\frac{g^2f_E E''}{fE_p}-\frac{gf_E g_E E'^2}{fE_p^2}+\frac{2g^2f_E^2E'^2}{f^2E_p^2}-\frac{2gg_{EE}E'^2}{E_p^2}-\frac{2gg_E E''}{E_p}+\frac{2g_E^2E'^2}{E_p^2}, \tag{38}$$

$$R_2^2=\frac{g^2A'B'}{4AB}-\frac{gA'g_E E'}{2AE_p}+\frac{g^2B''}{2B}+\frac{g^2B'C'}{4BC}-\frac{g^2B'f_E E'}{2BfE_p}-\frac{gB'g_E E'}{BE_p}-\frac{g^2B'^2}{4B^2}-\frac{gC'g_E E'}{2CE_p}+\frac{gf_E g_E E'^2}{fE_p^2}-\frac{gg_{EE}E'^2}{E_p^2}-\frac{gg_E E''}{E_p}+\frac{2g'^2E'^2}{E_p^2}, \tag{39}$$

$$R_3^3=\frac{g^2A'C'}{4AC}-\frac{gA'g_E E'}{2AE_p}+\frac{g^2B'C'}{4BC}-\frac{gB'g_E E'}{2BE_p}+\frac{g^2C''}{2C}-\frac{g^2C'f_E E'}{2CfE_p}-\frac{gC'g_E E'}{CE_p}-\frac{g^2C'^2}{4C^2}+\frac{gf_E g_E E'^2}{fE_p^2}-\frac{gg_{EE}E'^2}{E_p^2}-\frac{gg_E E''}{E_p}+\frac{2g_E^2E'^2}{E_p^2}. \tag{40}$$

The Ricci scalar is defined by

$$R=g^{ij}R_{ij} \tag{41}$$

We utilize above definition and get the following expression for Ricci scalar:

$$R=\frac{g^2A''}{A}+\frac{g^2A'B'}{2AB}+\frac{g^2A'C'}{2AC}-\frac{2g^2A'f_E E'}{AfE_p}-\frac{gA'g_E E'}{AE_p}-\frac{g^2A'^2}{2A^2}+\frac{g^2B''}{B}+\frac{g^2B'C'}{2BC}-\frac{g^2B'f_E E'}{BfE_p}-\frac{2gB'g_E E'}{BE_p}-\frac{g^2B'^2}{2B^2}+\frac{g^2C''}{C}-\frac{g^2C'f_E E'}{CfE_p}-\frac{2gC'g_E E'}{CE_p}-\frac{g^2C'^2}{2C^2}-\frac{2g^2f_{EE}E'^2}{fE_p^2}-\frac{2g^2f_E E''}{fE_p}+\frac{2gf_E g_E E'^2}{fE_p^2}+\frac{4g^2f_E^2E'^2}{f^2E_p^2}-\frac{4gg_{EE}E'^2}{E_p^2}-\frac{4gg_E E''}{E_p}+\frac{6g_E^2E'^2}{E_p^2}. \tag{42}$$

The covariant derivative for a vector B_μ is defined by

$$\nabla_\mu B_\nu=\partial_\mu B_\nu-\Gamma^\alpha_{\mu\nu}B_\alpha. \tag{43}$$

From the above definition, we can compute the components of $\nabla_\mu\nabla_\mu F$

$$\nabla_t\nabla_t F=\ddot{F}-\Gamma^r_{tt}F', \tag{44}$$

$$\nabla_r\nabla_r F=F''-\Gamma^r_{rr}F'=F'', \tag{45}$$

$$\nabla_\varphi\nabla_\varphi F=F_{\varphi\varphi}-\Gamma^r_{\varphi\varphi}F'=-\Gamma^r_{\varphi\varphi}F', \tag{46}$$

$$\nabla_z\nabla_z F=F_{zz}-\Gamma^r_{zz}F'=-\Gamma^r_{zz}F'. \tag{47}$$

Now, we are able to calculate the all four components of quantity A_μ defined in (28) as

$$A_t=\frac{g^2F'A'}{2A}-\frac{f^2\ddot{F}}{A}-\frac{g^2F'f_E E'}{fE_p}+\frac{Fg^2A''}{2A}+\frac{Fg^2A'B'}{4AB}+\frac{Fg^2A'C'}{4AC}-\frac{Fg^2A'f_E E'}{AfE_p}-\frac{FgA'g_E E'}{2AE_p}-\frac{Fg^2A'^2}{4A^2}-\frac{Fg^2B'f_E E'}{2BfE_p}-\frac{Fg^2C'f_E E'}{2CfE_p}-\frac{Fg^2f_{EE}E'}{fE_p^2}-\frac{Fg^2f_E E''}{fE_p}$$

$$+\frac{Fgf_Eg_EE'^2}{fE_p^2}+\frac{2Fg^2f_E^2E'^2}{f^2E_p^2}, \tag{48}$$

$$\begin{aligned}A_r={}&g^2F''+\frac{gF'g_EE'}{E_p}+\frac{Fg^2A''}{2A}-\frac{Fg^2A'f_EE'}{AfE_p}+\frac{FgA'g_EE'}{2AE_p}-\frac{Fg^2A'^2}{4A^2}+\frac{Fg^2B''}{2B}\\&-\frac{FgB'g_EE'}{2BE_p}-\frac{Fg^2B'^2}{4B^2}+\frac{Fg^2C''}{2C}-\frac{FgC'g_EE'}{2CE_p}-\frac{Fg^2C'^2}{4C^2}-\frac{Fg^2f''E'^2}{fE_p^2}-\frac{Fg^2f_EE''}{fE_p}\\&-\frac{Fgf_Eg_EE'^2}{fE_p^2}+\frac{2Fg^2f_E^2E'^2}{f^2E_p^2}-\frac{2Fgg_{EE}E'^2}{E_p^2}-\frac{2Fgg_EE''}{E_p}+\frac{2Fg_{EE}^2E'^2}{E_p^2},\end{aligned} \tag{49}$$

$$\begin{aligned}A_\varphi={}&\frac{g^2F'B'}{2B}-\frac{gF'g_EE'}{E_p}+\frac{Fg^2A'B'}{4AB}-\frac{FgA'g_EE'}{2AE_p}+\frac{Fg^2B''}{2B}+\frac{Fg^2B'C'}{4BC}-\frac{Fg^2B'f_EE'}{2BfE_p}\\&-\frac{FgB'g_EE'}{BE_p}-\frac{Fg^2B'^2}{4B^2}-\frac{FgC'g_EE'}{2CE_p}+\frac{Fgf_Eg_EE'^2}{fE_p^2}-\frac{Fgg_{EE}E'^2}{E_p^2}-\frac{Fgg_EE''}{E_p}+\frac{2Fg_E^2E'^2}{E_p^2},\end{aligned} \tag{50}$$

$$\begin{aligned}A_z={}&\frac{g^2F'C'}{2C}-\frac{gF'g_EE'}{E_p}+\frac{Fg^2A'C'}{4AC}-\frac{FgA'g_EE'}{2AE_p}+\frac{Fg^2B'C'}{4BC}-\frac{FgB'g_EE'}{2BE_p}+\frac{Fg^2C''}{2C}\\&-\frac{Fg^2C'f_EE'}{2CfE_p}-\frac{FgC'g_EE'}{CE_p}-\frac{Fg^2C'^2}{4C^2}+\frac{Fgf_Eg_EE'^2}{fE_p^2}-\frac{Fgg_{EE}E'^2}{E_p^2}-\frac{Fgg_EE''}{E_p}+\frac{2Fg_E^2E'^2}{E_p^2}.\end{aligned} \tag{51}$$

This enables us to write the following independent field equations:

$$\begin{aligned}&\frac{g^2F'A'}{2A}-\frac{f^2\ddot{F}}{A}-\frac{g^2F'f_EE'}{fE_p}-g^2F''-\frac{gF'g_EE'}{E_p}+\frac{Fg^2A'B'}{4AB}+\frac{Fg^2A'C'}{4AC}\\&-\frac{FgA'g_EE'}{AE_p}-\frac{Fg^2B''}{2B}-\frac{Fg^2B'f_EE'}{2BfE_p}+\frac{FgB'g_EE'}{2BE_p}+\frac{Fg^2B'^2}{4B^2}-\frac{Fg^2C''}{2C}-\frac{Fg^2C'f_EE'}{2CfE_p}\\&+\frac{FgC'g_EE'}{2CE_p}+\frac{Fg^2C'^2}{4C^2}+\frac{2Fgf_Eg_EE'^2}{fE_p^2}+\frac{2Fgg_{EE}E'^2}{E_p^2}+\frac{2Fgg_EE''}{E_p}-\frac{2Fg_E^2E'^2}{E_p^2}=0,\end{aligned} \tag{52}$$

$$\begin{aligned}&\frac{g^2F'A'}{2A}-\frac{f^2\ddot{F}}{A}-\frac{g^2F'B'}{2B}-\frac{g^2F'f_EE'}{fE_p}+\frac{gF'g_EE'}{E_p}+\frac{Fg^2A''}{2A}+\frac{Fg^2A'C'}{4AC}-\frac{Fg^2A'f_EE'}{AfE_p}\\&-\frac{Fg^2A'^2}{4A^2}-\frac{Fg^2B''}{2B}-\frac{Fg^2B'C'}{4BC}+\frac{FgB'g_EE'}{BE_p}+\frac{Fg^2B'^2}{4B^2}-\frac{Fg^2C'f_EE'}{2CfE_p}+\frac{FgC'g_EE'}{2CE_p}\\&-\frac{Fg^2f_{EE}E'^2}{fE_p^2}-\frac{Fg^2f_EE''}{fE_p}+\frac{2Fg^2f_E^2E'^2}{f^2E_p^2}+\frac{Fgg_{EE}E'^2}{E_p^2}+\frac{Fgg_EE''}{E_p}-\frac{2Fg_E^2E'^2}{E_p^2}=0,\end{aligned} \tag{53}$$

$$\begin{aligned}&\frac{g^2F'A'}{2A}-\frac{f^2\ddot{F}}{A}-\frac{g^2F'C'}{2C}-\frac{g^2F'f_EE'}{fE_p}+\frac{gF'g_EE'}{E_p}+\frac{Fg^2A''}{2A}+\frac{Fg^2A'B'}{4AB}\\&-\frac{Fg^2A'f_EE'}{AfE_p}-\frac{Fg^2A'^2}{4A^2}-\frac{Fg^2B'C'}{4BC}-\frac{Fg^2B'f_EE'}{2BfE_p}+\frac{FgB'g_EE'}{2BE_p}-\frac{Fg^2C''}{2C}+\frac{FgC'g_EE'}{CE_p}\\&+\frac{Fg^2C'^2}{4C^2}-\frac{Fg^2f_{EE}E'^2}{fE_p^2}-\frac{Fg^2f_EE''}{fE_p}+\frac{2Fg^2f_E^2E'^2}{f^2E_p^2}+\frac{Fgg_{EE}E'^2}{E_p^2}+\frac{Fgg_EE''}{E_p}-\frac{2Fg_E^2E'^2}{E_p^2}=0.\end{aligned} \tag{54}$$

corresponding to $A_t=A_r$, $A_t=A_\varphi$ and $A_t=A_z$ respectively.

6 Time-dependent cosmic strings

In this section, we discuss the time-dependent solutions (cosmic strings) in gravity's rainbow. Although many different mass configurations lead to static and time-independent metrics, there are some examples with time-dependent results and, therefore, it is worth studying. In this non-static case, the most important difference with the static one is the structure of the spacetime, as in the former case there are only two parameters in the metric $g_{\varphi\varphi}(E/E_p)$ reducing to cosmic strings. One more reason to study the cylindrical solutions with time-dependent fields could be existence of a challenge between spherically and cylindrically symmetries. In GR, according to the Birkhoff theorem, we know that there always exist a timelike Killing vector ∂_t in the spherically symmetric vacuum metrics. Consequently, we easily conclude that the spherically symmetric vacuum gravitating system is necessarily static, i.e., time independent. However, a dramatic change occurs when one considers the cylindrically symmetric systems because there is no such theorem in case of cylindrical symmetry analogous to Birkhoff's theorem. Propagation of gravitational waves during the gravitational collapse of a cylindrically symmetric system could be a reason to study time-dependent cylindrical objects (Delice 2006c). In the gravity's rainbow scenario, if we can find a static solution with time dependent mass (energy) factor $f^2(E(t)/E_p)$, $g^2(E(t)/E_p)$, then our solution could be a subset of the most general class of Einstein-Rosen (ER) gravitational wave solutions in gravity's rainbow in comparison with the similar GR solutions obtained in Akyar and Delice (2014), Delice (2006c).

Let us to start our analysis by writing the field equations for the following time-dependent metric,

$$ds^2 = \frac{A(t,r)}{f^2(E/E_p)}dt^2 - \frac{1}{g^2(E/E_p)}dr^2 - \frac{B(t,r)}{g^2(E/E_p)}d\varphi^2 - \frac{C(t,r)}{g^2(E/E_p)}dz^2. \tag{55}$$

By implementing metric (55) to (8), we obtain the following set of field equations:

$$\frac{g^2}{4}\left[\left(\frac{B'}{B}\right)^2 + \left(\frac{C'}{C}\right)^2 - \frac{B'C'}{BC} - 2\frac{B''}{B} - 2\frac{C''}{C}\right] + \frac{1}{4}\frac{\dot{B}\dot{C}}{BC} = \rho, \tag{56}$$

$$\frac{A'\dot{B}}{AB} + \frac{A'\dot{C}}{AC} - 2\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} - 2\frac{\dot{C}'}{C} + \frac{C'\dot{C}}{C^2} = 0, \tag{57}$$

$$\frac{1}{4}\left(\frac{A'B'}{AB} + \frac{1}{4}\frac{A'C'}{AC} + \frac{1}{4}\frac{B'C'}{BC}\right) + \frac{1}{4}\frac{f^2}{A}\left[\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC}\right. \tag{58}$$

$$\left. - \frac{\dot{B}\dot{C}}{BC} + \left(\frac{\dot{B}}{B}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2 - 2\frac{\ddot{B}}{B} - 2\frac{\ddot{C}}{C}\right] = p_r, \tag{59}$$

$$\frac{1}{4}\left[\frac{A'C'}{AC} + 2\frac{A''}{A} + 2\frac{C''}{C} - \left(\frac{A'}{A}\right)^2 - \left(\frac{C'}{C}\right)^2\right] + \frac{1}{4}\frac{f^2}{Ag^2}\left[\frac{\dot{A}\dot{C}}{AC} - 2\frac{\ddot{C}}{C} + \left(\frac{\dot{C}}{C}\right)^2\right] = \frac{p_\varphi}{g^2}, \tag{60}$$

$$\frac{1}{4}\left[\frac{A'B'}{AB} + 2\frac{A''}{A} + 2\frac{B''}{B} - \left(\frac{A'}{A}\right)^2 - \left(\frac{B'}{B}\right)^2\right] + \frac{1}{4}\frac{f^2}{Ag^2}\left[\frac{\dot{A}\dot{B}}{AB} - 2\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2\right] = \frac{p_z}{g^2}. \tag{61}$$

Here "prime" and "dot" indicate the derivative with respect to r and t respectively. A solution can possibly be obtained in vacuum when $T_\mu^\nu = 0$ with the assumption that all metric functions depend on time only, this means that all primed functions will vanish. With these assumptions, we can show that the second field equation is satisfied identically and other equations reduce to have following form:

$$\frac{\dot{B}\dot{C}}{BC} = 0, \tag{62}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{B}\dot{C}}{BC} + \left(\frac{\dot{B}}{B}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2 - 2\frac{\ddot{B}}{B} - 2\frac{\ddot{C}}{C} = 0, \tag{63}$$

$$\frac{\dot{A}\dot{C}}{AC} - 2\frac{\ddot{C}}{C} + \left(\frac{\dot{C}}{C}\right)^2 = 0, \tag{64}$$

$$\frac{\dot{A}\dot{B}}{AB} - 2\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 = 0. \tag{65}$$

Here, we found three class of exact solutions for time-dependent gravity's rainbow.

- The first class of exact solutions is given by

$$ds^2 = \frac{A(t)}{f^2(E/E_p)}dt^2 - \frac{1}{g^2(E/E_p)}dr^2 - \frac{B_0}{g^2(E/E_p)}d\varphi^2 - \frac{C_0}{g^2(E/E_p)}dz^2, \tag{66}$$

  where $A(t)$ is an arbitrary function of time and B_0, C_0 are arbitrary constants.
- The second family of exact solutions is as following:

$$ds^2 = \frac{c_1\dot{C}^2}{C(t)f^2(E/E_p)}dt^2 - \frac{1}{g^2(E/E_p)}dr^2 - \frac{B(t)}{g^2(E/E_p)}d\varphi^2 - \frac{C(t)}{g^2(E/E_p)}dz^2, \tag{67}$$

  where $C(t)$, $B(t)$ are arbitrary time functions and c_1 is a constant.
- The last member of exact solutions is given by the following metric:

$$ds^2 = \frac{c_1\dot{B}^2}{B(t)f^2(E/E_p)}dt^2 - \frac{1}{g^2(E/E_p)}dr^2 - \frac{B(t)}{g^2(E/E_p)}d\varphi^2 - \frac{C_0}{g^2(E/E_p)}dz^2. \tag{68}$$

Importantly, we note that here $E = E(t)$ and metrics are completely different from any other GR solutions.

7 When $E = E(t)$

The line element in this case of gravity's rainbow is given by

$$ds^2 = \frac{A(t,r)}{f^2(E(t)/E_p)}dt^2 - \frac{1}{g^2(E(t)/E_p)}dr^2 - \frac{B(t,r)}{g^2(E(t)/E_p)}d\varphi^2 - \frac{C(t,r)}{g^2(E(t)/E_p)}dz^2. \quad (69)$$

It is clearly evident that the expressions of the nonzero components of the metric tensor are

$$g_{00} = \frac{A(t,r)}{f^2(E(t)/E_p)}, \qquad g_{11} = -\frac{1}{g^2(E(r)/E_p)},$$
$$g_{22} = -\frac{B(t,r)}{g^2(E(t)/E_p)}, \qquad g_{33} = -\frac{C(t,r)}{g^2(E(t)/E_p)}.$$

The inverse metric components are,

$$g^{00} = \frac{f^2(E(t)/E_p)}{A(t,r)}, \qquad g^{11} = -g^2(E(t)/E_p),$$
$$g^{22} = -\frac{g^2(E(t)/E_p)}{B(t,r)}, \qquad g^{33} = -\frac{g^2(E(t)/E_p)}{C(t,r)}.$$

With these values of metric components, it is easy to calculate the Christoffel symbols of the second kind

$$\Gamma^i_{jl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mj}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^m}\right). \quad (70)$$

The calculation leads to the following expressions:

$$\Gamma^0_{00} = \frac{\dot{A}}{2A} - \frac{f_E\dot{E}}{fE_p}, \qquad \Gamma^1_{10} = \frac{A'}{2A}, \qquad \Gamma^0_{11} = -\frac{f^2 g_E\dot{E}}{Ag^3E_p},$$
$$\Gamma^1_{22} = \frac{f^2\dot{B}}{2Ag^2} - \frac{Bf^2g_E\dot{E}}{Ag^3E_p}, \qquad \Gamma^1_{33} = \frac{f^2\dot{C}}{2Ag^2} - \frac{Cf^2g_E\dot{E}}{Ag^3E_p},$$
$$\Gamma^1_{00} = \frac{g^2A'}{2f^2}, \qquad \Gamma^1_{10} = -\frac{g_E\dot{E}}{gE_p}, \qquad \Gamma^1_{22} = -\frac{1}{2}B', \quad (71)$$
$$\Gamma^1_{33} = -\frac{1}{2}C', \qquad \Gamma^2_{20} = \frac{\dot{B}}{2B} - \frac{g_E\dot{E}}{gE_p}, \qquad \Gamma^2_{21} = \frac{B'}{2B},$$
$$\Gamma^3_{30} = \frac{\dot{C}}{2C} - \frac{g_E\dot{E}}{gE_p}, \qquad \Gamma^3_{31} = \frac{C'}{2C}.$$

Exploiting definition of Ricci tensor (32), the components are calculated by

$$R_{00} = \frac{g^2A'B'}{4Bf^2} + \frac{\dot{A}\dot{B}}{4AB} + \frac{g^2A'C'}{4Cf^2} + \frac{\dot{A}\dot{C}}{4AC} + \frac{g^2A''}{2f^2} - \frac{g^2A'^2}{4Af^2} - \frac{3\dot{A}g_E\dot{E}}{2AgE_p} - \frac{\dot{B}f_E\dot{E}}{2BfE_p} + \frac{\dot{B}g_E\dot{E}}{BgE_p} - \frac{\ddot{B}}{2B} + \frac{\dot{B}^2}{4B^2} - \frac{\dot{C}f_E\dot{E}}{2CfE_p} + \frac{\dot{C}g_E\dot{E}}{CgE_p} - \frac{\ddot{C}}{2C} + \frac{\dot{C}^2}{4C^2} + \frac{3f_Eg_E\dot{E}^2}{fgE_p^2} + \frac{3g_{EE}\dot{E}^2}{gE_p^2} + \frac{3g_E\ddot{E}}{gE_p} - \frac{6g_E^2\dot{E}^2}{g^2E_p^2}, \quad (72)$$

$$R_{10} = \frac{A'B'}{4AB} + \frac{A'\dot{C}}{4AC} - \frac{A'g_E\dot{E}}{AgE_p} - \frac{\dot{B}'}{2B} + \frac{B'\dot{B}}{4B^2} - \frac{\dot{C}'}{2C} + \frac{C'\dot{C}}{4C^2}, \quad (73)$$

$$R_{11} = -\frac{A''}{2A} + \frac{f^2\dot{A}g_E\dot{E}}{2A^2g^3E_p} + \frac{A'^2}{4A^2} - \frac{f^2\dot{B}g_E\dot{E}}{2ABg^3E_p} - \frac{f^2\dot{C}g_E\dot{E}}{2ACg^3E_p} - \frac{B''}{2B} + \frac{B'^2}{4B^2} - \frac{C''}{2C} + \frac{C'^2}{4C^2} + \frac{4f^2g_E^2\dot{E}^2}{Ag^4E_p^2} - \frac{f^2g_{EE}\dot{E}^2}{Ag^3E_p^2} - \frac{f^2g_E\dot{E}}{Ag^3E_p} - \frac{ff_Eg_E\dot{E}^2}{Ag^3E_p^2}, \quad (74)$$

$$R_{22} = -\frac{A'B'}{4A} - \frac{f^2\dot{A}\dot{B}}{4A^2g^2} + \frac{Bf^2\dot{A}g_E\dot{E}}{2A^2g^3E_p} + \frac{f^2\dot{B}\dot{C}}{4ACg^2} + \frac{f^2\ddot{B}}{2Ag^2} - \frac{f^2\dot{B}^2}{4ABg^2} - \frac{2f^2\dot{B}g_E\dot{E}}{Ag^3E_p} + \frac{f\dot{B}f_E\dot{E}}{2Ag^2E_p} - \frac{Bf^2\dot{C}g_E\dot{E}}{2ACg^3E_p} - \frac{B'C'}{4C} + \frac{B'^2}{4B} - \frac{1}{2}B'' + \frac{4Bf^2g_E^2\dot{E}}{Ag^4E_p^2} - \frac{Bf^2g_{EE}\dot{E}^2}{Ag^3E_p^2} - \frac{Bf^2g_E\ddot{E}}{Ag^3E_p} - \frac{Bff_Eg_E\dot{E}^2}{Ag^3E_p^2}, \quad (75)$$

$$R_{33} = -\frac{A'C'}{4A} - \frac{f^2\dot{A}\dot{C}}{4A^2g^2} + \frac{Cf^2\dot{A}g_E\dot{E}}{2A^2g^3E_p} + \frac{f^2\dot{B}\dot{C}}{4ABg^2} - \frac{Cf^2\dot{B}g_E\dot{E}}{2ABg^3E_p} + \frac{f^2\ddot{C}}{2Ag^2} - \frac{f^2\dot{C}^2}{4ACg^2} - \frac{2f^2\dot{C}g_E\dot{E}}{Ag^3E_p} + \frac{f\dot{C}f_E\dot{E}}{2Ag^2E_p} - \frac{B'C'}{4B} + \frac{C'^2}{4C} - \frac{1}{2}C'' + \frac{4Cf^2g_E^2\dot{E}^2}{Ag^4E_p^2} - \frac{Cf^2g_{EE}\dot{E}^2}{Ag^3E_p^2} - \frac{Cf^2g_E\ddot{E}}{Ag^3E_p} - \frac{Cff_Eg_E\dot{E}^2}{Ag^3E_p^2}. \quad (76)$$

Now, we compute the expression for Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ as following:

$$R = \frac{g^2A'B'}{2AB} + \frac{g^2A'C'}{2AC} + \frac{g^2A''}{A} + \frac{f^2\dot{A}\dot{B}}{2A^2B} + \frac{f^2\dot{A}\dot{C}}{2A^2C} - \frac{3f^2\dot{A}g_E\dot{E}}{A^2gE_p} - \frac{g^2A'^2}{2A^2} - \frac{f^2\dot{B}\dot{C}}{2ABC}$$
$$+ \frac{4f^2\dot{B}g_E\dot{E}}{ABgE_p} - \frac{f^2\ddot{B}}{AB} - \frac{f\dot{B}f_E\dot{E}}{ABE_p} + \frac{f^2\dot{B}^2}{2AB^2} + \frac{4f^2\dot{C}g_E\dot{E}}{ACgE_p} - \frac{f^2\ddot{C}}{AC} - \frac{f\dot{C}f_E\dot{E}}{ACE_p} + \frac{f^2\dot{C}^2}{2AC^2}$$
$$+ \frac{g^2B'C'}{2BC} + \frac{g^2B''}{B} - \frac{g^2B'^2}{2B^2} + \frac{g^2C''}{C} - \frac{g^2C'^2}{2C^2} + \frac{6f^2g_{EE}\dot{E}^2}{AgE_p^2} + \frac{6f^2g_E\ddot{E}}{AgE_p}$$
$$- \frac{18f^2g_E^2\dot{E}^2}{Ag^2E_p^2} + \frac{6ff_Eg_E\dot{E}^2}{AgE_p^2}. \tag{77}$$

Now, following Sect. 5, the independent field equation corresponding to $A_t = A_r$ is

$$\frac{4Fg_E^2\dot{E}^2f^2}{Ag^6E_p^2} + \frac{F\dot{B}^2f^2}{4AB^2} + \frac{F\dot{C}^2f^2}{4AC^2} + \frac{3F\dot{E}^2g_{EE}f^2}{AgE_p^2} + \frac{3Fg_E\ddot{E}f^2}{AgE_p} + \frac{Fg_E\dot{E}\dot{A}f^2}{2A^2g^5E_p}$$
$$+ \frac{Fg_E\dot{E}\dot{B}f^2}{ABgE_p} + \frac{F\dot{A}\dot{B}f^2}{4A^2B} + \frac{Fg_E\dot{E}\dot{C}f^2}{ACgE_p} + \frac{F\dot{A}\dot{C}f^2}{4A^2C} + \frac{\dot{A}\dot{F}f^2}{2A^2} - \frac{\ddot{F}f^2}{A} - \frac{F\ddot{B}f^2}{2AB}$$
$$- \frac{F\ddot{C}f^2}{2AC} - \frac{3Fg_E\dot{E}\dot{A}f^2}{2A^2gE_p} - \frac{Fg_E\ddot{E}f^2}{Ag^5E_p} - \frac{Fg_E\dot{E}\dot{B}f^2}{2ABg^5E_p} - \frac{Fg_E\dot{E}\dot{C}f^2}{2ACg^5E_p}$$
$$- \frac{6Fg_E^2\dot{E}^2f^2}{Ag^2E_p^2} - \frac{F\dot{E}^2g_{EE}f^2}{Ag^5E_p^2} - \frac{g_E\dot{E}\dot{F}f^2}{Ag^5E_p^2} + \frac{3Ff_Eg_E\dot{E}^2f}{AgE_p^2} - \frac{Ff_E\dot{E}\dot{B}f}{2ABE_p}$$
$$- \frac{Ff_E\dot{E}\dot{C}f}{2ACE_p} - \frac{Ff_Eg_E\dot{E}^2f}{Ag^5E_p^2} + \frac{FA'^2}{4A^2g^2} + \frac{FB'^2}{4B^2g^2} + \frac{FC'^2}{4C^2g^2} + \frac{Fg^2A'B'}{4AB} + \frac{Fg^2A'C'}{4AC}$$
$$+ \frac{g^2A'F'}{2A} + \frac{Fg^2A''}{2A} - \frac{Fg^2A'^2}{4A^2} - \frac{F''}{g^2} - \frac{FA''}{2Ag^2} - \frac{FB''}{2Bg^2} - \frac{FC''}{2Cg^2} = 0, \tag{78}$$

The independent equations corresponding to $A_t = A_\varphi$ and $A_t = A_z$ are given respectively by,

$$\frac{4Fg_E^2\dot{E}^2f^2}{Ag^6E_p^2} + \frac{F\dot{B}^2f^2}{4AB^2} + \frac{F\dot{C}^2f^2}{4AC^2} + \frac{3F\dot{E}^2g_{EE}f^2}{AgE_p^2} + \frac{3Fg_E\ddot{E}f^2}{AgE_p} + \frac{Fg_E\dot{E}\dot{A}f^2}{2A^2g^5E_p}$$
$$+ \frac{Fg_E\dot{E}\dot{B}f^2}{ABgE_p} + \frac{F'\dot{B}f^2}{2ABg^4} + \frac{F\dot{A}\dot{B}f^2}{4A^2B} + \frac{Fg_E\dot{E}\dot{C}f^2}{ACgE_p} + \frac{F\dot{A}\dot{C}f^2}{4A^2C} + \frac{F\dot{B}\dot{C}f^2}{4ABCg^4} + \frac{\dot{A}\dot{F}f^2}{2A^2}$$
$$+ \frac{F\ddot{B}f^2}{2ABg^4} - \frac{\ddot{F}f^2}{A} - \frac{F\ddot{B}f^2}{2AB} - \frac{F\ddot{C}f^2}{2AC} - \frac{F\dot{A}\dot{B}f^2}{4A^2Bg^4} - \frac{F\dot{B}^2f^2}{4AB^2g^4} - \frac{3Fg_E\dot{E}\dot{A}f^2}{2A^2gE_p} - \frac{Fg_E\ddot{E}f^2}{Ag^5E_p}$$
$$- \frac{2Fg_E\dot{E}\dot{B}f^2}{ABg^5E_p} - \frac{Fg_E\dot{E}\dot{C}f^2}{2ACg^5E_p} - \frac{6Fg_E^2\dot{E}^2f^2}{Ag^2E_p^2} - \frac{F\dot{E}^2g_{EE}f^2}{Ag^5E_p^2} + \frac{3Ff_Eg_E\dot{E}^2f}{AgE_p^2} + \frac{Ff_E\dot{E}\dot{B}f}{2ABg^4E_p}$$
$$- \frac{Ff_E\dot{E}\dot{B}f}{2ABE_p} - \frac{Ff_E\dot{E}\dot{C}f}{2ACE_p} - \frac{Ff_Eg_E\dot{E}^2f}{Ag^5E_p^2} + \frac{FB'^2}{4B^2g^2} + \frac{Fg^2A'B'}{4AB} + \frac{Fg^2A'C'}{4AC}$$
$$+ \frac{g^2A'F'}{2A} + \frac{Fg^2A''}{2A} - \frac{Fg^2A'^2}{4A^2} - \frac{FB''}{2Bg^2} - \frac{FA'B'}{4ABg^2} - \frac{FB'C'}{4BCg^2} = 0, \tag{79}$$

$$\frac{4Fg_E^2\dot{E}^2f^2}{Ag^6E_p^2} + \frac{F\dot{B}^2f^2}{4AB^2} + \frac{F\dot{C}^2f^2}{4AC^2} + \frac{3F\dot{E}^2g_{EE}f^2}{AgE_p^2} + \frac{3Fg_E\ddot{E}f^2}{AgE_p} + \frac{Fg_E\dot{E}\dot{A}f^2}{2A^2g^5E_p}$$
$$+ \frac{Fg_E\dot{E}\dot{B}f^2}{ABgE_p} + \frac{F\dot{A}\dot{B}f^2}{4A^2B} + \frac{Fg_E\dot{E}\dot{C}f^2}{ACgE_p} + \frac{F\dot{A}\dot{C}f^2}{4A^2C} + \frac{F\dot{B}\dot{C}f^2}{4ABCg^4} + \frac{\dot{A}\dot{F}f^2}{2A^2} + \frac{F\ddot{C}f^2}{2ACg^4}$$
$$- \frac{\ddot{F}f^2}{A} - \frac{F\ddot{B}f^2}{2AB} - \frac{F\ddot{C}f^2}{2AC} - \frac{F\dot{A}\dot{C}f^2}{4A^2Cg^4} - \frac{F\dot{C}^2f^2}{4AC^2g^4} - \frac{3Fg_E\dot{E}\dot{A}f^2}{2A^2gE_p} - \frac{Fg_E\ddot{E}f^2}{Ag^5E_p}$$
$$- \frac{Fg_E\dot{E}\dot{B}f^2}{2ABg^5E_p} - \frac{2Fg_E\dot{E}\dot{C}f^2}{ACg^5E_p} - \frac{6Fg_E^2\dot{E}^2f^2}{Ag^2E_p^2} - \frac{F\dot{E}^2g_{EE}f^2}{Ag^5E_p^2} + \frac{3Ff_Eg_E\dot{E}^2f}{AgE_p^2}$$

$$+\frac{Ff_E\dot{E}\dot{C}f}{2ACg^4E_p}-\frac{Ff_E\dot{E}\dot{B}f}{2ABE_p}-\frac{Ff_E\dot{E}\dot{C}f}{2ACE_p}-\frac{Ff_Eg_E\dot{E}^2f}{Ag^5E_p^2}+\frac{F\dot{C}^2}{4C^2g^2}+\frac{Fg^2A'B'}{4AB}$$
$$+\frac{Fg^2A'C'}{4AC}+\frac{g^2A'F'}{2A}+\frac{Fg^2A''}{2A}-\frac{Fg^2A'^2}{4A^2}-\frac{C'F'}{2Cg^2}-\frac{FC''}{2Cg^2}-\frac{FA'C'}{4ACg^2}-\frac{FB'C'}{4BCg^2}=0. \quad (80)$$

Here, any set of functions satisfying the above equations would be a solution of the modified Einstein field equations for a given $F(r)$ in gravity's rainbow. Here, we see that to find a general solution to the above equations are not an easy task.

8 Concluding remarks

The exact solutions play a central role in gravity theory. However, a deformed formalism of special relativity, which modifies the standard dispersion relations in the order of Planck length (commonly known as DSR), is generalized to curved-spacetime. This generalization is known as gravity's rainbow and has found lots of attention recent days. Keeping these points in mind, in this work, we have investigated the static cylindrical solutions for Einstein's field equations in gravity's rainbow. The cosmic string metric is supposed to be static (i.e. with vanishing off diagonal components and time independent) and cylindrically symmetric. In this setting, we have discussed cosmic strings in energy-dependent background. The fields equations following this metric lead to various energy-dependent differential equations. In order to solve these differential equations, we have considered the possibility of Kasner's, quasi-Kasner and non-Kasner solutions. It is well-known that the Kasner solutions are two parametric metric and unique exact solutions for the Einstein equations with cylindrical symmetry. It is shown that the quasi-Kasner solutions cannot be realized in gravity's rainbow. Also, we have found that the gravity's rainbow cosmic strings follow same behavior (singularities) to that of standard GR theory. We also analysed the time-dependent solutions (cosmic strings) in gravity's rainbow. Here, to discuss the time-dependent solutions, we have assumed the vanishing energy-momentum tensor together with only time-dependent metric functions. Here, we have observed that the metric structures are completely different to that of the other GR solutions.

Appendix: Mathematical details

In this appendix, we present explicit forms of different geometrical quantities which are used to derive field equations.

A.1 Case $A = A(r)$, $B = B(r)$ and $C = C(r)$

The nonzero components of the metric tensor are as following:

$$g_{00}=\frac{A(r)}{f^2(E/E_p)},\qquad g_{11}=-\frac{1}{g^2(E/E_p)},$$
$$g_{22}=-\frac{B(r)}{g^2(E/E_p)},\qquad g_{33}=-\frac{C(r)}{g^2(E/E_p)}.$$

The inverse of these metric components are given by

$$g^{00}=\frac{f^2(E/E_p)}{A(r)},\qquad g^{11}=-g^2(E/E_p),$$
$$g^{22}=-\frac{g^2(E/E_p)}{B(r)},\qquad g^{33}=-\frac{g^2(E/E_p)}{C(r)}.$$

Utilizing definition (30), the nonzero values of the Christoffel symbols are as follows,

$$\Gamma^0_{10}=\frac{1}{2}\frac{A'}{A},\qquad \Gamma^1_{00}=\frac{1}{2}\frac{A'}{f^2}g^2,\qquad \Gamma^1_{22}=-\frac{1}{2}B',$$
$$\Gamma^1_{33}=-\frac{1}{2}C',\qquad \Gamma^2_{21}=\frac{1}{2}\frac{B'}{B},\qquad \Gamma^3_{31}=\frac{1}{2}\frac{C'}{C}. \quad (81)$$

Exploiting the definition (32), the covariant components of Ricci tensor are calculated as

$$R_{00}=-\frac{1}{4}\frac{g^2}{f^2}\frac{A'^2}{A}+\frac{1}{4}\frac{g^2}{f^2}\frac{A'B'}{B}+\frac{1}{4}\frac{g^2}{f^2}\frac{A'C'}{C}+\frac{1}{2}\frac{g^2}{f^2}A'', \quad (82)$$

$$R_{11}=\frac{1}{4}\frac{A'^2}{A^2}+\frac{1}{4}\frac{B'^2}{B^2}+\frac{1}{4}\frac{C'^2}{C^2}-\frac{1}{2}\frac{A''}{A}-\frac{1}{2}\frac{B''}{B}-\frac{1}{2}\frac{C''}{C}, \quad (83)$$

$$R_{22}=-\frac{1}{4}\frac{A'B'}{A}+\frac{1}{4}\frac{B'^2}{B}-\frac{1}{4}\frac{B'C'}{C}-\frac{1}{2}B'', \quad (84)$$

$$R_{33}=-\frac{1}{4}\frac{A'C'}{A}-\frac{1}{4}\frac{B'C'}{C}+\frac{1}{4}\frac{C'^2}{C}-\frac{1}{2}C'', \quad (85)$$

and similarly the mixed components are computed as

$$R^0_0=-\frac{g^2}{4}\frac{A'^2}{A^2}+\frac{g^2}{4}\left(\frac{A'B'}{AB}\right)^2+\frac{g^2}{4}\left(\frac{A'C'}{AC}\right)^2+\frac{g^2}{2}\frac{A''}{A}, \quad (86)$$

$$R^1_1=-\frac{g^2}{4}\frac{A'^2}{A^2}-\frac{g^2}{4}\frac{B'^2}{B^2}-\frac{g^2}{4}\frac{C'^2}{C^2}+\frac{g^2}{2}\frac{A''}{A}+\frac{g^2}{2}\frac{B''}{B}+\frac{g^2}{2}\frac{C''}{C}, \quad (87)$$

$$R^2_2=\frac{g^2}{4}\frac{A'B'}{AB}-\frac{g^2}{4}\frac{B'^2}{B^2}+\frac{g^2}{4}\frac{B'C'}{BC}+\frac{g^2}{2}\frac{B''}{B}, \quad (88)$$

$$R^3_3=\frac{g^2}{4}\frac{A'C'}{AC}-\frac{g^2}{4}\frac{C'^2}{C^2}+\frac{g^2}{4}\frac{B'C'}{BC}+\frac{g^2}{2}\frac{C''}{C}, \quad (89)$$

Finally, using definition $R = g^{\mu\nu} R_{\mu\nu}$, the expression for Ricci scalar is given by,

$$R = -\frac{g^2}{2}\frac{A'^2}{A^2} + \frac{g^2}{2}\frac{A'B'}{AB} - \frac{g^2}{2}\frac{B'^2}{B^2} + \frac{g^2}{2}\frac{A'C'}{AC} + \frac{g^2}{2}\frac{B'C'}{BC} - \frac{g^2}{2}\frac{C'^2}{C^2} + g^2\frac{A''}{A} + g^2\frac{B''}{B} + g^2\frac{C''}{C}. \tag{90}$$

A.2 Case $A = A(t,r)$, $B = B(t,r)$ and $C = C(t,r)$

The nonzero components of time-dependent metric tensor are

$$g_{00} = \frac{A(t,r)}{f^2(E/E_p)}, \qquad g_{11} = -\frac{1}{g^2(E/E_p)},$$
$$g_{22} = -\frac{B(t,r)}{g^2(E/E_p)}, \qquad g_{33} = -\frac{C(t,r)}{g^2(E/E_p)}.$$

The inverse of these metric components are

$$G^{00} = \frac{f^2(E/E_p)}{A(t,r)}, \qquad g^{11} = -g^2(E/E_p),$$
$$g^{22} = -\frac{g^2(E/E_p)}{B(t,r)}, \qquad g^{33} = -\frac{g^2(E/E_p)}{C(t,r)}.$$

The Christoffel symbols of the second kind are,

$$\Gamma^0_{00} = \frac{1}{2}\frac{\dot A}{A}, \qquad \Gamma^0_{10} = \frac{1}{2}\frac{A'}{A}, \qquad \Gamma^0_{22} = \frac{1}{2}\frac{f^2}{g^2}\frac{\dot B}{A},$$
$$\Gamma^0_{33} = \frac{1}{2}\frac{f^2}{g^2}\frac{\dot C}{A}, \qquad \Gamma^1_{00} = \frac{1}{2}\frac{g^2}{f^2}A', \qquad \Gamma^1_{22} = -\frac{1}{2}B',$$
$$\Gamma^1_{33} = -\frac{1}{2}C', \qquad \Gamma^2_{20} = \frac{1}{2}\frac{\dot B}{B}, \qquad \Gamma^2_{21} = \frac{1}{2}\frac{B'}{B},$$
$$\Gamma^3_{30} = \frac{1}{2}\frac{\dot C}{C}, \qquad \Gamma^3_{31} = \frac{1}{2}\frac{C'}{C}. \tag{91}$$

These induce the following forms of Ricci tensor:

$$R_{00} = \frac{1}{4}\frac{g^2}{f^2}\frac{A'B'}{B} + \frac{1}{4}\frac{\dot A\dot B}{AB} + \frac{1}{4}\frac{g^2}{f^2}\frac{A'C'}{C} + \frac{1}{4}\frac{\dot A\dot C}{AC} + \frac{1}{2}\frac{g^2}{f^2}A'' - \frac{1}{4}\frac{g^2}{f^2}\frac{A'^2}{A} - \frac{1}{2}\frac{\ddot B}{B} + \frac{1}{4}\frac{\dot B^2}{B^2} - \frac{1}{2}\frac{\ddot C}{C} + \frac{1}{4}\frac{\dot C^2}{C^2}, \tag{92}$$

$$R_{10} = \frac{1}{4}\frac{A'B'}{AB} + \frac{1}{4}\frac{A'\dot C}{AC} - \frac{1}{2}\frac{\dot B'}{B} + \frac{1}{4}\frac{B'\dot B}{B^2} - \frac{1}{2}\frac{\dot C'}{C} + \frac{1}{4}\frac{C'\dot C}{C^2}, \tag{93}$$

$$R_{11} = -\frac{1}{2}\frac{A''}{A} + \frac{1}{4}\frac{A'^2}{A^2} - \frac{1}{2}\frac{B''}{B} + \frac{1}{4}\frac{B'^2}{B^2} - \frac{1}{2}\frac{C''}{C} + \frac{1}{4}\frac{C'^2}{C^2}, \tag{94}$$

$$R_{22} = -\frac{1}{4}\frac{A'B'}{A} - \frac{1}{4}\frac{f^2}{g^2}\frac{\dot A\dot B}{A^2} + \frac{1}{4}\frac{f^2}{g^2}\frac{\dot B\dot C}{AC} + \frac{1}{2}\frac{f^2}{g^2}\frac{\ddot B}{A} - \frac{1}{4}\frac{f^2}{g^2}\frac{\dot B^2}{AB} - \frac{1}{4}\frac{B'C'}{C} + \frac{1}{4}\frac{B'^2}{B} - \frac{1}{2}B'', \tag{95}$$

$$R_{33} = -\frac{A'C'}{4A} - \frac{f^2\dot A\dot C}{4A^2g^2} + \frac{f^2\dot B\dot C}{4ABg^2} + \frac{f^2\ddot C}{2Ag^2} - \frac{f^2\dot C^2}{4ACg^2} - \frac{B'C'}{4B} + \frac{C'^2}{4C} - \frac{1}{2}C''. \tag{96}$$

The Ricci scalar is this case is given by

$$R = \frac{g^2}{2}\frac{A'B'}{AB} + \frac{g^2}{2}\frac{A'C'}{AC} + g^2\frac{A''}{A} + \frac{f^2}{2}\frac{\dot A\dot B}{A^2B} + \frac{f^2}{2}\frac{\dot A\dot C}{A^2C} - \frac{g^2}{2}\frac{A'^2}{A^2} - \frac{f^2}{2}\frac{\dot B\dot C}{ABC} - f^2\frac{\ddot B}{AB} + \frac{f^2}{2}\frac{\dot B^2}{AB^2} - f^2\frac{\ddot C}{AC} + \frac{f^2}{2}\frac{\dot C^2}{AC^2} + \frac{g^2}{2}\frac{B'C'}{BC} + g^2\frac{B''}{B} - \frac{g^2}{2}\frac{B'^2}{B^2} + g^2\frac{C''}{C} - \frac{g^2}{2}\frac{C'^2}{C^2}. \tag{97}$$

References

Abbott, B.P., et al.: Phys. Rev. Lett. **118**, 221101 (2017)
Adler, R.J., Chen, P., Santiago, D.I.: Gen. Relativ. Gravit. **33**, 2101 (2001)
Akyar, L., Delice, O.: Eur. Phys. J. Plus **129**, 226 (2014)
Alfaro, J., Morales-Tecotl, H.A., Urrutia, L.F.: Phys. Rev. D **65**, 103509 (2002)
Ali, A.F.: Phys. Rev. D **89**, 104040 (2014)
Ali, A.F., Faizal, M., Khalil, M.M.: J. High Energy Phys. **12**, 159 (2014)
Amati, D., Ciafaloni, M., Veneziano, G.: Phys. Lett. B **216**, 41 (1989)
Ambjorn, J., Loll, R.: Nucl. Phys. B **536**, 407 (1998)
Ambjorn, J., Jurkiewicz, J., Loll, R.: Phys. Rev. Lett. **85**, 924 (2000)
Ambjorn, J., Jurkiewicz, J., Loll, R.: Nucl. Phys. B **610**, 347 (2001)
Amelino-Camelia, G., Ellis, J.R., Mavromatos, N.E., Nanopoulos, D.V., Sarkar, S.: Nature (London) **393**, 763 (1998)
Awad, A., Ali, A.F., Majumder, B.: J. Cosmol. Astropart. Phys. **10**, 052 (2013)
Azadi, A., Momeni, D., Nouri-Zonoz, M.: Phys. Lett. B **670**, 210 (2008)
Baker, W.M.: Class. Quantum Gravity **7**, 717 (1990)
Barros, A., Romero, C.: J. Math. Phys. **36**, 5800 (1995)
Baykal, A., Delice, O.: (2005). gr-qc/0512143
Baykal, A., Ciftci, D.K., Delice, O.: J. Math. Phys. **51**, 072505 (2010)
Bezerra, V.B., Ferreira, C.N., Fonseca-Neto, J.B., Sobreira, A.A.R.: Phys. Rev. D **68**, 124020 (2003)
Carlip, S.: Rep. Prog. Phys. **64**, 885 (2001)
Carroll, S.M., Harvey, J.A., Kostelecky, V.A., Lane, C.D., Okamoto, T.: Phys. Rev. Lett. **87**, 141601 (2001)
Cavaglia, M., Das, S., Maartens, R.: Class. Quantum Gravity **20**, L205 (2003). Cavaglia, M., Das, S.: Class. Quant. Grav. **21**, 4511 (2004)
Chatrabhuti, A., Yingcharoenrat, V., Channuie, P.: Phys. Rev. D **93**, 043515 (2016)
Cheng, H.-B., Liu, Y.-Q.: Chin. Phys. Lett. **25**, 1160 (2008)
Çiftci, D.K., Delice, Ö.: J. Math. Phys. **56**, 072502 (2015)
Corda, C.: Int. J. Mod. Phys. D **18**, 2275 (2009)
Delice, O.: Phys. Rev. D **74**, 124001 (2006a)
Delice, O.: Phys. Rev. D **74**, 067703 (2006b)
Delice, O.: Acta Phys. Pol. B **37**, 2445 (2006c)
Dvali, G., Kogan, I.I., Shifman, M.: Phys. Rev. D **62**, 106001 (2000)
Emília, M., Guimarães, X.: Class. Quantum Gravity **14**, 435 (1997)
Feng, Z.W., Yang, S.Z., Li, H.L., Zu, X.T.: (2016). arXiv:1608.06824
Ferraro, R., Fiorini, F.: J. Phys. Conf. Ser. **314**, 012114 (2011)

Ferreira, C.N., Guimarães, M.E.X., Helayël-Neto, J.A.: Nucl. Phys. B **581**, 165 (2000)
Furtado, C., Moraes, F., Bezerra, V.B.: Phys. Rev. D **59**, 107504 (1999)
Gambini, R., Pullin, J.: Phys. Rev. D **59**, 124021 (1999)
Garattini, R.: J. Cosmol. Astropart. Phys. **06**, 017 (2013)
Garattini, R., Mandanici, G.: Phys. Rev. D **85**, 023507 (2012)
Garattini, R., Sakellariadou, M.: Phys. Rev. D **90**, 043521 (2014)
Gibbons, G.W., Herdeiro, C.A.R.: Phys. Rev. D **63**, 064006 (2001)
Gim, Y., Kim, W.: Eur. Phys. J. C **76**, 166 (2016)
Girelli, F., Livine, E.R., Oriti, D.: Nucl. Phys. B **708**, 411 (2005)
Gregory, R., Santos, C.: Phys. Rev. D **56**, 1194 (1997)
Gundlach, C., Ortiz, M.E.: Phys. Rev. D **42**, 2521 (1990)
Hackett, J.: Class. Quantum Gravity **23**, 3833 (2006)
Harko, T., Lake, M.J.: Eur. Phys. J. C **75**, 60 (2015a)
Harko, T., Lake, M.J.: Phys. Rev. D **91**, 045012 (2015b)
Hayward, S.A.: Class. Quantum Gravity **17**, 1749 (2000)
Hendi, S.H., Bordbar, G.H., Eslam Panah, B., Panahiyan, S.: J. Cosmol. Astropart. Phys. **09**, 013 (2016a)
Hendi, S.H., Eslam Panah, B., Panahiyan, S., Momennia, M.: Adv. High Energy Phys. **2016**, 9813582 (2016b)
Hendi, S.H., Momennia, M., Eslam Panah, B., Faizal, M.: Astrophys. J. **827**, 153 (2016c)
Hendi, S.H., Panahiyan, S., Eslam Panah, B., Faizal, M., Momennia, M.: Phys. Rev. D **94**, 024028 (2016d)
Hendi, S.H., Panahiyan, S., Upadhyay, S., Eslam Panah, B.: Phys. Rev. D **95**, 084036 (2017). arXiv:1611.02937
Hindmarsh, M., Kibble, T.: Rep. Prog. Phys. **58**, 477 (1995)
Horava, P.: Phys. Rev. D **79**, 084008 (2009a)
Horava, P.: Phys. Rev. Lett. **102**, 161301 (2009b)
Houndjo, M.J.S., Momeni, D., Myrzakulov, R.: Int. J. Mod. Phys. D **21**, 1250093 (2012)
Houndjo, M.J.S., Rodrigues, M.E., Momeni, D., Myrzakulov, R.: Can. J. Phys. **92**, 1528 (2014)
Kasner, E.: J. Math. **43**, 217 (1921)
Kasner, E.: Trans. Am. Math. Soc. **27**, 155 (1925)
Khodadi, M., Nozari, K., Vakili, B.: Gen. Relativ. Gravit. **48**, 64 (2016)
Kim, Y.W., Kim, S.K., Park, Y.J.: Eur. Phys. J. C **76**, 557 (2016)
Kirezli, P., Çiftci, D.K., Delice, Ö.: Gen. Relativ. Gravit. **45**, 2251 (2013)
Kostelecky, V.A., Samuel, S.: Phys. Rev. D **39**, 683 (1989)
Kramer, D., Stephani, H., Herlt, E., MacCallum, M.: Exact Solutions of Einstein's Field Equations. Cambridge University Press, Cambridge (1980)
Li, H., Ling, Y., Han, X.: Class. Quantum Gravity **26**, 065004 (2009)
Linet, B.: Gen. Relativ. Gravit. **17**, 1109 (1985)
Linet, B.: J. Math. Phys. **27**, 1817 (1986)
Ling, Y.: J. Cosmol. Astropart. Phys. **0708**, 017 (2007)
Ling, Y., Hu, B., Li, X.: Phys. Rev. D **73**, 087702 (2006)
Ling, Y., Li, X., Zhang, H.: Mod. Phys. Lett. A **22**, 2749 (2007)
Maggiore, M.: Phys. Lett. B **304**, 65 (1993)
Magueijo, J., Smolin, L.: Phys. Rev. Lett. **88**, 190403 (2002a). Phys. Rev. D **67**, 044017 (2003)
Magueijo, J., Smolin, L.: Phys. Rev. Lett. **88**, 190403 (2002b)
Magueijo, J., Smolin, L.: Class. Quantum Gravity **21**, 1725 (2004)
Maluf, J.W., Goya, A.: Class. Quantum Gravity **18**, 5143 (2001)
Momeni, D.: Int. J. Theor. Phys. **50**, 1493 (2011)
Momeni, D., Gholizade, H.: Int. J. Mod. Phys. D **18**, 1719 (2009)
Momeni, D., Myrzakulov, K., Myrzakulov, R., Raza, M.: Eur. Phys. J. C **76**, 301 (2016)
Park, M.I.: Phys. Lett. B **659**, 698 (2008)
Reddy, D.R.K., Adhao, K.S., Katore, S.D.: Astrophys. Space Sci. **301**, 149 (2006)
Rodrigues, M.E., Houndjo, M.J.S., Momeni, D., Myrzakulov, R.: Can. J. Phys. **92**, 173 (2014)
Rovelli, C.: Living Rev. Relativ. **1**, 1 (1998)
Sahlmann, H., Thiemann, T.: Class. Quantum Gravity **23**, 909 (2006)
Simon, J.Z.: Phys. Rev. D **41**, 3720 (1990)
Smolin, L.: Nucl. Phys. B **742**, 142 (2006)
t'Hooft, G.: Class. Quantum Gravity **13**, 1023 (1996)
Tian, Q.: Phys. Rev. D **33**, 3549 (1986)
Tseytlin, A.A., Vafa, C.: Nucl. Phys. B **372**, 443 (1992)
Vilenkin, A., Shellard, E.S.: Cosmic Strings and Other Topological Defects. Cambridge University Press, Cambridge (2000)
Will, C.M.: Living Rev. Relativ. **17**, 4 (2014)
Yadav, G., Komal, B., Majhi, B.R.: (2016). arXiv:1605.01499
Zel'dovich, Ya.B.: Mon. Not. R. Astron. Soc. **192**, 663 (1980)