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# Darboux transformation and soliton solution for generalized Konno-Oono equation

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**Abstract.** In this paper, the generalized Konno-Oono equation is investigated. Using the Lax pair, we obtain a new Darboux transformation for the dispersionless equation. A soliton solution of the generalized Konno-Oono equation is obtained on the basis of the Darboux transformation. Corresponding graphs are also built.

## 1. Introduction

Nowadays dispersionless equations are equations that describes various problems of quantum mechanics, fluid mechanics, plasma physics, propagation of shallow water waves, optical fibers, electricity, etc. Therefore, investigations of dispersionless equations are becoming more and more interesting in nonlinear sciences day by day. However, not all equations posed of these models are solvable [1,2]. As a result, many new techniques have been successfully developed by diverse groups of mathematicians and physicists, such as: the Hirota's bilinear transformation method, the Darboux transformation (DT), the sine-cosine method, the tanh-function method and so forth. [3-6]. DT is one of the most useful method to get soliton solutions of the many developed equations from a trivial seed [7-14]. In this paper, we constructed DT for the generalized Konno-Oono equation [15] and we get single soliton, double soliton and three soliton solutions are derived by assuming a trivial seed solution by the DT.

## 2. The generalized Konno-Oono equation

The Konno-Oono equation (KOE) [15] in form

$$q_{xt} - \rho q = 0, \quad (1)$$

$$\rho_x + 0.5(q^2)_t = 0, \quad (2)$$

was proposed by Konno and Oono. The KOE is also known as the coupled dispersionless equation (CDE). Later, Konno and Kakuata proposed a generalization of this equation having



the following form

$$q_{xt} - \rho q = 0, \quad (3)$$

$$r_{xt} - \rho r = 0, \quad (4)$$

$$\rho_x + 0.5(rq)_t = 0. \quad (5)$$

This set of equations we call the generalized Konno-Oono equation (GKOE) or the generalized CDE (GCDE). GKOE is integrable by IST and its Lax representation (LR) can be expressed as

$$\Psi_x = U\Psi, \quad (6)$$

$$\Psi_t = V\Psi, \quad (7)$$

where

$$U = \begin{pmatrix} \frac{i}{4\lambda} & -0.5q \\ 0.5r & -\frac{i}{4\lambda} \end{pmatrix}, \quad V = -i\lambda \begin{pmatrix} \rho & q_t \\ r_t & -\rho \end{pmatrix}. \quad (8)$$

The compatibility condition of equations (3)-(5) given us

$$U_t - V_x + [U, V] = 0,$$

by direct calculation of above equation, we can yield the generalized Konno-Oono equation.

### 3. Darboux transformation

In this section, firstly, we consider the transformation of linear function  $\Psi$  by

$$\Psi' = T\Psi. \quad (9)$$

New function  $\Psi'$  is supposed to satisfy

$$\begin{aligned} \Psi'_x &= U'\Psi', \\ \Psi'_t &= V'\Psi'. \end{aligned}$$

Here

$$U' = \begin{pmatrix} \frac{i}{4\lambda} & -0.5q' \\ 0.5r' & -\frac{i}{4\lambda} \end{pmatrix}, \quad V' = -i\lambda \begin{pmatrix} \rho' & q'_t \\ r'_t & -\rho' \end{pmatrix}.$$

Then matrix  $T$  must satisfy following identities

$$T_x = U'T - TU, \quad (10)$$

$$T_t = V'T - TV. \quad (11)$$

Taking

$$T = \frac{1}{\lambda}I - M,$$

where  $I$  is an unit matrix and  $S = H\Lambda H^{-1}$ .

Now we come to construct a concrete transformation. Let  $\lambda = \lambda_1$ , the solution of Lax pairs (6) and (7) is  $(h_1, h_2)^T$ . It is easy to see that  $(h_2, -h_1)^T$  is the solution of  $\lambda = -\lambda_1$ . Suppose that

$$H = \begin{pmatrix} h_1 & h_2 \\ h_2 & -h_1 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & -\frac{1}{\lambda_1} \end{pmatrix}.$$

Let  $\sigma = \frac{h_2}{h_1}$  and

$$S = \frac{1}{\lambda_1(1 + \sigma^2)} \begin{pmatrix} 1 - \sigma^2 & 2\sigma \\ 2\sigma & \sigma^2 - 1 \end{pmatrix}. \quad (12)$$

In simplicity, noting  $\tan \frac{\theta}{2} = \sigma$ , then Eq. (12) will become

$$S = \frac{1}{\lambda_1} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.$$

Then our Darboux matrix have the form

$$T = \begin{pmatrix} \frac{1}{\lambda} - \frac{1}{\lambda_1} \cos \theta & -\frac{1}{\lambda_1} \sin \theta \\ -\frac{1}{\lambda_1} \sin \theta & \frac{1}{\lambda} + \frac{1}{\lambda_1} \cos \theta \end{pmatrix}. \quad (13)$$

With the help of

$$\sigma_x = 0.5(r - q\sigma^2) - \frac{i}{2\lambda}\sigma,$$

we find  $\theta_x$  for eq. (10)

$$\theta_x = \frac{r(1 + \cos \theta) - q(1 - \cos \theta)}{2} - \frac{i}{2\lambda} \sin \theta.$$

Then by using of the above transformation  $T$  and eq. (11), we can get

$$\begin{aligned} \sigma_t &= -i\lambda r_t + 2i\lambda\rho\sigma + i\lambda q_t\sigma^2, \\ \theta_t &= i\lambda q_t(1 - \cos \theta) - i\lambda r_t(1 + \cos \theta) + 2i\lambda\rho \sin \theta. \end{aligned}$$

Then the Darboux transformation on  $q$ ,  $r$  and  $\rho$  can be get by eq. (10)-(11), and we obtain

$$q' = q - \frac{i}{\lambda_1} \frac{2\sigma}{(1 + \sigma^2)^2}, \quad (14)$$

$$r' = r - \frac{i}{\lambda_1} \frac{2\sigma^3}{(1 + \sigma^2)^2}, \quad (15)$$

$$\rho' = -\rho - \frac{2\lambda(1 - \sigma^2)(q_t\sigma^2 - r_t + 2i\lambda\rho\sigma)}{\lambda_1(1 + \sigma^2)^2}. \quad (16)$$

or

$$q' = q - \frac{i}{2\lambda_1} \sin \theta (\cos \theta + 1), \quad (17)$$

$$r' = r + \frac{i}{2\lambda_1} \sin \theta (\cos \theta - 1), \quad (18)$$

$$\rho' = -\rho - \frac{i}{\lambda_1} (\cos \theta)_t. \quad (19)$$

Using the this DT we can found soliton solution of the generalized Konno-Oono equation.

#### 4. The soliton solutions

For a given solution of the coupled dispersionless Eqs.(3)-(5), suppose we have known the compatible basic solution old Lax pairs (6)-(7)

$$\Psi(x, t, \lambda) = \begin{pmatrix} \psi_{11}(x, t, \lambda) & \psi_{12}(x, t, \lambda) \\ \psi_{21}(x, t, \lambda) & \psi_{22}(x, t, \lambda) \end{pmatrix},$$

$\lambda_1, \mu_1$  are the arbitrary constants, let

$$\sigma = \frac{\psi_{22}(x, t, \lambda_1) + \mu_1 \psi_{21}(x, t, \lambda_1)}{\psi_{12}(x, t, \lambda_1) + \mu_1 \psi_{11}(x, t, \lambda_1)},$$

be the proportion of the two component of a solution for Lax pairs (6)-(7).

Now we take the trivial solution  $q = 0, r = 0, \rho = 1$ , the corresponding compatible basic solution of the Lax pairs (6)-(7) can be written as

$$\Psi(x, t, \lambda) = \begin{pmatrix} \exp(\frac{i}{4\lambda}x - i\lambda t) & 0 \\ 0 & \exp(-\frac{i}{4\lambda}x + i\lambda t) \end{pmatrix},$$

for the constant  $\lambda_1 \neq 0$ , and  $\mu_1 = \exp(2\alpha_1) > 0$ . We know

$$\sigma = \exp\left(-\frac{i}{2\lambda}x + 2i\lambda t - 2\alpha_1\right),$$

end

$$T = \frac{1}{\lambda}I - \frac{1}{\lambda_1 \cosh \nu_1} \begin{pmatrix} \sinh \nu_1 & 1 \\ 1 & -\sinh \nu_1 \end{pmatrix},$$

with

$$\nu_1 = \exp\left(\frac{i}{2\lambda}x - 2i\lambda t + 2\alpha_1\right).$$

By using eqs. (14)-(16), we get the one soliton solution of the generalized Konno-Oono equation.

$$q' = -\frac{i}{\lambda_1} \operatorname{sech}^2\left(-\frac{i}{2\lambda}x + 2i\lambda t - 2\alpha_1\right), \quad (20)$$

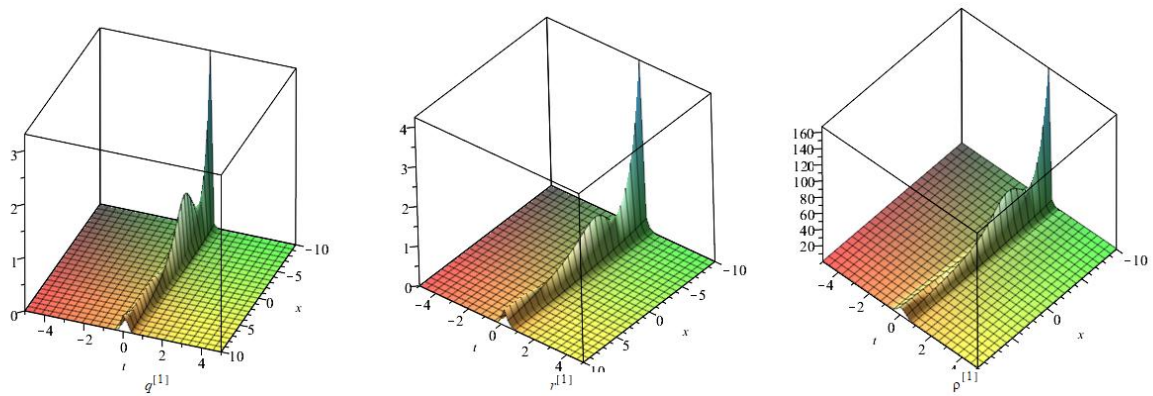
$$r' = -\frac{i}{\lambda_1} \operatorname{sech}^2\left(-\frac{i}{2\lambda}x + 2i\lambda t - 2\alpha_1\right) \exp\left(-\frac{i}{2\lambda}x + 2i\lambda t - 2\alpha_1\right), \quad (21)$$

$$\rho' = -1 - \frac{2i\lambda^2}{\lambda_1} \frac{\operatorname{sh}\left(\exp\left(-\frac{i}{2\lambda}x + 2i\lambda t - 2\alpha_1\right)\right)}{\operatorname{ch}^2\left(\exp\left(-\frac{i}{2\lambda}x + 2i\lambda t - 2\alpha_1\right)\right)}. \quad (22)$$

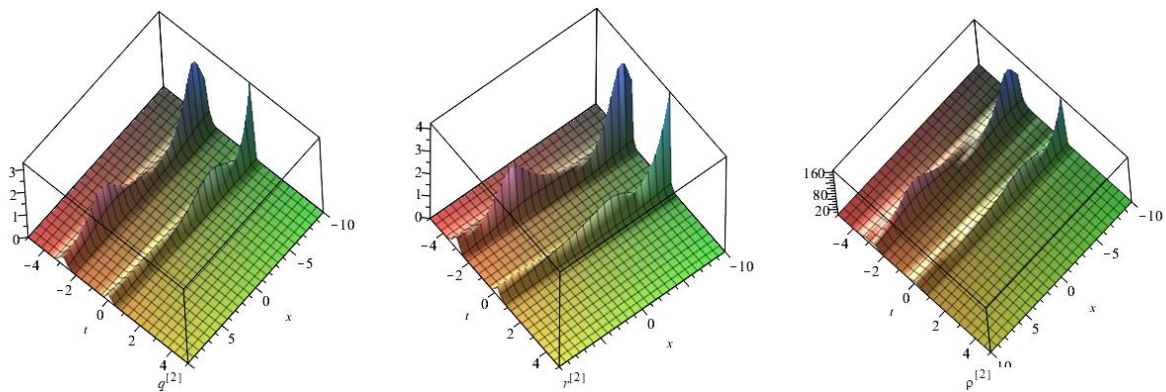
Visualization of single soliton solutions (20)-(22) of the GKOE is shown in figure 1.

#### 5. Conclusion

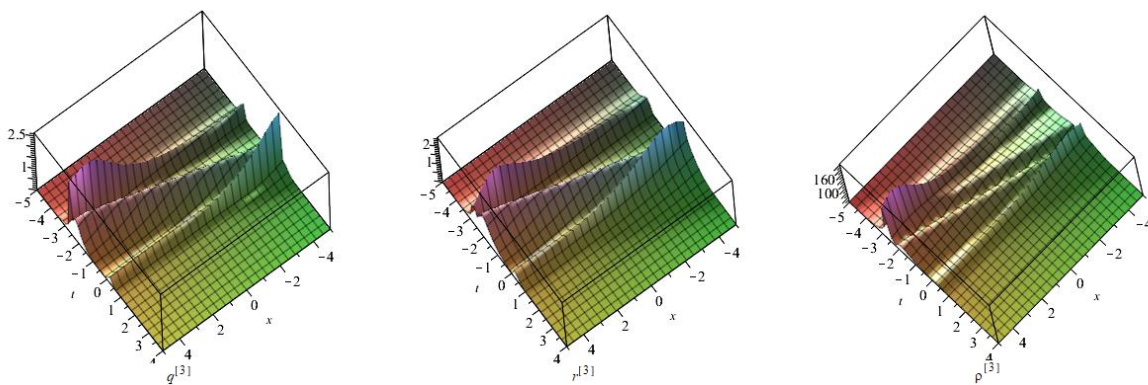
In this paper, using the Lax pair, we obtained a new Darboux transformation for the generalized Konno-Oono equation (3)-(5). The single soliton solution for the GKOE have been constructed by the DT. We provided graphical representation not only for all obtained solutions, but also for double and three soliton solutions (see figure 2, 3).



**Figure 1.** The propagation of one-soliton solution  $q^{[1]}$ ,  $r^{[1]}$  and  $\rho^{[1]}$  when  $\lambda_1 = -3i + 1$ ,  $\alpha_1 = 1$ .



**Figure 2.** The propagation of two-soliton solutions  $q^{[2]}$ ,  $r^{[2]}$  and  $\rho^{[2]}$  when  $\lambda_2 = 2i - 2$ ,  $\alpha_2 = 5$ .



**Figure 3.** The propagation of two-soliton solutions  $q^{[3]}$ ,  $r^{[3]}$  and  $\rho^{[3]}$  when  $\lambda_3 = -i$ ,  $\alpha_3 = -1$ .

## 6. Acknowledgments

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