Journal of Physics: Conference Series

PAPER • OPEN ACCESS

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Journal of Physics: Conference Series **1391** (2019) 012035 doi:10.1088/1742-6596/1391/1/012035

Eight-vertex model over Grassmann algebra

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Abstract. An eight-vertex model on a square lattice over a Grassmann algebra is investigated using an equation that is a three-dimensional generalization of the Yang-Baxter equation. Anticommuting quantum spin systems are studied, where the quasiclassical limit leads to some abstract classical physics with anticommuting variables. The solution of the quantum Yang-Baxter equation is the R - matrix, which corresponds to the transfer R - matrix of the eight-vertex model of statistical mechanics.

1. Introduction

This article is devoted to the consideration of the eight-vertex model on a square lattice over a Grassmann algebra using the tetrahedron equation, or the Zamolodchikov equation, which is a three-dimensional generalization of the well-known Yang-Baxter equation. We will conduct a small historical review. A.B. Zamolodchikov investigated the permutation of transition matrices for a statistical model on a cubic lattice with spin variables on the faces and with Boltzmann weights equal to the elements of its S - matrix. Bazhanov proved that the elements of the Zamolodchikov S - matrices really satisfy the equation of the tetrahedron. A little later in [1], Bazhanov and Stroganov studied several versions of the tetrahedron equation for statistical models with different types of interaction. Both scientists also proposed a generalization of the tetrahedron equation for models in a space of higher dimension, calling them n - simplex equations. After 10 years later, in 1984, a solution was found to the equation of a tetrahedron with Grassmann variables on the edges of a cubic lattice [2, 3]. 3 - the simplex equation of a tetrahedron has the form

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}, \tag{1}$$

and in index form

$$r^{l_1l_2l_3}_{k_1k_2k_3}r^{k_1l_4l_5}_{j_1k_4k_5}r^{k_2k_4l_6}_{j_2j_4k_6}r^{k_3k_5k_6}_{j_3j_5j_6} = r^{l_3l_5l_6}_{k_3k_5k_6}r^{l_2l_4k_6}_{k_2k_4j_6}r^{l_1k_4k_5}_{k_1j_4j_5}r^{k_1k_2k_3}_{j_1j_2j_3}, \tag{2}$$

recently has become intensely considered by leading scientists [3]. This equation first arose in the study of solvable vertex models in statistical mechanics [4] and was subsequently recognized as a key equation. The constant form (1) is also important for quantum groups [5], knot theory [6], and so on. For quantum systems with fermion fields, the quasiclassical limit naturally leads to some abstract classical physics with anticommuting variables. It is precisely to study such unusual classical systems that an algebra is needed that operates on functions anticommuting variables. It is important to emphasize that in normal classical physics problems of this type do not arise naturally; they appear, as explained above, only as quasiclassical limit for quantum problems with fermion fields. But this already justifies their study, since the study the quasi-classical approximation for various quantum systems is a very important task for theoretical physicists [7].

The importance of a thorough study of the solutions of the Yang-Baxter equation is associated with its key role in accurately solvable models of statistical mechanics [4] and [8] and field theory in small dimensions [9], conformal field theory [10] and in quantum integrable systems [11]. From the group-theoretic point of view, while the classical Yang-Baxter equation is closely related to the theory of classical (semisimple) groups, the quantum Yang-Baxter equation is the basis of the modern theory of quantum groups [12, 13, 14, 15]. There are different types of quantum Yang-Baxter equation: constant, one-parameter and two-parameter forms [16]. The necessary constant (and permutation) solutions of the Yang-Baxter equation [17] are applied in quantization of integrable nonlinear evolution equations, the theory of quantum groups [18, 19, 20, 21], and the theory of knots [22, 23]. The solution of the quantum Yang-Baxter equation is the R - matrix [24, 25] (the corresponding transfer matrix in lattice statistical models [4]) [26].

2. The tetrahedron equation for an eight-vertex model over a Grassmann algebra

We compare the operator R - a numerical matrix with n pairs of indices

$$R(e_{i_1},\otimes\cdots\otimes e_{i_n}) = R^{j_1\cdots j_n}_{i_1\cdots i_n}(e_{j_1},\otimes\cdots\otimes e_{j_n}), \tag{3}$$

where summation is performed over repeated indices.

Consider the 3-simplex tetrahedron equation for an eight-vertex model on a tensor product $V^{\otimes}[n(n+1)/2]$, where the linear operators R act trivially, for example $R_{123}(e_{i_1}\otimes e_{i_2}\otimes e_{i_3}) = r^{j_1 j_2}_{i_1 i_2}(e_{i_1}\otimes e_{i_2}\otimes e_{i_3})$. In the general case, with $K_\alpha \in \{1,\ldots,N\}$,

$$N = \frac{n(n+1)}{2}, \tag{4}$$

where $n=3$ and the value of N forms 6 Grassmann generators; R operators are

$$(R_{K_1\cdots K_N})^{j_1\cdots j_n}_{i_1\cdots i_N} = r^{j_{K_1}\cdots j_{K_n}}_{i_{K_1}\cdots i_{K_n}} \prod_{k=1,k=K_\alpha,\forall_\alpha}^{N} \delta^{j_k}_{i_k}, \tag{5}$$

where $r^{j_{K_1}\cdots j_{K_n}}_{i_{K_1}\cdots i_{K_n}}$ - element R - matrices [26].

Consider the R - matrix over the even part of a Grassmann algebra with 6 generators, we write its decomposition into numerical and nilponent parts

$$\begin{aligned}R = R^{(0)} + R^{(123)}\xi_1\xi_2\xi_3 + R^{(145)}\xi_1\xi_4\xi_5 + R^{(246)}\xi_2\xi_4\xi_6 + R^{(356)}\xi_3\xi_5\xi_6 + R^{(124)}\xi_1\xi_2\xi_4 +\\ +R^{(125)}\xi_1\xi_2\xi_5 + R^{(126)}\xi_1\xi_2\xi_6 + R^{(134)}\xi_1\xi_3\xi_4 + R^{(135)}\xi_1\xi_3\xi_5 + R^{(136)}\xi_1\xi_3\xi_6 + R^{(146)}\xi_1\xi_4\xi_6 +\\ +R^{(156)}\xi_1\xi_5\xi_6 + R^{(234)}\xi_2\xi_3\xi_4 + R^{(235)}\xi_2\xi_3\xi_5 + R^{(236)}\xi_2\xi_3\xi_6 + R^{(245)}\xi_2\xi_4\xi_5 + R^{(256)}\xi_2\xi_5\xi_6 +\\ +R^{(345)}\xi_3\xi_4\xi_5 + R^{(346)}\xi_3\xi_4\xi_6 + R^{(456)}\xi_4\xi_5\xi_6 + R^{(123456)}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6,\end{aligned} \tag{6}$$

the components of the Yang-Baxter equation are presented in the same form

$$\begin{aligned}R_{123} = R^{(0)}_{123} + R^{(123)}_{123}\xi_1\xi_2\xi_3 + R^{(145)}_{123}\xi_1\xi_4\xi_5 + R^{(246)}_{123}\xi_2\xi_4\xi_6 + R^{(356)}_{123}\xi_3\xi_5\xi_6 + R^{(124)}_{123}\xi_1\xi_2\xi_4 +\\ +R^{(125)}_{123}\xi_1\xi_2\xi_5 + R^{(126)}_{123}\xi_1\xi_2\xi_6 + R^{(134)}_{123}\xi_1\xi_3\xi_4 + R^{(135)}_{123}\xi_1\xi_3\xi_5 + R^{(136)}_{123}\xi_1\xi_3\xi_6 + R^{(146)}_{123}\xi_1\xi_4\xi_6 +\\ +R^{(156)}_{123}\xi_1\xi_5\xi_6 + R^{(234)}_{123}\xi_2\xi_3\xi_4 + R^{(235)}_{123}\xi_2\xi_3\xi_5 + R^{(236)}_{123}\xi_2\xi_3\xi_6 + R^{(245)}_{123}\xi_2\xi_4\xi_5 + R^{(256)}_{123}\xi_2\xi_5\xi_6 +\\ +R^{(345)}_{123}\xi_3\xi_4\xi_5 + +R^{(346)}_{123}\xi_3\xi_4\xi_6 + R^{(346)}_{123}\xi_3\xi_4\xi_6 + R^{(456)}_{123}\xi_4\xi_5\xi_6 + R^{(123456)}_{123}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6,\end{aligned} \tag{7}$$

Journal of Physics: Conference Series **1391** (2019) 012035 doi:10.1088/1742-6596/1391/1/012035

$$R_{145} = R_{145}^{(0)} + R_{145}^{(123)}\xi_1\xi_2\xi_3 + R_{145}^{(145)}\xi_1\xi_4\xi_5 + R_{145}^{(246)}\xi_2\xi_4\xi_6 + R_{145}^{(356)}\xi_3\xi_5\xi_6 + R_{145}^{(124)}\xi_1\xi_2\xi_4 + \\ +R_{145}^{(125)}\xi_1\xi_2\xi_5 + R_{145}^{(126)}\xi_1\xi_2\xi_6 + R_{145}^{(134)}\xi_1\xi_3\xi_4 + R_{145}^{(135)}\xi_1\xi_3\xi_5 + R_{145}^{(136)}\xi_1\xi_3\xi_6 + R_{145}^{(146)}\xi_1\xi_4\xi_6 + \\ +R_{145}^{(156)}\xi_1\xi_5\xi_6 + R_{145}^{(234)}\xi_2\xi_3\xi_4 + R_{145}^{(235)}\xi_2\xi_3\xi_5 + R_{145}^{(236)}\xi_2\xi_3\xi_6 + R_{145}^{(245)}\xi_2\xi_4\xi_5 + R_{145}^{(256)}\xi_2\xi_5\xi_6 + \\ +R_{145}^{(345)}\xi_3\xi_4\xi_5 + R_{145}^{(346)}\xi_3\xi_4\xi_6 + R_{145}^{(456)}\xi_4\xi_5\xi_6 + R_{145}^{(123456)}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6, \quad (8)$$

$$R_{246} = R_{246}^{(0)} + R_{246}^{(123)}\xi_1\xi_2\xi_3 + R_{246}^{(145)}\xi_1\xi_4\xi_5 + R_{246}^{(246)}\xi_2\xi_4\xi_6 + R_{246}^{(356)}\xi_3\xi_5\xi_6 + R_{246}^{(124)}\xi_1\xi_2\xi_4 + \\ +R_{246}^{(125)}\xi_1\xi_2\xi_5 + R_{246}^{(126)}\xi_1\xi_2\xi_6 + R_{246}^{(134)}\xi_1\xi_3\xi_4 + R_{246}^{(135)}\xi_1\xi_3\xi_5 + R_{246}^{(136)}\xi_1\xi_3\xi_6 + R_{246}^{(146)}\xi_1\xi_4\xi_6 + \\ +R_{246}^{(156)}\xi_1\xi_5\xi_6 + R_{246}^{(234)}\xi_2\xi_3\xi_4 + R_{246}^{(235)}\xi_2\xi_3\xi_5 + R_{246}^{(236)}\xi_2\xi_3\xi_6 + R_{246}^{(245)}\xi_2\xi_4\xi_5 + R_{246}^{(256)}\xi_2\xi_5\xi_6 + \\ +R_{246}^{(345)}\xi_3\xi_4\xi_5 + R_{246}^{(346)}\xi_3\xi_4\xi_6 + R_{246}^{(456)}\xi_4\xi_5\xi_6 + R_{246}^{(123456)}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6, \quad (9)$$

$$R_{356} = R_{356}^{(0)} + R_{356}^{(123)}\xi_1\xi_2\xi_3 + R_{356}^{(145)}\xi_1\xi_4\xi_5 + R_{356}^{(246)}\xi_2\xi_4\xi_6 + R_{356}^{(356)}\xi_3\xi_5\xi_6 + R_{356}^{(124)}\xi_1\xi_2\xi_4 + \\ +R_{356}^{(125)}\xi_1\xi_2\xi_5 + R_{356}^{(126)}\xi_1\xi_2\xi_6 + R_{356}^{(134)}\xi_1\xi_3\xi_4 + R_{356}^{(135)}\xi_1\xi_3\xi_5 + R_{356}^{(136)}\xi_1\xi_3\xi_6 + R_{356}^{(146)}\xi_1\xi_4\xi_6 + \\ +R_{356}^{(156)}\xi_1\xi_5\xi_6 + R_{356}^{(234)}\xi_2\xi_3\xi_4 + R_{356}^{(235)}\xi_2\xi_3\xi_5 + R_{356}^{(236)}\xi_2\xi_3\xi_6 + R_{356}^{(245)}\xi_2\xi_4\xi_5 + R_{356}^{(256)}\xi_2\xi_5\xi_6 + \\ +R_{356}^{(345)}\xi_3\xi_4\xi_5 + R_{356}^{(346)}\xi_3\xi_4\xi_6 + R_{356}^{(456)}\xi_4\xi_5\xi_6 + R_{356}^{(123456)}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6. \quad (10)$$

We substitute (7), (8), (9), (10) expressions in (5) and obtain the system of equations for the components $124, 125, 126, 134, 135, 136, 145, 146, 156, 456, 356, 346, 345, 256, 246, 245, 236, 235, 234, 123456$. The final solutions of all equations due to their bulkiness are not given. We give their explicit form for numerical and nilponent components

$$R_{123}^{(0)}R_{145}^{(0)}R_{246}^{(0)}R_{356}^{(0)} = R_{356}^{(0)}R_{246}^{(0)}R_{145}^{(0)}R_{123}^{(0)}, \quad (11)$$

$$R_{123}^{(0)}R_{145}^{(0)}R_{246}^{(0)}R_{356}^{(123)} + R_{123}^{(0)}R_{145}^{(0)}R_{246}^{(123)}R_{356}^{(0)} + R_{123}^{(0)}R_{145}^{(123)}R_{246}^{(0)}R_{356}^{(0)} + R_{123}^{(123)}R_{145}^{(0)}R_{246}^{(0)}R_{356}^{(0)} = \\ = R_{356}^{(123)}R_{246}^{(0)}R_{145}^{(0)}R_{123}^{(0)} + R_{356}^{(0)}R_{246}^{(123)}R_{145}^{(0)}R_{123}^{(0)} + R_{356}^{(0)}R_{246}^{(0)}R_{145}^{(123)}R_{123}^{(0)} + R_{356}^{(0)}R_{246}^{(0)}R_{145}^{(0)}R_{123}^{(123)} \quad (12)$$

3. Classification of solutions of the tetrahedron equation with eight vertices

The eight-vertex solution of the Yang-Baxter equation is the R matrix in the form

$$R = \begin{pmatrix} p & \cdot & \cdot & f \\ \cdot & c & d & \cdot \\ \cdot & a & b & \cdot \\ g & \cdot & \cdot & q \end{pmatrix}, \quad (13)$$

where a, b, c, d, p, f, g, q are elements of the Grassmann algebra. From (5) follows the explicit form of matrices

$$R_{123} = \begin{pmatrix} p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & f \\ \cdot & p & \cdot & \cdot & \cdot & \cdot & f & \cdot \\ \cdot & \cdot & c & \cdot & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c & \cdot & d & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & b & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a & \cdot & b & \cdot & \cdot \\ \cdot & g & \cdot & \cdot & \cdot & \cdot & q & \cdot \\ g & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q \end{pmatrix}, R_{145} = \begin{pmatrix} p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & f \\ \cdot & c & \cdot & \cdot & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & p & \cdot & \cdot & f & \cdot & \cdot \\ \cdot & \cdot & \cdot & c & \cdot & \cdot & d & \cdot \\ \cdot & a & \cdot & \cdot & b & \cdot & \cdot & \cdot \\ \cdot & \cdot & g & \cdot & \cdot & q & \cdot & \cdot \\ \cdot & \cdot & \cdot & a & \cdot & \cdot & b & \cdot \\ g & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q \end{pmatrix}, \quad (14)$$

$$R_{246} = \begin{pmatrix} p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & f \\ \cdot & c & d & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & q & g & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & f & p & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & d & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a & b & \cdot \\ g & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q \end{pmatrix}, R_{356} = \begin{pmatrix} p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & f \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & d & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a & b & \cdot \\ \cdot & \cdot & \cdot & q & g & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & f & p & \cdot & \cdot & \cdot \\ \cdot & c & d & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ g & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q \end{pmatrix}. \quad (15)$$

Substitute $R_{123}, R_{145}, R_{246}, R_{356}$ in (2) and get the following system of equations

$$cda = 0, \quad bda = 0, \quad qgf = 0, \quad pgf = 0, \quad (16)$$
$$da(d-a) = 0, \quad gf(g-f) = 0, \quad (17)$$
$$pd(d-p) + cbd = 0, \quad qd(d-q) + cbd = 0, \quad pa(a-p) + cba = 0, \quad (18)$$
$$qa(a-q) + cba = 0, \quad bg(g-b) + qpg = 0, \quad bf(f-b) + qpf = 0, \quad (19)$$
$$cg(g-c) + qpg = 0, \quad cf(f-c) + qpf = 0, \quad gq(q-g) + fgq = 0, \quad (20)$$
$$gp(p-g) + fgp = 0, \quad fp(p-f) + gfp = 0, \quad fq(q-f) + gfq = 0. \quad (21)$$

3.1. Classification by numerical part of the solution

1. $a_0, d_0 \neq 0, \quad b_0, c_0 = 0 \quad a_0 = d_0, \quad \{p_0, q_0, g_0, f_0\} = \{0, a_0\}$. Then for the numerical part we get the 6 - vertex solution.

$$R = \begin{pmatrix} \{0, a_0\} & \cdot & \cdot & \{0, a_0\} \\ \cdot & \cdot & d_0 & \cdot \\ \cdot & a_0 & \cdot & \cdot \\ \{0, a_0\} & \cdot & \cdot & \{0, a_0\} \end{pmatrix}. \quad (22)$$

2. $a_0 = 0, \quad d_0 \neq 0 \longrightarrow \{p_0, q_0, g_0, f_0\}$. Then for the numerical part we get the 7 - vertex solution.

$$R = \begin{pmatrix} \frac{d_0}{2} \pm \sqrt{\frac{d_0^2}{4} + b_0 c_0} & \cdot & \cdot & b_0 - \frac{p_0 q_0}{b_0} \\ \cdot & c_0 & d_0 & \cdot \\ \cdot & \cdot & b_0 & \cdot \\ b_0 - \frac{p_0 q_0}{b_0} & \cdot & \cdot & \frac{d_0}{2} \pm \sqrt{\frac{d_0^2}{4} + b_0 c_0} \end{pmatrix}, \quad (23)$$

where $\{p_0, q_0\} = \frac{d_0}{2} \pm \sqrt{\frac{d_0^2}{4} + b_0 c_0}, \quad d_0 = 0, \ a_0 \neq 0 \longrightarrow \{p_0, q_0, g_0, f_0\}$, then vice versa.

3. $a_0 = d_0 = 0, \{p_0, q_0, g_0, f_0\}$ - any items. Then for the numerical part we get the 6 - vertex solution.

$$R = \begin{pmatrix} p_0 & \cdot & \cdot & f_0 \\ \cdot & c_0 & \cdot & \cdot \\ \cdot & \cdot & b_0 & \cdot \\ g_0 & \cdot & \cdot & q_0 \end{pmatrix}. \quad (24)$$

8th International Conference on Mathematical Modeling in Physical Science — IOP Publishing
Journal of Physics: Conference Series **1391** (2019) 012035 doi:10.1088/1742-6596/1391/1/012035

4. Solving a tetrahedron equation over a Grassmann algebra with 6 generators

Consider the specific type of solutions of the eight-vertex model. We decompose each element of the R - matrix (13) into 6 Grassmann generators in the form

$$\begin{aligned} c = c_0 + c_{123}\xi_1\xi_2\xi_3 + c_{145}\xi_1\xi_4\xi_5 + c_{246}\xi_2\xi_4\xi_6 + c_{356}\xi_3\xi_5\xi_6 + c_{124}\xi_1\xi_2\xi_4 + \\ + c_{125}\xi_1\xi_2\xi_5 + c_{126}\xi_1\xi_2\xi_6 + c_{134}\xi_1\xi_3\xi_4 + c_{135}\xi_1\xi_3\xi_5 + c_{136}\xi_1\xi_3\xi_6 + c_{145}\xi_1\xi_4\xi_5 + \\ + c_{146}\xi_1\xi_4\xi_6 + c_{156}\xi_1\xi_5\xi_6 + c_{234}\xi_2\xi_3\xi_4 + c_{235}\xi_2\xi_3\xi_5 + c_{236}\xi_2\xi_3\xi_6 + c_{245}\xi_2\xi_4\xi_5 + \\ + c_{256}\xi_2\xi_5\xi_6 + c_{345}\xi_3\xi_4\xi_5 + c_{346}\xi_3\xi_4\xi_6 + c_{456}\xi_4\xi_5\xi_6 + c_{123456}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6, \end{aligned} \tag{25}$$

$$\begin{aligned} p = p_0 + p_{123}\xi_1\xi_2\xi_3 + p_{145}\xi_1\xi_4\xi_5 + p_{246}\xi_2\xi_4\xi_6 + p_{356}\xi_3\xi_5\xi_6 + p_{124}\xi_1\xi_2\xi_4 + \\ + p_{125}\xi_1\xi_2\xi_5 + p_{126}\xi_1\xi_2\xi_6 + p_{134}\xi_1\xi_3\xi_4 + p_{135}\xi_1\xi_3\xi_5 + p_{136}\xi_1\xi_3\xi_6 + p_{145}\xi_1\xi_4\xi_5 + \\ + p_{146}\xi_1\xi_4\xi_6 + p_{156}\xi_1\xi_5\xi_6 + p_{234}\xi_2\xi_3\xi_4 + c_{235}\xi_2\xi_3\xi_5 + p_{236}\xi_2\xi_3\xi_6 + p_{245}\xi_2\xi_4\xi_5 + \\ + p_{256}\xi_2\xi_5\xi_6 + p_{345}\xi_3\xi_4\xi_5 + p_{346}\xi_3\xi_4\xi_6 + p_{456}\xi_4\xi_5\xi_6 + p_{123456}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6, \end{aligned} \tag{26}$$

$$\begin{aligned} b = b_0 + b_{123}\xi_1\xi_2\xi_3 + b_{145}\xi_1\xi_4\xi_5 + b_{246}\xi_2\xi_4\xi_6 + b_{356}\xi_3\xi_5\xi_6 + b_{124}\xi_1\xi_2\xi_4 + \\ + b_{125}\xi_1\xi_2\xi_5 + b_{126}\xi_1\xi_2\xi_6 + b_{134}\xi_1\xi_3\xi_4 + b_{135}\xi_1\xi_3\xi_5 + b_{136}\xi_1\xi_3\xi_6 + b_{145}\xi_1\xi_4\xi_5 + \\ + b_{146}\xi_1\xi_4\xi_6 + b_{156}\xi_1\xi_5\xi_6 + b_{234}\xi_2\xi_3\xi_4 + b_{235}\xi_2\xi_3\xi_5 + b_{236}\xi_2\xi_3\xi_6 + b_{245}\xi_2\xi_4\xi_5 + \\ + b_{256}\xi_2\xi_5\xi_6 + b_{345}\xi_3\xi_4\xi_5 + b_{346}\xi_3\xi_4\xi_6 + b_{456}\xi_4\xi_5\xi_6 + b_{123456}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6, \end{aligned} \tag{27}$$

$$\begin{aligned} q = q_0 + q_{123}\xi_1\xi_2\xi_3 + q_{145}\xi_1\xi_4\xi_5 + q_{246}\xi_2\xi_4\xi_6 + q_{356}\xi_3\xi_5\xi_6 + q_{124}\xi_1\xi_2\xi_4 + \\ + q_{125}\xi_1\xi_2\xi_5 + q_{126}\xi_1\xi_2\xi_6 + q_{134}\xi_1\xi_3\xi_4 + q_{135}\xi_1\xi_3\xi_5 + q_{136}\xi_1\xi_3\xi_6 + q_{145}\xi_1\xi_4\xi_5 + \\ + q_{146}\xi_1\xi_4\xi_6 + q_{156}\xi_1\xi_5\xi_6 + q_{234}\xi_2\xi_3\xi_4 + q_{235}\xi_2\xi_3\xi_5 + q_{236}\xi_2\xi_3\xi_6 + q_{245}\xi_2\xi_4\xi_5 + \\ + q_{256}\xi_2\xi_5\xi_6 + q_{345}\xi_3\xi_4\xi_5 + q_{346}\xi_3\xi_4\xi_6 + q_{456}\xi_4\xi_5\xi_6 + q_{123456}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6, \end{aligned} \tag{28}$$

$$\begin{aligned} a = a_0 + a_{123}\xi_1\xi_2\xi_3 + a_{145}\xi_1\xi_4\xi_5 + a_{246}\xi_2\xi_4\xi_6 + a_{356}\xi_3\xi_5\xi_6 + a_{124}\xi_1\xi_2\xi_4 + \\ + a_{125}\xi_1\xi_2\xi_5 + a_{126}\xi_1\xi_2\xi_6 + a_{134}\xi_1\xi_3\xi_4 + a_{135}\xi_1\xi_3\xi_5 + a_{136}\xi_1\xi_3\xi_6 + a_{145}\xi_1\xi_4\xi_5 + \\ + a_{146}\xi_1\xi_4\xi_6 + a_{156}\xi_1\xi_5\xi_6 + a_{234}\xi_2\xi_3\xi_4 + a_{235}\xi_2\xi_3\xi_5 + a_{236}\xi_2\xi_3\xi_6 + a_{245}\xi_2\xi_4\xi_5 + \\ + a_{256}\xi_2\xi_5\xi_6 + a_{345}\xi_3\xi_4\xi_5 + a_{346}\xi_3\xi_4\xi_6 + a_{456}\xi_4\xi_5\xi_6 + a_{123456}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6, \end{aligned} \tag{29}$$

$$\begin{aligned} g = g_0 + g_{123}\xi_1\xi_2\xi_3 + g_{145}\xi_1\xi_4\xi_5 + g_{246}\xi_2\xi_4\xi_6 + g_{356}\xi_3\xi_5\xi_6 + g_{124}\xi_1\xi_2\xi_4 + \\ + g_{125}\xi_1\xi_2\xi_5 + g_{126}\xi_1\xi_2\xi_6 + g_{134}\xi_1\xi_3\xi_4 + g_{135}\xi_1\xi_3\xi_5 + g_{136}\xi_1\xi_3\xi_6 + g_{145}\xi_1\xi_4\xi_5 + \\ + g_{146}\xi_1\xi_4\xi_6 + g_{156}\xi_1\xi_5\xi_6 + g_{234}\xi_2\xi_3\xi_4 + g_{235}\xi_2\xi_3\xi_5 + g_{236}\xi_2\xi_3\xi_6 + g_{245}\xi_2\xi_4\xi_5 + \\ + g_{256}\xi_2\xi_5\xi_6 + g_{345}\xi_3\xi_4\xi_5 + g_{346}\xi_3\xi_4\xi_6 + g_{456}\xi_4\xi_5\xi_6 + g_{123456}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6, \end{aligned} \tag{30}$$

$$\begin{aligned} d = d_0 + d_{123}\xi_1\xi_2\xi_3 + d_{145}\xi_1\xi_4\xi_5 + d_{246}\xi_2\xi_4\xi_6 + d_{356}\xi_3\xi_5\xi_6 + c_{124}\xi_1\xi_2\xi_4 + \\ + d_{125}\xi_1\xi_2\xi_5 + d_{126}\xi_1\xi_2\xi_6 + d_{134}\xi_1\xi_3\xi_4 + d_{135}\xi_1\xi_3\xi_5 + d_{136}\xi_1\xi_3\xi_6 + d_{145}\xi_1\xi_4\xi_5 + \\ + d_{146}\xi_1\xi_4\xi_6 + d_{156}\xi_1\xi_5\xi_6 + d_{234}\xi_2\xi_3\xi_4 + d_{235}\xi_2\xi_3\xi_5 + d_{236}\xi_2\xi_3\xi_6 + d_{245}\xi_2\xi_4\xi_5 + \\ + d_{256}\xi_2\xi_5\xi_6 + d_{345}\xi_3\xi_4\xi_5 + d_{346}\xi_3\xi_4\xi_6 + d_{456}\xi_4\xi_5\xi_6 + d_{123456}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6, \end{aligned} \tag{31}$$

$$\begin{aligned} f = f_0 + c_{123}\xi_1\xi_2\xi_3 + f_{145}\xi_1\xi_4\xi_5 + f_{246}\xi_2\xi_4\xi_6 + f_{356}\xi_3\xi_5\xi_6 + f_{124}\xi_1\xi_2\xi_4 + \\ + f_{125}\xi_1\xi_2\xi_5 + f_{126}\xi_1\xi_2\xi_6 + f_{134}\xi_1\xi_3\xi_4 + f_{135}\xi_1\xi_3\xi_5 + f_{136}\xi_1\xi_3\xi_6 + f_{145}\xi_1\xi_4\xi_5 + \\ + f_{146}\xi_1\xi_4\xi_6 + f_{156}\xi_1\xi_5\xi_6 + f_{234}\xi_2\xi_3\xi_4 + f_{235}\xi_2\xi_3\xi_5 + f_{236}\xi_2\xi_3\xi_6 + f_{245}\xi_2\xi_4\xi_5 + \\ + f_{256}\xi_2\xi_5\xi_6 + f_{345}\xi_3\xi_4\xi_5 + f_{346}\xi_3\xi_4\xi_6 + f_{456}\xi_4\xi_5\xi_6 + f_{123456}\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6. \end{aligned} \tag{32}$$

4.1. Case I. $d_0 = 0$. From the system of equations (16)-(21) it follows that

$$\begin{aligned}
&p_0^2 d_{123} = 0,\ \ p_0^2 d_{145} = 0,\ \ p_0^2 d_{246} = 0,\ \ p_0^2 d_{356} = 0,\ \ p_0^2 d_{124} = 0,\ \ p_0^2 d_{125} = 0,\\
&p_0^2 d_{126} = 0,\ \ p_0^2 d_{134} = 0,\ \ p_0^2 d_{135} = 0,\ \ p_0^2 d_{136} = 0,\ \ p_0^2 d_{146} = 0,\ \ p_0^2 d_{156} = 0,\\
&p_0^2 d_{234} = 0,\ \ p_0^2 d_{235} = 0,\ \ p_0^2 d_{236} = 0,\ \ p_0^2 d_{245} = 0,\ \ p_0^2 d_{256} = 0,\ \ p_0^2 d_{345} = 0,\\
&p_0^2 d_{346} = 0,\ \ p_0^2 d_{456} = 0,\ \ p_0^2 d_{123456} = 0.
\end{aligned} \tag{33}$$

$$\begin{aligned}
&q_0^2 q_{123} = 0,\ \ q_0^2 q_{145} = 0,\ \ q_0^2 q_{246} = 0,\ \ q_0^2 q_{356} = 0,\ \ q_0^2 q_{124} = 0,\ \ q_0^2 q_{125} = 0,\\
&q_0^2 q_{126} = 0,\ \ q_0^2 q_{134} = 0,\ \ q_0^2 q_{135} = 0,\ \ q_0^2 q_{136} = 0,\ \ q_0^2 q_{146} = 0,\ \ q_0^2 q_{156} = 0,\\
&q_0^2 q_{234} = 0,\ \ q_0^2 q_{235} = 0,\ \ q_0^2 q_{236} = 0,\ \ q_0^2 q_{245} = 0,\ \ q_0^2 q_{256} = 0,\ \ q_0^2 q_{345} = 0,\\
&q_0^2 q_{346} = 0,\ \ q_0^2 q_{456} = 0,\ \ q_0^2 q_{123456} = 0.
\end{aligned} \tag{34}$$

$$\begin{aligned}
&a_0^2 a_{123} = 0,\ \ a_0^2 a_{145} = 0,\ \ a_0^2 a_{246} = 0,\ \ a_0^2 a_{356} = 0,\ \ a_0^2 a_{124} = 0,\ \ a_0^2 a_{125} = 0,\\
&a_0^2 a_{126} = 0,\ \ a_0^2 a_{134} = 0,\ \ a_0^2 a_{135} = 0,\ \ a_0^2 a_{136} = 0,\ \ a_0^2 a_{146} = 0,\ \ a_0^2 a_{156} = 0,\\
&a_0^2 a_{234} = 0,\ \ a_0^2 a_{235} = 0,\ \ a_0^2 a_{236} = 0,\ \ a_0^2 a_{245} = 0,\ \ a_0^2 a_{256} = 0,\ \ a_0^2 a_{345} = 0,\\
&a_0^2 a_{346} = 0,\ \ a_0^2 a_{456} = 0,\ \ a_0^2 a_{123456} = 0.
\end{aligned} \tag{35}$$

4.2. Case II. From the case I $p_0^2 = 0,\ \ q_0^2 = 0,\ \ a_0^2 = 0$, means $p_0 = 0,\ \ q_0 = 0,$ $a_0 = 0$, then the system of equations (16)-(21) takes the form

$$\begin{aligned}
&g_0^2 q_{123} - f_0 g_0 q_{123} = 0,\ \ g_0^2 q_{145} - f_0 g_0 q_{145} = 0,\ \ g_0^2 q_{246} - f_0 g_0 q_{246} = 0,\\
&g_0^2 q_{356} - f_0 g_0 q_{356} = 0,\ \ g_0^2 q_{124} - f_0 g_0 q_{124} = 0,\ \ g_0^2 q_{125} - f_0 g_0 q_{125} = 0,\\
&g_0^2 q_{126} - f_0 g_0 q_{126} = 0,\ \ g_0^2 q_{134} - f_0 g_0 q_{134} = 0,\ \ g_0^2 q_{135} - f_0 g_0 q_{135} = 0,\\
&g_0^2 q_{136} - f_0 g_0 q_{136} = 0,\ \ g_0^2 q_{146} - f_0 g_0 q_{146} = 0,\ \ g_0^2 q_{156} - f_0 g_0 q_{156} = 0,\\
&g_0^2 q_{234} - f_0 g_0 q_{234} = 0,\ \ g_0^2 q_{235} - f_0 g_0 q_{235} = 0,\ \ g_0^2 q_{236} - f_0 g_0 q_{236} = 0,\\
&g_0^2 q_{245} - f_0 g_0 q_{245} = 0,\ \ g_0^2 q_{256} - f_0 g_0 q_{256} = 0,\ \ g_0^2 q_{345} - f_0 g_0 q_{345} = 0,\\
&g_0^2 q_{346} - f_0 g_0 q_{346} = 0,\ \ g_0^2 q_{456} - f_0 g_0 q_{456} = 0,\ \ g_0^2 q_{123456} - f_0 g_0 q_{123456} = 0.
\end{aligned} \tag{36}$$

$$\begin{aligned}
&g_0^2 p_{123} - f_0 g_0 p_{123} = 0,\ \ g_0^2 p_{145} - f_0 g_0 p_{145} = 0,\ \ g_0^2 p_{246} - f_0 g_0 p_{246} = 0,\\
&g_0^2 p_{356} - f_0 g_0 p_{356} = 0,\ \ g_0^2 p_{124} - f_0 g_0 p_{124} = 0,\ \ g_0^2 p_{125} - f_0 g_0 p_{125} = 0,\\
&g_0^2 p_{126} - f_0 g_0 p_{126} = 0,\ \ g_0^2 p_{134} - f_0 g_0 p_{134} = 0,\ \ g_0^2 p_{135} - f_0 g_0 p_{135} = 0,\\
&g_0^2 p_{136} - f_0 g_0 p_{136} = 0,\ \ g_0^2 p_{146} - f_0 g_0 p_{146} = 0,\ \ g_0^2 p_{156} - f_0 g_0 p_{156} = 0,\\
&g_0^2 p_{234} - f_0 g_0 p_{234} = 0,\ \ g_0^2 p_{235} - f_0 g_0 p_{235} = 0,\ \ g_0^2 p_{236} - f_0 g_0 p_{236} = 0,\\
&g_0^2 p_{245} - f_0 g_0 p_{245} = 0,\ \ g_0^2 p_{256} - f_0 g_0 p_{256} = 0,\ \ g_0^2 p_{345} - f_0 g_0 p_{345} = 0,\\
&g_0^2 p_{346} - f_0 g_0 p_{346} = 0,\ \ g_0^2 p_{456} - f_0 g_0 p_{456} = 0,\ \ g_0^2 p_{123456} - f_0 g_0 p_{123456} = 0.
\end{aligned} \tag{37}$$

$$\begin{aligned}
&f_0^2 q_{123} - g_0 f_0 q_{123} = 0,\ \ f_0^2 q_{145} - g_0 f_0 q_{145} = 0,\ \ f_0^2 q_{246} - g_0 f_0 q_{246} = 0,\\
&f_0^2 q_{356} - g_0 f_0 q_{356} = 0,\ \ f_0^2 q_{124} - g_0 f_0 q_{124} = 0,\ \ f_0^2 q_{125} - g_0 f_0 q_{125} = 0,\\
&f_0^2 q_{126} - g_0 f_0 q_{126} = 0,\ \ f_0^2 q_{134} - g_0 f_0 q_{134} = 0,\ \ f_0^2 q_{135} - g_0 f_0 q_{135} = 0,\\
&f_0^2 q_{136} - g_0 f_0 q_{136} = 0,\ \ f_0^2 q_{146} - g_0 f_0 q_{146} = 0,\ \ f_0^2 q_{156} - g_0 f_0 q_{156} = 0,\\
&f_0^2 q_{234} - g_0 f_0 q_{234} = 0,\ \ f_0^2 q_{235} - g_0 f_0 q_{235} = 0,\ \ f_0^2 q_{236} - g_0 f_0 q_{236} = 0,\\
&f_0^2 q_{245} - g_0 f_0 q_{245} = 0,\ \ f_0^2 q_{256} - g_0 f_0 q_{256} = 0,\ \ f_0^2 q_{345} - g_0 f_0 q_{345} = 0,\\
&f_0^2 q_{346} - g_0 f_0 q_{346} = 0,\ \ f_0^2 q_{456} - g_0 f_0 q_{456} = 0,\ \ f_0^2 q_{123456} - g_0 f_0 q_{123456} = 0.
\end{aligned} \tag{38}$$

$$\begin{gathered}
f_0^2 p_{123} - g_0 f_0 p_{123} = 0, \quad f_0^2 p_{145} - g_0 f_0 p_{145} = 0, \quad f_0^2 p_{246} - g_0 f_0 p_{246} = 0, \\
f_0^2 p_{356} - g_0 f_0 p_{356} = 0, \quad f_0^2 p_{124} - g_0 f_0 p_{124} = 0, \quad f_0^2 p_{125} - g_0 f_0 p_{125} = 0, \\
f_0^2 p_{126} - g_0 f_0 p_{126} = 0, \quad f_0^2 p_{134} - g_0 f_0 p_{134} = 0, \quad f_0^2 p_{135} - g_0 f_0 p_{135} = 0, \\
f_0^2 p_{136} - g_0 f_0 p_{136} = 0, \quad f_0^2 p_{146} - g_0 f_0 p_{146} = 0, \quad f_0^2 p_{156} - g_0 f_0 p_{156} = 0, \\
f_0^2 p_{234} - g_0 f_0 p_{234} = 0, \quad f_0^2 p_{235} - g_0 f_0 p_{235} = 0, \quad f_0^2 p_{236} - g_0 f_0 p_{236} = 0, \\
f_0^2 p_{245} - g_0 f_0 p_{245} = 0, \quad f_0^2 p_{256} - g_0 f_0 p_{256} = 0, \quad f_0^2 p_{345} - g_0 f_0 p_{345} = 0, \\
f_0^2 p_{346} - g_0 f_0 p_{346} = 0, \quad f_0^2 p_{456} - g_0 f_0 p_{456} = 0, \quad f_0^2 p_{123456} - g_0 f_0 p_{123456} = 0, \qquad (39)
\end{gathered}$$

it follows that $g_0 = f_0$.

4.3. Case III. From the case of II it is clear $b_0^2 = 0$, $c_0^2 = 0$, then $b_0 = 0$, $c_0 = 0$

$$\begin{aligned}
& f_0(q_{123}g_{456} + q_{456}g_{123} + q_{124}g_{356} + q_{356}g_{124} + q_{125}g_{346} + q_{346}g_{125} + q_{126}g_{345} + q_{345}g_{126} + \\
& + q_{134}g_{256} + q_{256}g_{134} + q_{135}g_{246} + q_{246}g_{135} + q_{136}g_{245} + q_{245}g_{136} + q_{145}g_{236} + q_{236}g_{145} + \\
& + q_{146}g_{235} + q_{235}g_{146} + q_{156}g_{234} + q_{234}g_{156} + q_{123456}g_0 + q_0 g_{123456}) - \\
& - g_0(f_{125}q_{346} + f_{346}q_{125} + f_{124}q_{356} + f_{356}q_{124} + f_{125}q_{346} + f_{346}q_{125} + f_{126}q_{345} + f_{345}q_{126} + \\
& + f_{134}q_{256} + f_{256}q_{134} + f_{135}q_{246} + f_{246}q_{135} + f_{136}q_{245} + f_{245}q_{136} - f_{145}q_{236} + f_{236}q_{145} + \\
& + f_{146}q_{235} + f_{235}q_{146} + f_{156}q_{234} + f_{156}q_{234} + f_{123456}q_0 + f_0 q_{123456}) = 0, \qquad (40) \\
& f_0(p_{123}g_{456} + p_{456}g_{123} + p_{124}g_{356} + p_{356}g_{124} + p_{125}g_{346} + p_{346}g_{125} + p_{126}g_{345} + p_{345}g_{126} + \\
& + p_{134}g_{256} + p_{256}g_{134} + p_{135}g_{246} + p_{246}g_{135} + p_{136}g_{245} + p_{245}g_{136} + p_{145}g_{236} + p_{236}g_{145} + \\
& + p_{146}g_{235} + p_{235}g_{146} + p_{156}g_{234} + p_{234}g_{156} + p_{123456}g_0 + p_0 g_{123456}) - \\
& - g_0(f_{125}p_{346} + f_{346}p_{125} + f_{124}p_{356} + f_{356}p_{124} + f_{125}p_{346} + f_{346}p_{125} + f_{126}p_{345} + f_{345}p_{126} + \\
& + f_{134}p_{256} + f_{256}p_{134} + f_{135}p_{246} + f_{246}p_{135} + f_{136}p_{245} + f_{245}p_{136} - f_{145}p_{236} + f_{236}p_{145} + \\
& + f_{146}p_{235} + f_{235}p_{146} + f_{156}p_{234} + f_{156}p_{234} + f_{123456}p_0 + f_0 p_{123456}) = 0, \qquad (41)
\end{aligned}$$

it follows that

$$f_0 = \frac{1}{\widetilde{f}_n \widetilde{p}_k + \widetilde{f}_k \widetilde{p}_n - \widetilde{p}_n \widetilde{g}_k - \widetilde{p}_k \widetilde{g}_n}, \qquad (42)$$

$$g_0 = \frac{1}{\widetilde{q}_n \widetilde{g}_k + \widetilde{q}_k \widetilde{g}_n - \widetilde{f}_n \widetilde{q}_k - \widetilde{f}_k \widetilde{q}_n)}. \qquad (43)$$

Substituting (42) and (43) in the remaining equations and as a result we get a series of expressions

$$f_0 = g_0 = \frac{1}{\widetilde{a}_n \widetilde{g}_k + \widetilde{a}_k \widetilde{g}_n - \widetilde{f}_n a_k - \widetilde{f}_k \widetilde{a}_n}, \qquad (44)$$

$$f_0 = g_0 = \frac{1}{\widetilde{f}_n \widetilde{a}_k + \widetilde{f}_k \widetilde{a}_n - \widetilde{a}_n \widetilde{g}_k - \widetilde{a}_k \widetilde{g}_n}, \qquad (45)$$

$$f_0 = g_0 = \widetilde{b}_n \widetilde{g}_k - \widetilde{b}_k \widetilde{g}_n, \qquad (46)$$

$$f_0 = g_0 = \widetilde{b}_n \widetilde{f}_k - \widetilde{f}_n \widetilde{b}_k, \qquad (47)$$

$$f_0 = g_0 = \widetilde{c}_n \widetilde{g}_k - \widetilde{g}_n \widetilde{c}_k, \qquad (48)$$

$$f_0 = g_0 = \widetilde{b}_n \widetilde{g}_k - \widetilde{b}_k \widetilde{c}_n. \qquad (49)$$

From the cases of I, II, III it is clear that $a_0 = b_0 = c_0 = d_0 = p_0 = q_0 = 0$, then the coefficients remain uncertain $f_0, g_0, a_n, b_n, c_n, d_n, p_n, q_n, f_n, g_n, a_k, b_k, c_k, d_k, p_k, q_k, f_k, g_k,$ where

$\widetilde{f}, \widetilde{g}, \widetilde{q}, \widetilde{p}, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$ - even nilponent elements, $n = 123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 123456$; $k = 456, 356, 346, 345, 256, 246, 245, 236, 235, 234, 123456$. The solution will be 170 - parametric $f_0 = y, g_0 = w, a_n = h_1, b_n = m_1, c_n = z_1, d_n = s_1, p_n = y_1, q_n = \omega_1, f_n = \lambda_1, g_n = \eta_1, a_k = h_2, b_k = m_2, c_k = z_2, d_k = s_2, p_k = y_2, q_k = \omega_2, f_k = \lambda_2, g_k = \eta_2$ and R - matrix can be represented as

$$R = \begin{pmatrix} y\zeta & \cdot & \cdot & t+\zeta \\ \cdot & z\zeta & s\zeta & \\ \cdot & h\zeta & m\zeta & \cdot \\ r+\zeta & \cdot & \cdot & w\zeta \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \cdot & t \\ \cdot & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \cdot \\ r & \cdot & \cdot & 0 \end{pmatrix} + \begin{pmatrix} y & \cdot & \cdot & \lambda \\ \cdot & z & s & \cdot \\ \cdot & h & m & \cdot \\ \eta & \cdot & \cdot & w \end{pmatrix} \zeta, \quad (50)$$

where $\zeta = \xi_1\xi_2\xi_3 + \xi_1\xi_4\xi_5 + \xi_2\xi_4\xi_6 + \xi_3\xi_5\xi_6 + \xi_1\xi_2\xi_3\xi_4\xi_5\xi_6$. This solution is an 8 - vertex model with two reversible elements.

5. Conclusion

This article considered a three-dimensional generalization of the solution of the Yang-Baxter equation for the eight-vertex model, used to describe the two-parameter quantum plane [27] and a special kind of quantum gates [28, 29]. As a result, a series of solutions of the tetrahedron equation is given as a sufficient condition for the permutation of the transition matrices of the eight-vertex model on a simple square lattice. Special cases of the solution are considered, when the R - matrix over the ordinary number field can have at the same time no more than 7 nonzero elements [17, 30], in our case over the Grassmann algebra all 8 elements can be non-zero. A new kind of solution appears, which is absent in the standard case [17, 30] - the full 8 - vertex solution.

Acknowledgments The work was carried out with the financial support of the Ministry of Education and Science of the Republic of Kazakhstan, Grant No.0118RK00935.

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