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Geometric Flows of Curves, Two-Component Camassa-Holm Equation and Generalized Heisenberg Ferromagnet Equation

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Abstract. In this paper, we study the generalized Heisenberg ferromagnet equation, namely, the M-CVI equation. This equation is integrable. The integrable motion of the space curves induced by the M-CVI equation is presented. Using this result, the Lakshmanan (geometrical) equivalence between the M-CVI equation and the two-component Camassa-Holm equation is established.

1. Introduction

The celebrated Camassa-Holm equation (CHE) has the form

$$u_t + \kappa\, u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + u u_{xxx}, \tag{1.1}$$

where $u = u(x, t)$ is the fluid velocity in the x direction and $\kappa = const$ is related to the critical shallow water wave speed. The CHE shares most of the important properties of integrable equations like the N-soliton solutions, the bi-Hamiltonian structure, the Lax representation (LR) and so on. In the case, when $\kappa = 0$, the CHE (1.1) has the so-called peakon solutions.

Several important generalizations of the CHE including integrable cases but many other (non-integrable or whose integrability has not been determined) have been discovered [1]-[19]. In particular, the two-component CHE (2-CHE) was constructed. Our main interest in this paper is to go further with the investigation initiated in our previous papers (see, e.g. [20]-[21]). In this paper, we study the 2-CHE, its relation with the geometry of space curves and the equivalent spin system.

The paper is organized as follows. In Section 2 we present main fact for the M-CVI equation. Basic information on the 2-CHE we give in Section 3. The integrable motion of space curves induced by the M-CVI equation and the 2-CHE is studied in Section 4. In Section 5, we consider the gauge equivalence between the M-CVI equation and the 2-CHE. Finally, in Section 6 we present a discussion of our achievements and how they impact some recent results found in the recent literature.

2. M-CVI equation

There are several integrable and non-integrable generalized Heisenberg ferromagnet equations (gHFE) (see, e.g. [20]-[21]). In this paper, we consider one of the gHFE, namely, the M-CVI equation. The M-CVI equation is integrable. It shares most of the important properties of integrable systems like the Lax representation (LR), the bi-hamiltonian structure, the N-soliton solutions, infinite hierarchy of symmetries and conservation laws and so on. The M-CVI equation also can admits the so-called peakon solutions.

2.1. Equation

Consider the M-CVI equation

$$[A, A_{xt} + (uA_x)_x] - \frac{1}{\beta^2}A_x - 4\beta\rho\rho_x Z = 0. \tag{2.1}$$

Here $m = \frac{\det(A_x^2)}{4\beta^2} = u - u_{xx}$, $\rho^2 = -\frac{tr(A_x^2) + 2det(A_x)}{8\beta^4}$, $u = 0.25\beta^{-2}(1 - \partial_x^2)^{-1}\det(A_x^2)$ are some real functions, $\beta = const$ and

$$Z = \frac{0.5\beta}{u_x + u_{xx}}[A, A_t + (u - 0.5\beta^{-2})A_x], \quad \mathbf{A} = (A_1, A_2, A_3), \tag{2.2}$$

$$A = \begin{pmatrix} A_3 & A^- \\ A^+ & -A_3 \end{pmatrix}, \quad A^{\pm} = A_1 \pm iA_2, \quad A^2 = I, \quad \mathbf{A}^2 = 1. \tag{2.3}$$

2.2. Lax representation

The LR of the M-CVI equation reads as

$$\Psi_x = U_1\Psi, \tag{2.4}$$

$$\Psi_s = V_1\Psi. \tag{2.5}$$

Here

$$U_1 = \left(\frac{\lambda}{4\beta} - \frac{1}{4}\right)[A, A_x] + (\lambda^3 - \beta^2\lambda)\rho^2 Z, \tag{2.6}$$

$$V_1 = \left(\frac{1}{4\beta^2} - \frac{1}{4\lambda^2}\right)A + \frac{u}{4}\left(\frac{\beta}{\lambda} - \frac{\lambda}{\beta}\right)[A, A_x] + \left(\frac{\beta}{4\lambda} - \frac{1}{4}\right)[A, A_t] + v\rho^2 Z \tag{2.7}$$

where $v = \lambda(0.5 + \beta^2 u) - \lambda^3 u - 0.5\beta^2\lambda^{-1}$. The compatibility condition $U_{1t} - V_{1x} + [U_1, V_1] = 0$ is equivalent to the M-CVI equation (2.1).

2.3. Reductions

One of the reductions of the M-CVI equation is the so-called M-CIV equation. Let $\rho = 0$. Then the M-CVI equation takes the form

$$[A, A_{xt} + (uA_x)_x] - \frac{1}{\beta^2} A_x = 0. \tag{2.8}$$

It is nothing but the M-CIV equation (see, e.g. [?]-[21]). The LR of the M-CIV equation follows from the LR of the M-CVI equation (2.4)-(2.5) as $\rho = 0$. So that the LR of the M-CIV equation is given by [?]

$$\Psi_x = U_3\Psi, \tag{2.9}$$

$$\Psi_s = V_3\Psi, \tag{2.10}$$

where

$$U_3 = \left(\frac{\lambda}{4\beta} - \frac{1}{4}\right)[A, A_x], \tag{2.11}$$

$$V_3 = \left(\frac{1}{4\beta^2} - \frac{1}{4\lambda^2}\right)A + \frac{u}{4}\left(\frac{\beta}{\lambda} - \frac{\lambda}{\beta}\right)[A, A_x] + \left(\frac{\beta}{4\lambda} - \frac{1}{4}\right)[A, A_t]. \tag{2.12}$$

3. 2-CHE

3.1. Equation

The two-component CHE (2-CHE) is given by [1]

$$m_t + um_x + 2mu_x - \rho\rho_x = 0, \tag{3.1}$$

$$\rho_t + (\rho u)_x = 0, \tag{3.2}$$

where $m = u - u_{xx} + 0.5\kappa$. If $\rho = 0$, the 2-CHE reduces to the CHE (1.1).

3.2. Lax representation

The LR of the 2-CHE is given by (see, e.g. [1])

$$\phi_{xx} = (\frac{1}{4} - m\varsigma + \rho^2\varsigma^2)\phi, \tag{3.3}$$

$$\phi_t = -(\frac{1}{2\varsigma} + u)\phi_x + \frac{u_x}{2}\phi, \tag{3.4}$$

where ς is a spectral parameter and $m = u - u_{xx} + 0.5\kappa \quad (\kappa = const)$.

3.3. Reciprocal transformation

From the equation (3.2) follows that the 1-form

$$\omega = \rho\,dx - \rho\,u\,dt \tag{3.5}$$

is closed. This means that we can define a reciprocal transformation $(x, t) \mapsto (y, s)$ by the relation [1]

$$dy = \rho\,dx - \rho\,u\,dt, \quad ds = dt. \tag{3.6}$$

So we obtain

$$\frac{\partial}{\partial x} = \rho\,\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial s} - \rho\,u\,\frac{\partial}{\partial y}. \tag{3.7}$$

Then the spectral problem (3.3)-(3.4) takes the form [1]

$$\varphi_{yy} = \left(\lambda^2 - P\,\lambda - Q\right)\varphi, \tag{3.8}$$

$$\varphi_s = -\frac{\rho}{2\lambda}\varphi_y + \frac{\rho_y}{4\lambda}\varphi, \tag{3.9}$$

where

$$\varphi = \sqrt{\rho}\,\phi, \quad P = \frac{m}{\rho^2}, \quad Q = -\frac{1}{4\rho^2} - \frac{\rho_{yy}}{2\rho} + \frac{\rho_y^2}{4\rho^2}. \tag{3.10}$$

The compatibility condition of the equations (3.8)-(3.9) gives

$$P_s = \rho_y,, \tag{3.11}$$

$$Q_s + \frac{1}{2}\rho\,P_y + P\,\rho_y = 0, \tag{3.12}$$

$$\frac{1}{2}\rho\,Q_y + Q\rho_y + \frac{1}{4}\rho_{yyy} = 0. \tag{3.13}$$

Hence we get the equation [1]

$$\rho^2\,Q + \frac{1}{2}\rho\,\rho_{yy} - \frac{1}{4}\rho_y^2 = C = -\frac{1}{4}. \tag{3.14}$$

From the equation (3.11) follows

$$P = \frac{\partial f(y,s)}{\partial y}, \quad \rho = \frac{\partial f(y,s)}{\partial s}, \tag{3.15}$$

where $f(y,s)$ is some function. This function satisfies the equation [1]

$$\frac{f_{ss}}{2f_s^3} + f_y f_{ys} - \frac{f_{ss}f_{ys}^2}{2f_s^3} + \frac{f_{ys}f_{yss}}{2f_s^2} + \frac{1}{2}f_s f_{yy} + \frac{f_{ss}f_{yys}}{2f_s^2} - \frac{f_{yyss}}{2f_s} = 0. \tag{3.16}$$

Finally we come to the following theorems [1]

Theorem 3.1 *Let f be a solution of the equation (3.16), and*

$$u = f_y f_s^2 + \frac{f_{ss}f_{ys}}{f_s} - f_{yss}, \quad \rho = f_s. \tag{3.17}$$

If $x(y,s)$ is a solution of the following system of ODEs:

$$\frac{\partial x}{\partial y} = \frac{1}{\rho}, \quad \frac{\partial x}{\partial s} = u, \tag{3.18}$$

then $(u(y,t), \rho(y,t), x(y,t))$ is a parametric solution of the 2-CHE (3.1)-(3.2).

Theorem 3.2 *Let $f(y,s)$ be a solution of the equation (3.16). Define the functions $x = x(y,s), u = u(y,s), \rho = \rho(y,s)$ by*

$$x = f(s,y), \quad u = \frac{\partial x}{\partial s}, \quad \frac{1}{\rho} = \frac{\partial x}{\partial y}. \tag{3.19}$$

Then $(u(y,t), \rho(y,t), x(y,t))$ is a parametric solution of the 2-CHE (3.1)-(3.2).

3.4. Relations to the first negative flow of the AKNS hierarchy

Let us briefly present the well-known result about relation between the 2-CHE and the AKNS spectral problem following the paper [1]. The AKNS spectral problem reads as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_y = \begin{pmatrix} \lambda & -q \\ r & -\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{3.20}$$

The first negative flow of the AKNS problem is given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_s = \frac{1}{4\lambda} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{3.21}$$

The compatibility condition of the equations (3.20)-(3.21) gives

$$q_s = \frac{1}{2} b, \tag{3.22}$$

$$r_s = \frac{1}{2} c, \tag{3.23}$$

$$b_y = 2 a q, \tag{3.24}$$

$$c_y = 2 a r, \tag{3.25}$$

$$a_y + b r + c q = 0. \tag{3.26}$$

Hence we get the condition [1]

$$a^2 + b c = \varepsilon^2, \tag{3.27}$$

where $\varepsilon = const$. We have the following theorems [1]:

Theorem 3.3 *Let (a, b, c, q, r) be a solution of the equations (3.23)–(3.26) with $\varepsilon^2 = 1$, then any function $f(y, s)$ satisfying*

$$2a = b e^{-f} - c e^f \tag{3.28}$$

gives a primary solution of the 2-CH system.

Theorem 3.4 *If f is a primary solution of the 2-CHE system (3.1)-(3.2), then we can construct a solution of the first negative flow of the AKNS hierarchy by the following formulae*

$$q = \frac{e^f}{2} \left(f_y + \frac{\varepsilon - f_{ys}}{f_s} \right), \ r = \frac{e^{-f}}{2} \left(f_y - \frac{\varepsilon - f_{ys}}{f_s} \right), \ b = 2 q_s, \ c = 2 r_s, \ a = \frac{b e^{-f} - c e^f}{2}. \tag{3.29}$$

where $\varepsilon = 1$ or $\varepsilon = -1$.

3.5. Bi-Hamiltonian structure

Let us again briefly present some basic facts about the bi-Hamiltonian structure of the 2-CHE following the paper [1]. Note that both bi-Hamiltonian structures of the CHE and the KdV hierarchies are deformations of the following bi-Hamiltonian structure of hydrodynamic type [1]

$$\{u(x), u(y)\}_1 = \delta'(x - y),$$

$$\{u(x), u(y)\}_2 = u(x)\delta'(x - y) + \frac{1}{2} u(x)'\delta(x - y). \tag{3.30}$$

This means that the dispersionless limits of the CHE and KdV hierarchies have the same forms. One of the main features of the integrable hierarchies that correspond to bi-Hamiltonian structures with constant central invariants is the existence of τ-functions. It is well known that integrable hierarchies of nonlinear partial differential equations with one spatial variable possess bi-Hamiltonian structures that are deformations of bi-Hamiltonian structure of hydrodynamic type with constant central invariants, and the existence of τ-functions plays an important role in the study of these integrable systems. The CHE hierarchy is an exceptional example of integrable systems which does not possess τ-functions.

4. Motion of space curves induced by the M-CIV equation. Lakshmanan (geometrical) equivalence

In this section, we would like to find the integrable motion of the space curves induced by the M-CVI equation. To do that, let us consider a smooth space curve in R^3 given by

$$\gamma(x,t) : [0, X] \times [0, T] \to R^3, \tag{4.1}$$

where x is the arc length of the curve at each time t. Then the following three vectors

$$\mathbf{e}_1 = \gamma_x, \quad \mathbf{e}_2 = \frac{\gamma_{xx}}{|\gamma_{xx}|}, \quad \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2, \tag{4.2}$$

are the unit tangent vector, the principal normal vector and the binormal vector of the curve, respectively. The corresponding Frenet-Serret equation is given by

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \tau \\ -\kappa_2 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \tag{4.3}$$

where τ, κ_1 and κ_2 are torsion, geodesic curvature and normal curvature of the curve, respectively. Let the deformation of the curves are given by

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = G \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \tag{4.4}$$

Here

$$C = -\tau L_1 + \kappa_2 L_2 - \kappa_1 L_3 = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \tau \\ -\kappa_2 & -\tau & 0 \end{pmatrix}, \tag{4.5}$$

$$G = -\omega_1 L_1 + \omega_2 L_2 - \omega_3 L_3 = \begin{pmatrix} 0 & \omega_3 & \omega_2 \\ -\omega_3 & 0 & \omega_1 \\ -\omega_2 & -\omega_1 & 0 \end{pmatrix}, \tag{4.6}$$

where

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.7}$$

are basis elements of $so(3)$ algebra. The compatibility condition of the equations (4.4) has the form

$$C_t - G_x + [C, G] = 0 \tag{4.8}$$

or in components

$$\kappa_{1t} - \omega_{3x} - \kappa_2\omega_1 + \tau\omega_2 = 0, \tag{4.9}$$

$$\kappa_{2t} - \omega_{2x} + \kappa_1\omega_1 - \tau\omega_3 = 0, \tag{4.10}$$

$$\tau_t - \omega_{1x} - \kappa_1\omega_2 + \kappa_2\omega_3 = 0. \tag{4.11}$$

As usual, we consider the following identification $\mathbf{A} \equiv \mathbf{e}_1$. We have

$$\kappa_1 = i, \quad \kappa_2 = \lambda(m-1) + \lambda^3\rho^2, \quad \tau = -i[\lambda(m+1) + \lambda^3\rho^2], \tag{4.12}$$

where $\kappa_2 + i\tau = 2m\lambda + 2\rho^2\lambda^3$ and $\kappa_2 - i\tau = -2\lambda$, $\lambda = const$. Finally we get the following expressions for the functions ω_j

$$\omega_1 = i[(u\lambda - 0.5\lambda^{-1})(m+1) - 0.5\lambda^{-1}(u_x + u_{xx}) - 0.5\lambda\rho^2 + \lambda^3 u\rho^2], \tag{4.13}$$

$$\omega_2 = [(0.5\lambda^{-1} - \lambda u)(m-1) + 0.5\lambda^{-1}(u_x + u_{xx}) + 0.5\lambda\rho^2 - \lambda^3 u\rho^2], \tag{4.14}$$

$$\omega_3 = i[0.5\lambda^{-2} - u - u_x]. \tag{4.15}$$

Substituting these expressions to Eqs.(4.9)-(4.11) we get the following equations for m, ρ:

$$m_t + 2u_x m + um_x - \rho\rho_x = 0, \tag{4.16}$$

$$\rho_t + (u\rho)_x = 0, \tag{4.17}$$

$$m - u + u_{xx} - 0.5\kappa = 0, \tag{4.18}$$

which is the 2-CHE. So, we have proved the Lakshmanan (geometrical) equivalence between the M-CVI equation (2.1) and the 2-CHE (3.1)-(3.2).

5. Gauge equivalence

In the previous section, we have shown that the M-CIV equation (2.1) and the 2-CHE (3.1)-(3.2) are the geometrical equivalent each to other. As it was established in [20], between these equations takes place also the gauge equivalence. In fact, consider the gauge transformation $\Phi = g\Psi$, where $g = \Phi|_{\lambda=\beta}$. Then we have

$$U_1 = g^{-1}U_2 g - g^{-1}g_x, \quad V_1 = g^{-1}V_2 g - g^{-1}g_t. \tag{5.1}$$

As result, we get the following LR for the 2-CHE

$$\Phi_x = U_2\Phi, \tag{5.2}$$

$$\Phi_t = V_2\Phi, \tag{5.3}$$

where

$$U_2 = \begin{pmatrix} -0.5 & \lambda \\ m\lambda + \rho^2\lambda^3 & 0.5 \end{pmatrix}, \tag{5.4}$$

$$V_2 = \begin{pmatrix} 0.5(u+u_x) - 0.25\lambda^{-2} & 0.5\lambda^{-1} - u\lambda \\ 0.5(m+u_x+u_{xx})\lambda^{-1} - um + 0.5\rho^2\lambda - u\rho^2\lambda^3 & 0.25\lambda^{-2} - 0.5(u+u_x) \end{pmatrix}. \tag{5.5}$$

The compatibility condition

$$U_{2t} - V_{2x} + [U_2, V_2] = 0 \tag{5.6}$$

gives the 2-CHE.

6. Conclusion

In this paper, we have considered the M-CVI equation and the 2-CHE. First, we have presented some well-known main facts on these equations. In particular, we briefly present the reciprocal transformation between the 2-CHE and the first negative flow of the AKNS hierarchy which includes in particular the well known sine-Grodon and the sinh-Gordon equations. This transformation gives the correspondence between solutions of the first negative flow of the AKNS hierarchy and the 2-CHE. Then we have studied the motion of the space curves induced by these equations. Using this result, we have proved that the M-CVI equation and the 2-CHE is the Lakshmanan (geometrical) equivalent each to other. Last but not least, we would like to note and believe that the "spinalization" of integrable systems gives some new informations on their nature.

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