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Geometric characteristics for Camassa–Holm equation

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Abstract. One of the actual problems of mathematical physics is to relate differential geometry and nonlinear differential equation. Research in this direction is very important, as the results are a theoretical and practical application. In this paper, we investigate the Camassa-Holm equation. It is well known that the integrable nonlinear Camassa-Holm equation play an important role in the study of wave propagation. We present the relationship between Camassa-Holm equation and soliton surfaces. The first and second fundamental forms, surface area and curvature for Camassa-Holm equation are found.

1. Introduction

Some nonlinear partial differential equations are integrable, allow physically interesting exact solutions, moreover, these integrable equations are solvable by the inverse scattering problem method. The study of integrable equations in (1+1)-, (2+1)-measures is relevant from the point of view of mathematical physics [1]-[7]. Integrable equations allow different types of solutions: soliton solution, domain wall solution, vortex solution, etc [8]. Moreover, solutions of integrable equations have geometric characteristics. To study the geometric properties of solutions, the theory of differential geometry of curves and surfaces is applied [9]-[11].

In 1993, Camassa and Holm derived an integrable generalization of the nonlinear equation, which later became known as the Camassa-Holm equation [12]:

$$q_t + 2u_x q + u q_x = 0, \tag{1}$$

where

$$q = u - u_{xx}.$$

Here u is the fluid velocity in the direction x [13].

2. Fundamental form.

In this section, we used the first and second fundamental form for finding soliton surfaces and used the Sym Tafel formula. The Sym Tafel formula gives the connection between the theory of solitons and classical geometry. Finding soliton surfaces is important when solving integrable geometry. Geometrical objects associated with soliton surfaces can be associated with the solutions of some nonlinear models [14]-[17].

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2.1. The first fundamental form

We give the first fundamental surface shape for the Camassa-Holm equation. Equation (1) is fully integrable and admits a Lax pair [18]

$$U = \begin{pmatrix} -\frac{1}{2} & \lambda \\ \lambda q & \frac{1}{2} \end{pmatrix}, \tag{2}$$

$$V = \begin{pmatrix} \frac{1}{2}(u+u_x) - \frac{1}{4\lambda^2} & \frac{1}{2\lambda} - \lambda u \\ \frac{1}{2\lambda}(q+u_x+u_{xx}) - \lambda uq & \frac{1}{4\lambda^2} - \frac{1}{2}(u+u_x) \end{pmatrix}, \tag{3}$$

where λ eigenvalue. The derivatives U from (2) and V from (3) with respect to λ are of the form

$$U_\lambda = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}, \tag{4}$$

$$V_\lambda = \begin{pmatrix} \frac{1}{2\lambda^3} & -\frac{1}{2\lambda^2} - u \\ -\frac{1}{2\lambda^2}(q+u_x+u_{xx}) - uq & -\frac{1}{2\lambda^3} \end{pmatrix}. \tag{5}$$

Zero curvature conditions for equation (1) is:

$$U_t - V_x + [U,V] = 0, \tag{6}$$

where $[U,V] = UV - VU$, the matrices U and V are given in (2)-(3) [19].

In addition, a nonlinear partial differential equation (6) is a compatibility condition for a system of linear equations:

$$\Phi_x = U\Phi,$$

$$\Phi_t = V\Phi.$$

Using the Sym-Tafel formula

$$r = \Phi^{-1}\Phi_\lambda,$$

we define the following formulas:

$$r_x = \Phi^{-1}U_\lambda\Phi, \tag{7}$$

$$r_t = \Phi^{-1}V_\lambda\Phi. \tag{8}$$

The first quadratic form defines the internal geometry of the surface in the vicinity of a given point. It is often denoted as I. For the Camassa-Holm equation, the first fundamental surface shape is defined as [20]

$$I = \vec{dr}\cdot\vec{dr} = Edx^2 + 2Fdxdt + Gdt^2. \tag{9}$$

Relations between the derivatives of a vector and the matrix form r with respect to x and t:

$$E = \vec{r}_x^2 = \frac{1}{2}tr\left(r_x^2\right), \tag{10}$$

$$F = \vec{r}_x\vec{r}_t = \frac{1}{2}tr\left(r_x r_t\right), \tag{11}$$

$$G = \vec{r}_t^2 = \frac{1}{2} tr\left(r_t^2\right). \tag{12}$$

where r_x and r_t are some matrices.
From (7) and (8) we can obtain

$$r_x^2 = \Phi^{-1} U_\lambda^2 \Phi, \tag{13}$$

$$r_t^2 = \Phi^{-1} V_\lambda^2 \Phi, \tag{14}$$

$$r_x r_t = \Phi^{-1} U_\lambda V_\lambda \Phi, \tag{15}$$

where the matrix trace has the form

$$trU_\lambda^2 = 2q,$$
$$trV_\lambda^2 = \frac{1}{2\lambda^6} + 2\left(-\frac{1}{2\lambda^2}\left(q + u_x + u_{xx}\right) - uq\right)\cdot\left(-\frac{1}{2\lambda^2} - u\right),$$
$$trU_\lambda V_\lambda = -\frac{1}{2\lambda^2}\left(q + u_x + u_{xx}\right) - uq + q\left(-\frac{1}{2\lambda^2} - u\right).$$

Taking into account (13), (14), (15), we have

$$\vec{r}_x^2 = q, \tag{16}$$

$$\vec{r}_x \vec{r}_t = -\frac{1}{4\lambda^2}\left(q + u_x + u_{xx}\right) - \frac{uq}{2} + \frac{q}{2}\left(-\frac{1}{2\lambda^2} - u\right), \tag{17}$$

$$\vec{r}_t^2 = \frac{1}{4\lambda^6} + \left(-\frac{1}{2\lambda^2} - u\right)\left(\frac{1}{2\lambda^2}(q + u_x + u_{xx}) - uq\right). \tag{18}$$

By substituting (16), (17), (18) into (9) we obtain the first fundamental form for the Camassa-Holm equation:

$$I = qdx^2 + \left(-\frac{1}{2\lambda^2}(q + u_x + u_{xx}) - uq + q\left(-\frac{1}{2\lambda^2} - u\right)\right)dxdt + \\ + \left(\frac{1}{4\lambda^6} + \left(-\frac{1}{2\lambda^2} - u\right)\left(\frac{1}{2\lambda^2}(q + u_x + u_{xx}) - uq\right)\right)dt^2. \tag{19}$$

Knowing the first quadratic shape of the surface, we can calculate the lengths of the curves on the surface, the angles between the curves and the area of the regions on the surface.

2.2. The second fundamental form

The second fundamental surface shape for the Camassa-Holm equation has the form [21]

$$II = \vec{dr}\cdot\vec{dn} = Ldx^2 + 2Mdxdt + Ndt^2. \tag{20}$$

where

$$L = \vec{r}_{xx}\cdot\vec{n},$$
$$M = \vec{r}_{xt}\cdot\vec{n},$$

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$$N = \vec{r}_{tt} \cdot \vec{n}.$$

Using the Sym-Tafel formula, we get

$$\begin{aligned} r_{xx} &= \Phi^{-1} U_{\lambda x} \Phi + \Phi^{-1} \left[U_\lambda, U\right] \Phi, && (21) \\ r_{xt} &= \Phi^{-1} U_{\lambda t} \Phi + \Phi^{-1} \left[U_\lambda, V\right] \Phi, && (22) \\ r_{tt} &= \Phi^{-1} V_{\lambda t} \Phi + \Phi^{-1} \left[V_\lambda, V\right] \Phi, && (23) \end{aligned}$$

where are the matrices

$$U_{\lambda x} = \begin{pmatrix} 0 & 0 \\ q_x & 0 \end{pmatrix},$$

$$U_{\lambda t} = \begin{pmatrix} 0 & 0 \\ q_t & 0 \end{pmatrix},$$

$$V_{\lambda t} = \begin{pmatrix} 0 & -u_t \\ -\frac{1}{2\lambda^2}\left(u_x + u_{xx}\right) - u_t q - u q_t & 0 \end{pmatrix},$$

$$\left[U_\lambda, U\right] = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix},$$

$$\left[U_\lambda, V\right] = \begin{pmatrix} \frac{1}{2\lambda}\left(u_x + u_{xx}\right) & \frac{1}{2\lambda^2} - \left(u + u_x\right) \\ q\left(u + u_x\right) - \frac{q}{2\lambda^2} & -\frac{1}{2\lambda}\left(u_x + u_{xx}\right) \end{pmatrix},$$

$$\left[V_\lambda, V\right] = \begin{pmatrix} -\frac{u}{\lambda}\left(2q + u_x + u_{xx}\right) & \frac{1}{2\lambda^4} - \frac{u}{\lambda^2} \\ \frac{1}{4\lambda^4}\left(q + u_x + u_{xx} + 2\lambda^2 u q\right)\left(2\lambda^2\left(u + u_x\right) + 1\right) & \frac{u}{\lambda}\left(2q + u_x + u_{xx}\right) \end{pmatrix}.$$

To determine the normal n to the surface, we use the following formula

$$n = \frac{\Phi^{-1}\left[U_\lambda, V_\lambda\right]\Phi}{\sqrt{\frac{1}{2} tr\left(\left[U_\lambda, V_\lambda\right]^2\right)}}, \qquad (24)$$

where

$$\left[U_\lambda, V_\lambda\right] = \begin{pmatrix} -\frac{1}{2\lambda^2}\left(u_x + u_{xx}\right) & -\frac{1}{\lambda^3} \\ \frac{q}{\lambda^3} & \frac{1}{2\lambda^2}\left(u_x + u_{xx}\right) \end{pmatrix},$$

$$\left(\left[U_\lambda, V_\lambda\right]\right)^2 = \begin{pmatrix} \frac{1}{4\lambda^4}\left(u_x + u_{xx}\right)^2 - \frac{q}{\lambda^6} & 0 \\ 0 & -\frac{q}{\lambda^6} + \frac{1}{4\lambda^4}\left(u_x + u_{xx}\right)^2 \end{pmatrix}.$$

The relations between the derivatives of the vector and the matrix form r with respect to x and t are of the form:

$$\begin{aligned} L = \vec{r}_{xx} \cdot \vec{n} &= \frac{1}{2} tr\left(r_{xx} \cdot n\right), && (25) \\ M = \vec{r}_{xt} \cdot \vec{n} &= \frac{1}{2} tr\left(r_{xt} \cdot n\right), && (26) \\ N = \vec{r}_{tt} \cdot \vec{n} &= \frac{1}{2} tr\left(r_{tt} \cdot n\right). && (27) \end{aligned}$$

From (25), (26), (27) we have

$$tr\left(r_{xx}\cdot n\right)=\frac{2\left(2q-q_x\right)}{\sqrt{\lambda^2\left(u_x+u_{xx}\right)^2-4q}}, \tag{28}$$

$$tr\left(r_{xt}\cdot n\right)=\frac{2q-4q\lambda^2\left(u+u_x\right)-\lambda^2(u_x^2+u_{xx}^2)-2\lambda^2\left(u_xu_{xx}+q_t\right)}{\lambda^2\sqrt{\lambda^2\left(u_x+u_{xx}\right)^2-4q}}, \tag{29}$$

$$tr\left(r_{tt}\cdot n\right)=\frac{4\left(\lambda^4u\cdot a+b\lambda^2+\frac{q}{4}+\frac{u_x}{8}+\frac{u_{xx}}{8}\right)}{\lambda^4\sqrt{\lambda^2\left(u_x+u_{xx}\right)^2-4q}}, \tag{30}$$

where

$$a=\frac{u_x^2}{2}+(q+u_{xx})u_x+uq+\frac{u_{xx}^2}{2}+\frac{q_t}{2},$$

$$b=\frac{u_x^2}{4}+\left(\frac{q}{2}+\frac{u}{4}+\frac{u_{xx}}{4}+\frac{1}{4}\right)u_x+\left(-q+\frac{u_{xx}}{4}\right)u+\frac{u_{xx}}{4}.$$

Substituting (28), (29), (30) into (20), we obtain the second fundamental surface shape for the Camassa-Holm equation

$$II=\frac{(2q-q_x)}{\sqrt{\lambda^2\left(u_x+u_{xx}\right)^2-4q}}dx^2+$$
$$+\frac{(2q-4q\lambda^2\left(u+u_x\right)-\lambda^2(u_x^2+u_{xx}^2)-2\lambda^2\left(u_xu_{xx}+q_t\right))}{\lambda^2\sqrt{\lambda^2\left(u_x+u_{xx}\right)^2-4q}}dxdt+$$
$$+\frac{2\left(\lambda^4u\cdot a+b\lambda^2+\frac{q}{4}+\frac{u_x}{8}+\frac{u_{xx}}{8}\right)}{\lambda^4\sqrt{\lambda^2\left(u_x+u_{xx}\right)^2-4q}}dt^2. \tag{31}$$

The second quadratic form is a very effective tool for studying the geometric properties of a regular surface.

3. Area of Surfaces

Surfaces area is given in the form [18]

$$S=\int\int\left|\vec{r_x}\times\vec{r_t}\right|dxdt. \tag{32}$$

where $\times$ - a vector product, r_x, r_t-private derivatives by x, t and

$$\left|\vec{r_x}\times\vec{r_t}\right|=\sqrt{\vec{r}_x^2\vec{r}_t^2-(\vec{r_x}\vec{r_t})^2}, \tag{33}$$

$$\left|\vec{r_x}\times\vec{r_t}\right|=\sqrt{EG-F^2}. \tag{34}$$

We can write

$$S=\int\int\sqrt{EG-F^2}dxdt, \tag{35}$$

where

$$E=\frac{1}{2}tr(r_x^2)=\frac{1}{2}tr(U_\lambda^2)=q, \tag{36}$$

$$F=\frac{1}{2}tr(r_{xt})=\frac{1}{2}tr(U_\lambda V_\lambda)=-\frac{1}{4\lambda^2}\left(q+u_x+u_{xx}\right)-\frac{uq}{2}+\frac{q}{2}\left(-\frac{1}{2\lambda^2}-u\right), \tag{37}$$

$$G=\frac{1}{2}tr(r_t^2)=\frac{1}{2}tr(V_\lambda^2)=\frac{1}{4\lambda^6}+\left(-\frac{1}{2\lambda^2}-u\right)\left(\frac{1}{2\lambda^2}(q+u_x+u_{xx})-uq\right). \tag{38}$$

4. Total and Mean Curvatures of a surface

In mathematics, curvature is any of a number of loosely related concepts in different areas of geometry. Intuitively, curvature is the amount by which a geometric object such as a surface deviates from being a flat plane, or a curve from being straight as in the case of a line, but this is defined in different ways depending on the context. In studying the properties of regular surfaces, the concepts of average surface curvature and Gaussian curvature are widely used. The average curvature of the surface at a given point is the half-sum of its main curvatures

$$H = \frac{1}{2}(k_1 + k_2). \tag{39}$$

The Gaussian curvature of a surface is the product of its principal curvatures

$$K = k_1 k_2, \tag{40}$$

using the properties of the roots of the quadratic equation, we obtain the following formulas for the average curvature H and the Gaussian curvature K:

$$K = \frac{\det II}{\det I} = \frac{LN - M^2}{EG - F^2}, \tag{41}$$

$$H = \frac{1}{2}\frac{EN + GL - 2FM}{EG - F^2}. \tag{42}$$

5. Conclusion

In this article, we examined the Camassa-Holm equation. For integrability, we have the Lax pair and investigated a one-dimensional surface. The first and second fundamental forms were found by the formula of Sym Tafel. We found the surface area, Gaussian and average surface curvature.

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7. References

[1] Yesmakhanova K, Bekova G., Shaikhova G 2018 *AIP Conf. Proc.* **1997** 1-6
[2] Yesmakhanova K. , Bekova G., Ybyraiymova S., Shaikhova G 2017 *J. Phys.: Conf. Series* **936** 1-6
[3] Bekova G., Yesmakhanova K., Ozat N., Shaikhova G 2018 *J. Phys.: Conf. Series* **965** 1-10
[4] Myrzakulov R, Bekova G., Yesmakhanova K., Shaikhova G 2017 *AIP Conf. Proc.* **1880** 1-7
[5] Yesmakhanova K., Bekova G., Myrzakulov R., Shaikhova G 2017 *J. Phys.: Conf. Series* **804** 1-8
[6] Bekova G., Yesmakhanova K., Myrzakulov R, Shaikhova G 2017 *J. Phys.: Conf. Series* **936** 1-9
[7] Myrzakulov R., Mamyrbekova G., Nugmanova G., Lakshmanan M 2015 *Symmetry* **7** 1352-1375
[8] Ablowitz M J, Clarkson P A 1992 *Cambridge University Press, Cambridge.* **513**
[9] Pressley A 2001 *Springer, London*
[10] Sym A 1982 *Lett. Nuovo Cim.* **33** 394-400.
[11] Sym A 1983 *Lett. Nuovo Cim.* **36** 307-312
[12] Camassa R, Holm D 1661 (1993) *Phys. Rev. Lett.* **71**
[13] Camassa R, Holm D, Hyman J 1994 *Advances in Applied Mechanics* **31** 1-33
[14] Sym A 1984 *Lett. Nuovo Cim.* **39** 193-196
[15] Sym A 1984 *Lett. Nuovo Cim.* **40** 225-231
[16] Myrzakulov R, Danlybaeva A K, Nugmanova G N, 1999 *TMF* **118** 441-451
[17] Qiao Z J 2003 *Commun. Math. Phys.* **239** 309341

[18] Ivanov R, Lyons T, Orr N 2017 *AIP Conf. Proc.* **1895**
[19] Mussatayeva A B 2019 *Conf. Proc.* **603** 298302
[20] Rogers C, Schief W K 2002 *Cambridge University Press* **530** 18-41
[21] Mussatayeva A B, Myrzakulov N A The first and second fundamental surface form for the Camassa-Holm equation *Bulletin of the Eurasian National University named after L.N. Gumilyov. Series Physics. Astronomy* preprint