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Noether symmetry approach in $f(T,B)$ teleparallel gravity with a fermionic field

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Abstract. In this work, we consider a homogeneous and isotropic cosmological model of the universe in $f(T,B)$ gravity with non-minimally coupled fermionic field. In order to find the form of the coupling function $F(\Psi)$, the potential function $V(\Psi)$ of the fermionic field and the function $f(T,B)$, we found through the Noether symmetry approach. The results obtain are coincide with the observational data that describe the late-time accelerated expansion of the universe.

1. Introduction

In modern cosmology, we apply various modified theories of gravity to describe the available observational data. Usually in the literatures we can see modifications of the components of the gravitational field or the field of matter separately, as well as their general modifications in the action of Einstein-Hilbert. The first type includes models where we modify only components of the gravity: $f(R)$ gravity, where R is the Ricci scalar, $f(T)$ gravity, where T is the torsion scalar, $f(G)$ gravity, where G is the Gauss-Bonnet invariant and etc [1]-[3]. The second type includes models with a matter and their modifications: quintessence, phantom field, tachyon field, k-essence and etc [4]-[7]. There are a number of useful reviews of dark energy that mainly focused on theory [8]-[11]. The interested readers should consult reviews with more complete reference lists, e.g. [12]-[15]. All these models can describe the dynamics of our universe in different ways, but the choice of the best model we can only be shown by future observational data.

The dynamic equations of such models are nonlinear differential equations of a higher order, usually obtaining their exact solutions is a very difficult problem. The Noether symmetry approach is one way of solving such dynamical equations in cosmology. This approach was consider in the following work [16]. The application of the Noether symmetry approach in cosmological models with scalar fields is considered in [17]-[19]. Interesting works are cosmological models with fermionic fields, where was also used this approach [20, 21]. Recently, in paper [22], where was used the Noether symmetry approach in $f(T,B)$ teleparallel cosmology.

This work is organized as follows. In Sect. 2, we give the field equations are derived from a point-like Lagrangian in a spatially flat and isotropic Friedman-Robertson-Walker metric, which is obtained from an action including the fermion field non-minimally coupled to the gravitational field in the framework of $f(T,B)$ teleparallel gravity. In Sect. 3, we search the Noether symmetry for the Lagrangian of theory and Sect. 4, we obtain the particle solution of the field equations by using the coupling function $F(\Psi)$, potential $V(\Psi)$ and the function $f(T,B)$ obtaining the Noether symmetry approach. In Sect. 5, we conclude with a brief summary of the obtained results. This work, we adopted the metric signature $(+,-,-,-)$ and the natural units $c=\hbar=1$. Also we shall use the indices i,j and k take on the values $1,2,3,4$.

2. The action and the equations of motion

The Ricci scalar R and the torsion scalar T differs by a boundary term B via

$$R=-T+\frac{2}{e}\partial_\mu\left(eT^\mu\right)=-T-B, \tag{1}$$

here, for simplicity we introduce $B=(2/e)\partial_\mu(eT^\mu)=\nabla_\mu T^\mu$. The action for a fermion field that is non-minimally couled with the torsion scalar T and a boundary term B

$$S=\int d^4x\mathbf{e}\left\{F(\Psi)f(T,B)+\frac{i}{2}\left[\bar{\psi}\Gamma^\mu\left(\overrightarrow{\partial_\mu}-\Omega_\mu\right)\psi-\bar{\psi}\left(\overleftarrow{\partial_\mu}+\Omega_\mu\right)\Gamma_\mu\psi\right]-V(\Psi)\right\}, \tag{2}$$

where $\mathbf{e}=det(e^a_\mu)=\sqrt{-g}$ that e^a_μ is tetrad (vierbein) basis, T is a torsion scalar, B is a boundary term, ψ and $\bar{\psi}=\psi^\dagger\gamma^0$ denote the spinor field and its adjoint, with the dagger representing complex conjugation. $F(\Psi)$ and $V(\Psi)$ are generic functions, representing the coupling with gravity and the self-interaction potential of the fermionic field respectively. We assume that F and V depend on only functions of the bilinear $\Psi=\bar{\psi}\psi$, $\Gamma^\mu=e^\mu_a\gamma^a$ are the generalized Dirac-Pauli matrices satisfying the Clifford algebra $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$, where the braces denote the anti-commutation relation, the covariant derivatives e^a_μ are given by

$$D_\mu\psi=\partial_\mu\psi-\Omega_\mu\psi \tag{3}$$

and

$$D_\mu\bar{\psi}=\partial_\mu\bar{\psi}+\bar{\psi}\Omega_\mu. \tag{4}$$

Above, the fermionic connection Ω_μ is defined by

$$\Omega_\mu=-\frac{1}{4}g_{\rho\sigma}\left[\Gamma^\rho_{\mu\delta}-e^\rho_b\partial_\mu e^b_\delta\right]\Gamma^\sigma\Gamma^\delta, \tag{5}$$

with $\Gamma^\rho_{\mu\delta}$ denoting the Christoffel symbols. We will consider here the simplest homogeneous and isotropic cosmological model, FRW, whose spatially flat metric is given by

$$ds^2=dt^2-a(t)^2(dx^2+dy^2+dz^2), \tag{6}$$

where $a(t)$ is the scale factor of the Universe. For this metric, the vierbein is chosen to be

$$(e^a_\mu)=diag(1,a,a,a),\quad (e^\mu_a)=diag(1,1/a,1/a,1/a). \tag{7}$$

The Dirac matrices of curved spacetime Γ^μ are

$$\Gamma^0=\gamma^0,\quad \Gamma^j=a^{-1}\gamma^j,\quad \Gamma^5=-i\sqrt{-g}\Gamma^0\Gamma^1\Gamma^2\Gamma^3=\gamma^5,\quad \Gamma_0=\gamma^0,\quad \Gamma_j=a\gamma^j(i=1,2,3). \tag{8}$$

Hence we get

$$\Omega_0=0,\quad \Omega_j=\frac{1}{2}\dot{a}\gamma^j\gamma^0. \tag{9}$$

Finally, we note that the gamma matrices we write in the Dirac basis that is as

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{10}$$

where $I = diag(1,1)$ and the σ^k are Pauli matrices having the following form

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

In fact selecting suitable Lagrange multipliers and integrating by parts to eliminate higher order derivatives, the Lagrangian L becomes canonical. In physical units, the action is

$$S = \int d^4xe\left[Ff - \lambda_1\left(T + 6\frac{\dot{a}^2}{a^2}\right) - \lambda_2\left(B + 6\frac{\ddot{a}}{a} + 12\frac{\dot{a}^2}{a^2}\right) + \frac{i}{2}\left(\bar{\psi}\gamma^0\dot{\psi} - \dot{\bar{\psi}}\gamma^0\psi\right) - V\right]. \tag{12}$$

Here the definitions of torsion scalar and a boundary term in FRW metric have been adopted, that is

$$T = -6\frac{\dot{a}^2}{a^2} \quad B = -6\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right) \tag{13}$$

It is worth stressing that the two Lagrange multipliers are comparable. By varying the action with respect to T and B, one obtains

$$\lambda_1 = F(\Psi)\frac{\partial f(T,B)}{\partial T} = Ff_T, \qquad \lambda_2 = F(\Psi)\frac{\partial f(T,B)}{\partial B} = Ff_B, \tag{14}$$

then the above action becomes

$$S = \int d^4x\Bigg[Fa^3f - Fa^3Tf_T - 6Fa\dot{a}^2f_T - Fa^3Bf_B + 6\dot{F}a^2\dot{a}f_B + 6Fa^2\dot{a}\dot{T}f_{BT} + \tag{15}$$
$$6Fa^2\dot{a}\dot{B}f_{BB} + a^3\left[\frac{i}{2}\left(\bar{\psi}\gamma^0\dot{\psi} - \dot{\bar{\psi}}\gamma^0\psi\right) - V\right]\Bigg].$$

After an integration by parts, the point-like Lagrangian assumes the following form

$$\begin{aligned}\mathcal{L} &= Fa^3f - Fa^3Tf_T - 6Fa\dot{a}^2f_T - Fa^3Bf_B + 6\dot{F}a^2\dot{a}f_B + 6Fa^2\dot{a}\dot{T}f_{BT} + \\ &\quad 6Fa^2\dot{a}\dot{B}f_{BB} + \frac{i}{2}a^3\left(\bar{\psi}\gamma^0\dot{\psi} - \dot{\bar{\psi}}\gamma^0\psi\right) - a^3V.\end{aligned} \tag{16}$$

here, because of homogeneity and isotropy of the metric it is assumed that the spinor field depends only on time, i.e. $\psi = \psi(t)$.

The Euler-Lagrange equations are

$$\begin{aligned}&\dot{T}\dot{B}f_{BTB} + \dot{B}\dot{T}f_{BBT} + \dot{T}^2f_{BTT} + \dot{B}^2f_{BBB} - 2\frac{\dot{a}}{a}\left(\dot{T}f_{TT} + \dot{B}f_{TB}\right) + \\ &\left(2\frac{\dot{F}}{F}\dot{T} + \ddot{T}\right)f_{BT} + \left(2\frac{\dot{F}}{F}\dot{B} + \ddot{B}\right)f_{BB} - \left(\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} + 2\frac{\dot{a}}{a}\frac{\dot{F}}{F} - \frac{1}{2}T\right)f_T + \\ &\left(\frac{\ddot{F}}{F} + \frac{1}{2}B\right)f_B - \frac{1}{2}f - \frac{1}{2F}\left[\frac{i}{2}\left(\bar{\psi}\gamma^0\dot{\psi} - \dot{\bar{\psi}}\gamma^0\psi\right) - V\right] = 0,\end{aligned} \tag{17}$$

$$\dot{\psi} + \frac{3}{2}\frac{\dot{a}}{a}\psi + iV'\gamma^0\psi - ifF'\gamma^0\psi = 0, \tag{18}$$

$$\dot{\bar{\psi}} + \frac{3}{2}\frac{\dot{a}}{a}\bar{\psi} - iV'\bar{\psi}\gamma^0 - ifF'\bar{\psi}\gamma^0 = 0, \tag{19}$$

with the energy condition

$$E = \frac{\partial L}{\partial \dot{a}}\dot{a} + \frac{\partial L}{\partial \dot{T}}\dot{T} + \frac{\partial L}{\partial \dot{B}}\dot{B} + \frac{\partial L}{\partial \dot{\psi}}\dot{\psi} + \dot{\bar{\psi}}\frac{\partial L}{\partial \dot{\bar{\psi}}} - L = 0, \tag{20}$$

or

$$6\frac{\dot{a}}{a}\left(\dot{T}f_{BT} + \dot{B}f_{BB}\right) + \left(T - 6\frac{\dot{a}^2}{a^2}\right)f_T + \left(B + 6\frac{\dot{F}}{F}\frac{\dot{a}}{a}\right)f_B - f + \frac{V}{F} = 0. \tag{21}$$

3. The Noether Symmetries Approach

Noether symmetry approach tells us that Lie derivative of the Lagrangian with respect to a given vector field **X** vanishes, i.e.

$$\mathbf{X}\mathcal{L} \;=\; 0. \tag{22}$$

We will search the Noether symmetries for our model. In terms of the components of the spinor field $\psi = (\psi_0, \psi_1, \psi_2, \psi_3)^T$ and its adjoint $\bar{\psi} = ({\psi_0}^\dagger, {\psi_1}^\dagger, -{\psi_2}^\dagger, -{\psi_3}^\dagger)$, the Lagrangian

$$\begin{aligned}\mathcal{L} \;=\;& Fa^3 f - Fa^3 T f_T - 6Fa\dot{a}^2 f_T - Fa^3 B f_B + 6\dot{F}a^2\dot{a}f_B + 6Fa^2\dot{a}\dot{T}f_{BT} + \\ & 6Fa^2\dot{a}\dot{B}f_{BB} + \frac{i}{2}a^3\left(\bar{\psi}_i^\dagger\dot{\psi}_i - \dot{\psi}_i^\dagger\psi_i\right) - a^3 V.\end{aligned} \tag{23}$$

Here a vector field **X** we can written as

$$\mathbf{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial T} + \gamma\frac{\partial}{\partial B} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{T}} + \dot{\gamma}\frac{\partial}{\partial \dot{B}} + \sum_{i=0}^{3}\left(\eta_i\frac{\partial}{\partial \psi_i} + \dot{\eta}_i\frac{\partial}{\partial \dot{\psi}_i} + \chi_i\frac{\partial}{\partial \psi_i^\dagger} + \dot{\chi}_i\frac{\partial}{\partial \dot{\psi}_i^\dagger}\right), \tag{24}$$

where

$$\dot{\alpha} \;=\; \frac{\partial \alpha}{\partial a}\dot{a} + \frac{\partial \alpha}{\partial T}\dot{T} + \frac{\partial \alpha}{\partial B}\dot{B} + \sum_{i=0}^{3}\left(\frac{\partial \alpha}{\partial \psi_i}\dot{\psi}_i + \frac{\partial \alpha}{\partial \psi_i^\dagger}\dot{\psi}_i^\dagger\right), \tag{25}$$

$$\dot{\beta} \;=\; \frac{\partial \beta}{\partial a}\dot{a} + \frac{\partial \beta}{\partial T}\dot{T} + \frac{\partial \beta}{\partial B}\dot{B} + \sum_{i=0}^{3}\left(\frac{\partial \beta}{\partial \psi_i}\dot{\psi}_i + \frac{\partial \beta}{\partial \psi_i^\dagger}\dot{\psi}_i^\dagger\right), \tag{26}$$

$$\dot{\gamma} \;=\; \frac{\partial \gamma}{\partial a}\dot{a} + \frac{\partial \gamma}{\partial T}\dot{T} + \frac{\partial \gamma}{\partial B}\dot{B} + \sum_{i=0}^{3}\left(\frac{\partial \gamma}{\partial \psi_i}\dot{\psi}_i + \frac{\partial \gamma}{\partial \psi_i^\dagger}\dot{\psi}_i^\dagger\right), \tag{27}$$

$$\dot{\eta}_i \;=\; \sum_{i=0}^{3}\left(\frac{\partial \eta_i}{\partial a}\dot{a} + \frac{\partial \eta_i}{\partial T}\dot{T} + \frac{\partial \eta_i}{\partial B}\dot{B} + \sum_{j=0}^{3}\left(\frac{\partial \eta_i}{\partial \psi_j}\dot{\psi}_j + \frac{\partial \eta_i}{\partial \psi_j^\dagger}\dot{\psi}_j^\dagger\right)\right), \tag{28}$$

$$\dot{\chi}_i \;=\; \sum_{i=0}^{3}\left(\frac{\partial \chi_i}{\partial a}\dot{a} + \frac{\partial \chi_i}{\partial T}\dot{T} + \frac{\partial \chi_i}{\partial B}\dot{B} + \sum_{j=0}^{3}\left(\frac{\partial \chi_i}{\partial \psi_j}\dot{\psi}_j + \frac{\partial \chi_i}{\partial \psi_j^\dagger}\dot{\psi}_j^\dagger\right)\right), \tag{29}$$

where $\alpha, \beta, \gamma, \eta_i$ and χ_i are unknown functions of the variables a, T, B, ψ_i and $\psi_i^\dagger$.

In general, the Noether symmetry condition leads to an expression of second degree in the velocities ($\dot{a}$, $\dot{T}$, $\dot{B}$, $\dot{\psi}_i$ and $\dot{\psi}_i^\dagger$) with coefficients being partial derivatives of α, β, γ, η_i and χ_i with respect to the variables a, T, B, ψ_i and $\psi_i^\dagger$. Thus, the resulting expression is identically equal to zero if and only if these coefficients are zero. This gives us a set of partial differential equations for α, β, γ, η_i and χ_i. For the Lagrangian (23), the Noether symmetry condition (22) yields the following system of partial differential equations.

$$\alpha f_T + 2af_T\frac{\partial\alpha}{\partial a} + \beta af_{TT} - a^2 f_{TB}\frac{\partial\beta}{\partial a} + \gamma af_{TB} - a^2 f_{BB}\frac{\partial\gamma}{\partial a} \\ +af_T\frac{F'}{F}\epsilon_j\left(\eta_j\psi_j^\dagger + \chi_j\psi_j\right) - a^2 f_B\frac{F'}{F}\epsilon_j\left(\frac{\partial\eta_j}{\partial a}\psi_j^\dagger + \frac{\partial\chi_j}{\partial a}\psi_j\right) = 0, \tag{30}$$

$$6a^2Ff_{TB}\frac{\partial\alpha}{\partial T} = 0, \quad 6a^2Ff_{BB}\frac{\partial\alpha}{\partial B} = 0, \tag{31}$$

$$6a^2F'f_B\epsilon_j\frac{\partial\alpha}{\psi_j}\psi_j^\dagger = 0, \quad 6a^2F'f_B\epsilon_j\frac{\partial\alpha}{\psi_j^\dagger}\psi_j = 0 \tag{32}$$

$$2\alpha f_{BT} - 2f_T\frac{\partial\alpha}{\partial T} + af_{BT}\frac{\partial\alpha}{\partial a} + \beta af_{BTT} + af_{BT}\frac{\partial\beta}{\partial T} + \gamma af_{BTB} + af_{BB}\frac{\partial\gamma}{\partial T} \\ +af_{BT}\frac{F'}{F}\epsilon_j\left(\eta_j\psi_j^\dagger + \chi_j\psi_j\right) + af_B\frac{F'}{F}\epsilon_j\left(\frac{\partial\eta_j}{\partial T}\psi_j^\dagger + \frac{\partial\chi_j}{\partial T}\psi_j\right) = 0, \tag{33}$$

$$2\alpha f_{BB} - 2f_T\frac{\partial\alpha}{\partial B} + af_{BB}\frac{\partial\alpha}{\partial a} + \beta af_{BBT} + af_{BT}\frac{\partial\beta}{\partial B} + \gamma af_{BBB} + af_{BB}\frac{\partial\gamma}{\partial B} \\ +af_{BB}\frac{F'}{F}\epsilon_j\left(\eta_j\psi_j^\dagger + \chi_j\psi_j\right) + af_B\frac{F'}{F}\epsilon_j\left(\frac{\partial\eta_j}{\partial B}\psi_j^\dagger + \frac{\partial\chi_j}{\partial B}\psi_j\right) = 0, \tag{34}$$

$$\left(2\alpha f_B + af_B\frac{\partial\alpha}{\partial a} + \beta af_{BT} + \gamma af_{BB}\right)F'\psi_j^\dagger\epsilon_j + af_BF''\epsilon_i\epsilon_j\left(\eta_i\psi_i^\dagger + \chi_i\psi_i\right)\psi_j^\dagger \\ +\epsilon_j\chi_jaf_BF' + F\left(af_{BT}\frac{\partial\beta}{\partial\psi_j} + af_{BB}\frac{\partial\gamma}{\partial\psi_j} - 2f_T\frac{\partial\alpha}{\partial\psi_j}\right) + af_BF'\epsilon_j\left(\frac{\partial\eta_j}{\partial\psi_i}\psi_j^\dagger + \frac{\partial\chi_j}{\partial\psi_i}\psi_j\right) = 0, \tag{35}$$

$$\left(2\alpha f_B + af_B\frac{\partial\alpha}{\partial a} + \beta af_{BT} + \gamma af_{BB}\right)F'\psi_j\epsilon_j + af_BF''\epsilon_i\epsilon_j\psi_j\left(\eta_i\psi_i^\dagger + \chi_i\psi_i\right) \\ +\epsilon_j\eta_jaf_BF' + F\left(af_{BT}\frac{\partial\beta}{\partial\psi_j^\dagger} + af_{BB}\frac{\partial\gamma}{\partial\psi_j^\dagger} - 2f_T\frac{\partial\alpha}{\partial\psi_j^\dagger}\right) + af_BF'\epsilon_j\left(\frac{\partial\eta_j}{\partial\psi_i^\dagger}\psi_j^\dagger + \frac{\partial\chi_j}{\partial\psi_i^\dagger}\psi_j\right) = 0, \tag{36}$$

$$Ff_{BT}\frac{\partial\alpha}{\partial\psi_j} + f_BF'\epsilon_j\frac{\partial\alpha}{\partial T}\psi_j^\dagger = 0, \quad Ff_{BT}\frac{\partial\alpha}{\partial\psi_j^\dagger} + f_BF'\epsilon_j\frac{\partial\alpha}{\partial T}\psi_j = 0, \tag{37}$$

$$Ff_{BB}\frac{\partial\alpha}{\partial\psi_j} + f_BF'\epsilon_j\frac{\partial\alpha}{\partial B}\psi_j^\dagger = 0, \quad Ff_{BB}\frac{\partial\alpha}{\partial\psi_j^\dagger} + f_BF'\epsilon_j\frac{\partial\alpha}{\partial B}\psi_j = 0, \tag{38}$$

$$f_{BT}\frac{\partial\alpha}{\partial B} + f_{BB}\frac{\partial\alpha}{\partial T} = 0, \tag{39}$$

$$\epsilon_j\left(\frac{\partial\alpha}{\partial\psi_j}\psi_i+\frac{\partial\alpha}{\partial\psi_j^\dagger}\psi_i^\dagger\right)=0, \tag{40}$$

$$\epsilon_j\left(\frac{\partial\eta_j}{\partial a}\psi_j^\dagger-\frac{\partial\chi_j}{\partial a}\psi_j\right)=0, \tag{41}$$

$$\epsilon_j\left(\frac{\partial\eta_j}{\partial T}\psi_j^\dagger-\frac{\partial\chi_j}{\partial T}\psi_j\right)=0, \tag{42}$$

$$\epsilon_j\left(\frac{\partial\eta_j}{\partial B}\psi_j^\dagger-\frac{\partial\chi_j}{\partial B}\psi_j\right)=0, \tag{43}$$

$$\dot{\psi}_j:\ 3\alpha\psi_j^\dagger\epsilon_j+a\chi_j\epsilon_j+a\epsilon_j\left(\frac{\partial\eta_j}{\partial\psi_i}\psi_j^\dagger-\frac{\partial\chi_j}{\partial\psi_i}\psi_j\right)=0, \tag{44}$$

$$3\alpha\psi_j\epsilon_j+a\eta_j\epsilon_j-a\epsilon_j\left(\frac{\partial\eta_j}{\partial\psi_i^\dagger}\psi_j^\dagger-\frac{\partial\chi_j}{\partial\psi_i^\dagger}\psi_j\right)=0, \tag{45}$$

$$(f-Tf_T-Bf_B)\left[3\alpha+a\frac{F'}{F}\epsilon_j\left(\eta_j\psi_j^\dagger+\chi_j\psi_j\right)\right]-\beta a\left(Tf_{TT}+Bf_{BT}\right)-\gamma a\left(Tf_{TB}+Bf_{BB}\right)=0, \tag{46}$$

$$3\alpha V+aV'\epsilon_j\left(\eta_j\psi_j^\dagger+\chi_j\psi_j\right)=0. \tag{47}$$

where $\epsilon_j=1$, if $i=0,1$ and $\epsilon_j=-1$, if $i=2,3$. This system are obtained by imposing the fact that the coefficients of $\dot{a}^2$, $\dot{T}^2$, $\dot{B}^2$, $\dot{\psi_i}^2$, $\left(\dot{\psi_i}^\dagger\right)^2$, $\dot{a}\dot{T}$, $\dot{a}\dot{B}$, $\dot{a}\dot{\psi}_i$, $\dot{a}\dot{\psi}_i^\dagger$, $\dot{T}\dot{B}$, $\dot{T}\dot{\psi}_i$, $\dot{T}\dot{\psi}_i^\dagger$, $\dot{B}\dot{\psi}_i$, $\dot{B}\dot{\psi}_i^\dagger$, $\dot{\psi}_i\dot{\psi}_j^\dagger$, $\dot{a}$, $\dot{T}$, $\dot{B}$, $\dot{\psi}_i$ and $\dot{\psi}_i^\dagger$ vanish.

After some mathematical calculations, we obtained particular solutions for the field generators α, β, γ, η_j and χ_j as

$$\alpha(a)=\alpha_0a^n, \tag{48}$$
$$\beta(a,T)=2\alpha_0(n-1)a^{n-1}T, \tag{49}$$
$$\gamma(a,B)=2\alpha_0(n-1)a^{n-1}B, \tag{50}$$
$$\eta_j(a,\psi_j)=-\left(\frac{3}{2}\alpha_0a^{n-1}+\epsilon_j\eta_0\right)\psi_j, \tag{51}$$
$$\chi_j\left(a,\psi_j^\dagger\right)=-\left(\frac{3}{2}\alpha_0a^{n-1}-\epsilon_j\eta_0\right)\psi_j^\dagger. \tag{52}$$

Here α_0, η_0 and n are some constants ($n\neq1$). We also obtained particular solutions for the coupling function $F(\Psi)$, potential $V(\Psi)$ and $f(T,B)$ in the form

$$F(\Psi)=F_0\Psi^{\frac{m}{3}}, \tag{53}$$
$$V(\Psi)=V_0\Psi, \tag{54}$$
$$f(T,B)=C_0T^{\frac{1}{2}\frac{m-2}{n-1}}+\frac{m(n-1)}{m-2n-1}T+B, \tag{55}$$

where C_0, F_0, V_0 and m are constants. In the next section, we will substitute these solutions (53)-(55) into the equations of motion (17)-(19) and (21).

4. Exact cosmological solutions

In this section, we attempt to integrate the dynamical system given by (17)-(19) and (21) analytically. Since the coupling and potential functions depend on the bilinear function Ψ, using the Dirac equations (18) and (19) one gets

$$\dot{\Psi} + 3\frac{\dot{a}}{a}\Psi = 0, \tag{56}$$

and integration gives

$$\Psi = \frac{\Psi_0}{a^3}, \tag{57}$$

where Ψ_0 is a constant of integration. We note that, since the field equations can be directly integrable, it is not necessary to calculate the constants of motion associated with the Noether symmetry. Also the constants of motion give no new constraint on the field equations. If we put solutions (53)-(55) into equations of motion (17) and (21), we obtain

$$\mu \dot{a}(t)^2 + \nu a^{m-1} \quad = \quad 0. \tag{58}$$

here μ, ν are constans and we take

$$\mu = -6\left[\frac{(m-n-1)\,C_0}{n-1} + \frac{m\,(n-1)}{m-2n-1} + m\right], \tag{59}$$

$$\nu = \frac{V_0 \Psi_0^{1-\frac{m}{3}}}{F_0}. \tag{60}$$

Then we find the general solution of the equation (58) as

$$a = 4^{\frac{1}{m-3}}\left(-\frac{\mu}{(m-3)^2\,(t+C_1)^2\,\nu}\right)^{\frac{1}{m-3}}. \tag{61}$$

where C_1 is integral constant.

The Hubble parameter

$$H = \frac{6-2m}{(m-3)^2\,(t-C_1)}. \tag{62}$$

Defining the energy density and pressure

$$\rho = \frac{12}{(m-3)^2\,(t-C_1)^2}, \tag{63}$$

$$p = -\frac{4m}{(m-3)^2\,(t-C_1)^2}. \tag{64}$$

The equation of state ω be written as

$$\omega = \frac{p}{\rho} = -\frac{1}{3}m, \tag{65}$$

And the deceleration parameter q is defined by:

$$q = -\frac{\ddot{a}a}{\dot{a}} = -\frac{1}{2}\,(m-1)\,. \tag{66}$$

In the case when $m = -1$, we get $\omega = 1/3$ that is radiation dominates. If $m = 0$, we have $\omega = 0$, i.e. that fermion field behaves as a standard matter. Case $m > 4$ we have $\omega < -1$ coorecponds to the phantom phase, in the interval $1 < m < 3$, state parameter is $-1 < \omega < -1/3$ described by the quintessence phase. Finally, when $m = 3$, we have $\omega = -1$, this case corresponding to the model of cosmological constant. Thus, these results can be applied to describe the dynamics of the Universe, they are in good agreement with the known observational data at the present time.

5. Conclusions

Thus, in this work we have considered a homogeneous and isotropic Universe with a fermion field in $f(T, B)$ teleparallel gravity, where fermionic field is non-minimally coupled to gravity. The corresponding equations of motion are obtained, which are nonlinear differential equations with partial derivatives. To solve these equations, we applied the Noether symmetry approach. Using this approach, we obtained particular solutions for the coupling $F(\Psi)$, potential $V(\Psi)$ and $f(T, B)$ functions. Substituting solutions (53)-(55) into equation (21), we obtained an equation depending on one variable $a(t)$. Integration of this equation is gives (61). We also found such cosmological parameters as: the Hubble parameter $H(t)$, the energy density ρ and pressure p of the fermion field, the parameter of the equation of state ω and the deceleration parameter $q(t)$. We have established that these parameters are capable of describing various phases of the evolution of the expansion of the Universe from radioactive to late-times.

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