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Realization of Holographic Entanglement Temperature for a Nearly-AdS Boundary

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Abstract Computing the holographic entanglement entropy proposed by Ryu-Takayanagi shows that thermal energy near boundary region in AdS_3 gain maximum of the temperature. The absolute maxima of temperature is $T_E^{Max} = \frac{4G_3\epsilon_\infty}{l}$. By simple physical investigations it has become possible to predict a phase transition of first order at critical temperature $T_c \leq T_E$. As they predict a tail or root towards which the AdS space ultimately tend, the boundary is considered thermalized. The Phase transitions of this form have received striking theoretical and experimental verifications so far.

Keywords Holographic entanglement · Entanglement entropy · Gauge-gravity dual theory

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1 Introduction

In a pioneering work [1], it was shown that the entropy to energy ratio of any closed system never exceeds the following bound (in which $G = c = \bar{h} = 1$):

$$\frac{S}{E} \leq 2\pi R. \tag{1}$$

In this formula, R denotes the size (effective size) of the system. From the statistical point of view, the existence of this upper bound is transformed to the value of $\beta_0 = T_0^{-1}$ such that $Z(\beta_0) = 0$, where $Z(\beta)$ is the partition function of the system. All our errors in explaining the origin of entropy arise from our obstinacy in believing that gravitational entropy is entirely similar to thermodynamical one, which can be realized statistically. The only possible reply is already given by treating the origin of gravitational entropy as area [2, 3]. It is evident that the area law is more or less corrected with quantum effects, having its origin in backreactions from the quantum process of matter with which the entropy is usually changed. Although area of origin and its appearance like the "boundary" of the spacetime is wholly modern, yet its constitutional origin is analogous to that of the weakly coupled gravitational models in bulk and the strongly coupled quantum systems on boundary. Thus one may wonder if there is an analogous relation for general quantum systems which are far from equilibrium and a corresponding gravitational dual. Most probably this story had its origin in a particular gauge-gravity dual theory as to the meaning of the entropy mistletoe. The origin of the area law for entropy as a special type is wholly known using the celebrated AdS/CFT correspondence [4–6]. The pioneering work of Ryu-Takayanagi showed vast development so far about the computation of an entanglement entropy of a quantum system holographically [7, 8] (in section (2), we'll review the idea and methodology). Their fame title, however, is their pioneering work in the application of the AdS/CFT to applied research. During these years we became convinced that the success of the Ryu-Takayanagi machinery in a field of condensed matter was not to be reckoned with the tale of doubtful conversions. It is opening up and starting its work on new grounds [9–17].

Our aim in this paper is the production of a holographic entanglement temperature $\frac{1}{T_E} = \frac{\Delta S}{\Delta E}$, by means of the undulations of holographic entanglement entropy ΔS on a near AdS region caused by energy ΔE, of an infinitesimal layer.

2 Holographic Entanglement Entropy (HEE)

We suppose there are some "entangled" quantum systems in $N - 1$ dimensions on boundary which would considered to be divided into two parts A and B. The word "total density" ρ_{tot}, which is the name used in statistical mechanics for the density operator of $A \cup B$, had originally a general meaning, and may have required qualifications when applied to this particular subsystems. But it has now become a specific label, and the prefix "total" should be dropped. We now define the reduced partial density matrix on a subsystem A as $\rho_A = \mathrm{Tr}_B \rho_{tot}$ [18, 19]. It is convenient here to define the entanglement entropy (EE) of entangled subsystem A which characterize entanglement between A and B of the quantum system $A \cup B$ by $S_A = -\mathrm{Tr}\rho_A \log \rho_A$. This definition satisfy the subadditivity of the entropy $S(A \cup B) \geq S_A + S_B$. It was worse than the AdS/CFT after Ryu-Takayanagi discovered that S_A of a quantum system in boundary can be computed in the bulk gravity dual via the simple minimal area functional (see figure

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}\,. \tag{2}$$

here γ_A is a minimal (in the case of non static bulks it must be replaced by the maximal) area surface which has the same boundary as A, i.e. $\partial\gamma_A = \partial A$ [7, 8].

3 Near AdS_3 Boundary Calculations

The following metric and coordinates (t, ρ, ϕ) have been officially adopted by global representation of AdS_3:

$$ds^2 = l^2\big(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2\big). \tag{3}$$

The global form of the AdS_3 metric given by (8) is the official standard of full covered AdS_3 and is in use into the holographic set-up of the Ryu-Takayanagi algorithm. The AdS boundary was located in the $\rho = \infty$. Instead, we replace it with the finite radius $\rho = \rho_0$ to avoid divergence. Such cutoff ρ_0, gives a high degree of simplicity to the computation, but leave it as an open question whether it has the exactitude of the expression of the HEE, or it is only the approximation. This cutoff is valuable as enabling us to fix approximately the expression of the HEE, which must have occurred somewhere about $\rho_0 \gg 1$. The metric in this limit is, approximately, $ds^2 \simeq l^2 e^{-2\rho_0}(-dt^2 + d\phi^2)$. It provides the topology $\mathcal{R} \times S^1$ for surface defined by $\rho = \rho_0 \sim -\ln(\frac{2\pi l}{L})$. We're assuming the entangled quantum system "living" near to this cylinder with total length L and characteristic length l. The physics are also described by CFT_2 as well as AdS_3 bulk geometry.

We parameterize the minimal bulk surface γ_A as $\gamma_A = \{t = t_0, \phi \in (0, \frac{2\pi l}{L}), 0 \le \rho \le \rho_0\}$. The first step is to calculate the wrapped surface (curve) $\rho(\phi)$ which represents the path between the points of the boundary separated by $\phi = 0, \phi = \frac{2\pi l}{L}$. Taking the Euler-Lagrange equation of motion for the function $\phi(\rho)$, we can calculate the following values for the extremal functions corresponding to certain assigned values of integration constants $\big(a, \phi_0\big)$:

$$\coth(\rho) = a\cos\Big((a^2-1)(\phi-\phi_0)\Big), \quad \rho(0) = \rho(\frac{2\pi l}{L}) = \rho_0. \tag{4}$$

Then by solving this equation, regarding the two elements (a, ϕ_0) as unknown quantities, the values of the parameters may be computed:

$$\text{class I}: \quad a^2 = 1 - \frac{nL}{l}, n \in [1,\infty), \quad \phi_0 = \frac{l}{nL}\arccos\left(\frac{\coth(\rho_0)}{1-\frac{nL}{l}}\right), \tag{5}$$

$$\text{class II}: \quad a^2 = 1 + \frac{2n\pi}{\frac{2\pi l}{L} - 2\phi_0}, n \in [1,\infty), \quad \sqrt{1 + \frac{2n\pi}{\frac{2\pi l}{L} - 2\phi_0}}\cos\left[\frac{2n\pi\phi_0}{\frac{2\pi l}{L} - 2\phi_0}\right] = \rho_0. \tag{6}$$

By substituting the "class I" parameters of (a, ϕ_0) in (9) we obtain:

$$\text{class I}: \quad \coth(\rho) = \sqrt{1 - \frac{nL}{l}}\cos\left[\frac{nL}{l}\phi - \arccos\left(\frac{\coth(\rho_0)}{1-\frac{nL}{l}}\right)\right], \quad n \in [1,\infty). \tag{7}$$

Hence, solution (12) is a disconnected surface. The difficulties confronted in the way of solving “class II” are very severe, and up to the present time we are not agreed to the result. One can solicit the possibility of solving “class II” by pure numerical methods. We’ll use the “class I” for further studies.

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Then by solving this equation, regarding the two elements (a, ϕ_0) as unknown quantities, the values of the parameters may be computed:

$$\text{class I}: \quad a^2 = 1 - \frac{nL}{l}, n \in [1, \infty), \quad \phi_0 = \frac{l}{nL} \arccos\left(\frac{\coth(\rho_0)}{1 - \frac{nL}{l}}\right) \tag{10}$$

$$\text{class II}: \quad a^2 = 1 + \frac{2n\pi}{\frac{2\pi l}{L} - 2\phi_0}, n \in [1, \infty), \quad \sqrt{1 + \frac{2n\pi}{\frac{2\pi l}{L} - 2\phi_0}} \cos\left[\frac{2n\pi\phi_0}{\frac{2\pi l}{L} - 2\phi_0}\right] = \rho_0. \tag{11}$$

By substituing the “class I” parameters of (a, ϕ_0) in (9) we obtain:

$$\text{class I}: \quad \coth(\rho) = \sqrt{1 - \frac{nL}{l}} \cos\left[\frac{nL}{l}\phi - \arccos\left(\frac{\coth(\rho_0)}{1 - \frac{nL}{l}}\right)\right], \quad n \in [1, \infty). \tag{12}$$

Hence, solution (12) is a disconnected surface. The difficulties confronted in the way of solving “class II” are very severe, and up to the present time we are not agreed to the result.

One can solicit the possibility of solving "class II" by pure numerical methods. We'll use the "class I" for further studies.

We have calculated the changes of HEE, for the domain denoted by $\mathcal{D} = \{\rho \in (\rho_0 - \Delta\rho, \rho_0), \phi(\rho_0) = \phi(\rho_0 - \Delta)\}$, and change in the radial coordinate by $\Delta\rho$, then the approximate changes of HEE and energy are respectively given by the following:

$$\Delta S \simeq \frac{\Delta\rho}{4G_3}\left(1 + 2c^2 e^{-2\rho_0} + \mathcal{O}(e^{-4\rho_0})\right), \tag{13}$$

$$\Delta E \simeq \frac{\epsilon_\infty \Delta\rho}{l}\left(1 - 2e^{-2\rho_0} + \mathcal{O}(e^{-4\rho_0})\right), \tag{14}$$

here $|c|^2 = |-\frac{l}{nL}|$. We used the Tolman law to compute the energy measured by a local observer:

$$E = \frac{\epsilon_\infty}{\sqrt{-g_{tt}}}. \tag{15}$$

We may define this energy by the terms of red-shifted energy ϵ_∞ at AdS infinity to the local observer. The local energy of AdS_3 is always decreasing, and never increasing; in contrast to the strictly decreasing.

From these expressions, by the help of near AdS approximation $\rho \simeq \rho_0 \gg 1$, we can calculate the ratio $\frac{\Delta S}{\Delta E}$ for any domain $\mathcal{D}$:

$$\frac{\Delta S}{\Delta E} \simeq \frac{l}{4G_3\epsilon_\infty}\left(1 + 2(c^2+1)e^{-2\rho_0}\right) \geq \frac{l}{4G_3\epsilon_\infty}. \tag{16}$$

Therefore there exist a minima in the value of the entanglement entropy to energy ratio, depending on the l, G_3, ϵ_∞. We can calculate the effective entanglement temperature of formation $\mathcal{D}$ from its CFT boundary for any substance dissolved in a given region $\mathcal{D}$, from the knowledge about $\frac{\Delta S}{\Delta E}$, by means of an application of the well-known thermodynamical process:

$$T_E \leq \frac{4G_3\epsilon_\infty}{l}. \tag{17}$$

Which also gives us a "Universal" upper bound of temperature of the near boundary region when the holographic entanglement entropy of its boundary is known. In this work, the holographic temperature (17), according to infinitesimal change of the holographic entanglement entropy, add energy for a near AdS boundary domain $\mathcal{D}$, we sought to demonstrate the upper bound of temperature. It is an attempt once more to demonstrate that entropy of the holographic system on boundary can be increased as a function of energy. In statistical mechanics, the entropy of a closed system $S(E)$ always increasing or remaining constant, and never decreasing. Indeed, we contrast this property of $S(E)$ with strictly increasing. The same patterns serve to demonstrate the upper bound condition of the quantum entanglement temperature of the system in dual CFT. It's a lot different than a bound proposed in [20], although, the range of the computed holographic temperature for the near-AdS region is both vast and subjective, but our bulk and its near boundary geometry is so different from [20] that trying to define the temperature for the excited states in the CFT. But in reality it is not different from the "Universality" relation $T_E \propto l^{-1}$. That is what we expect to be able to do for more general geometries in excited states, because it is theoretically possible in a systematic different way. Now we can have something completely different:

Universal upper bound on the entanglement temperature is proposed as an attempt to obtain a lower bound on the entanglement entropy to energy ratio of a near-AdS region of *space when the bound is demographically different from the one which was proposed for the excited states.*

As an illustrative of this upper bound temperature, it may be explained that any entangled quantum system such as investigated here, will go well upon phase transitions, while the free energy F set well. We compute the free energy F of AdS_3 for the layer in the limit of $e^{\rho_0} \gg 1$ using the saddle point approximation $e^{-\beta F} = \int \rho(E) e^{-\beta E} dE$, we obtain:

$$F(\beta, l) = -\frac{2G_3}{l}\beta - \frac{l}{2G_3\beta} - \frac{1}{2\beta}\ln\left(\frac{l\pi}{4G_3}\right) + \frac{\ln\beta}{\beta} - \frac{1}{\beta}\ln\left(Erfi\left[\sqrt{\frac{G_3}{l}}(a + \frac{l}{2G_3})\right]\right) \tag{18}$$

here $\rho(E)$ is the density of distinct energy states in the entangled system, $\beta = \frac{dS}{dE}|_{E=E^*}$, $E = a\beta^{-1}$, $S_A(E) = S_0 - \frac{l}{2G_3}\ln\left(\frac{lE}{2\epsilon_\infty}\right)$ [7, 8] and $Erfi[x]$ is the imaginary error function is defined by $Erfi[x] = -ierf(ix) = \frac{2}{\sqrt{\pi}}\int_0^x e^{t^2} dt$.

According to the reference [21], we can classify the phase transitions based on free energy. At quantum based phase transition point (QPTP) first or second order derivative of free energy respected to thermodynamics variables diverges [22]. In our problem the first derivative of free energy respected to temperature gives the entropy and second derivative is proportional to heat capacity. The entropy and heat capacity are:

$$S = -\frac{\partial F}{\partial T}|_V \tag{19}$$

$$C_v = T\frac{\partial S}{\partial T}|_V. \tag{20}$$

There is a critical temperature T_c, that entropy becomes zero at this temperature. Below the T_c the entropy has negative value and by approaching to zero temperature we may expect that the equation of state of system will change:

$$T_c = \frac{2\sqrt{\frac{G_3}{l}}}{\sqrt{W\left(\pi e^{\frac{l}{G_3}+2}\text{Erfi}\left(a\sqrt{\frac{G_3}{l}} + \frac{\sqrt{l}}{2\sqrt{G_3}}\right)^2\right)}} \tag{21}$$

W is W-Lambert function and is defined by the solutions of the equation $W(x)e^{W(x)} = x, \ \ x \in \mathcal{R}, T_c \leq T_E$.

The Phase transitions of this form have received a striking theoretical [23] *and experimental verifications in so far as they predict a tail or root towards which the AdS space ultimately tend when the boundary is considered to be thermalized.* The second part of the statement of our paper- the reality of the prediction by phase transitions - has been frequently called in question, chiefly on the ground that, in order to predict a strongly coupled system on CFT with any chance of success, one should have the command of certain thermodynamical facts which are known until today, and then merely approximately, and only employed with that object in the this paper. The question, however, is whether HEE could predict the phase of the entangled system in CFT with any chance of success - much less whether it could state beforehand at what temperatures the phase would be visible, as some have erroneously supposed, and which of estimates would have been quite impossible for us to do. It is different, however, with physical properties, density, etc, at present we have no fixed rules which enable us to predict quantitatively the differences in physical properties corresponding to a given difference in entangled system in CFT and its near boundary dual system, the only general rule being that those differences are not large. We cannot predict with any exactness about the characters of a single phase individual here, but if we consider mixing of large number of phases in this nearly boundary layer, we can predict with considerable accuracy the percentage of phases which will have the mean character proper to their thermalization, or will differ from that mean character within any assigned limits. Others, however, are random, that is to say, the sequence of phases is repeated at irregular intervals, and it is thus impossible to predict when the maximum and minimum HEE will occur. We have not heard anything predicting even the possibility of these random phases before they came upon near-AdS region.

5 Discussion

It may be useful to summarize here the main results which have been gained in this paper. If we define the holographic entanglement entropy S for a near boundary region of AdS_3, as the thermodynamic entropy in which a small change in the radius of the space charged with positive ΔS would tend to move, we can observe an upper bound on the holographic temperature T_E in a simple form $T_E \leq \frac{4G_3\epsilon_\infty}{l}$ by saying that, if we have any minimal surface described in any manner in the near boundary region, the excess of the amount of the entropy flow which leave the surface $\frac{dS}{d\rho}\Delta_\rho|_{\rho=\rho_0}$ over the lack of energy which enter it $\Delta E = E(\rho_0) - E(\rho_0 - \Delta\rho) \sim \Delta\rho\frac{dE}{d\rho}$ is equal to $\frac{l}{4G_3\epsilon_\infty}$-times the algebraic sum of all the positive infinitesimal terms included within the minimal surface. The study of such cases suggests that the statement in terms of critical phenomena of the transitions between the mixed phases of entangled systems in the dual CFT may be only a critical one at $T_c \leq T_E$ in the bulk, which, though it may describe the effect of the actual transitions between them sufficiently for some practical purposes, is not to be regarded as representing them purely. In the limits assigned to this paper it is impossible to enter further into the nature of the phase transitions, but an attempt may be made to summarize the holographic results so far as they bear upon the CFT picture, which has again been revived in some branches, as to the fundamental descriptions of strongly coupled systems.

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