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Soliton solution for the integrable spin model with self-consistent potential

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Soliton solution for the integrable spin model with self-consistent potential

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Abstract. Integrable spin systems with potential are of great interest from the point of view of theoretical and applied physics. They make it possible to obtain accurate analytical solutions and study the properties of solitons - nonlinear wave structures that can be stable and move without distortion. The study of solitons in spin systems is of great importance not only for developing new methods for transmitting and processing information but also for developing spintronics and magnetoelectronics in general. These areas of technology are based on the use of the properties and control of the spin moment of electrons. Understanding and controlling spin dynamics in various systems opens up new possibilities for creating more efficient and powerful devices such as magnetic memories, spintronic transistors, and logic elements. This paper considers occurence of soliton in magnetic medium described by Generalized Landau-Lifshitz equation with self-consistent potential.

1. Introduction

During the past decades, there has been an increasing interest in the study of nonlinear models, especially, integrable nonlinear differential equations. Such integrable equations admit in particular soliton or soliton-like solutions. The study of the solitons and related solutions have become one of active areas of research in physics and mathematics. There are several methods to find soliton and other exact solutions of integrable equations, for instance, Hirota method, inverse scattering transformation, bilinear method, Darboux transformation and so on. Among these methods, the Darboux transformation (DT) is a efficient method to construct the exact solutions of integrable equations [1]-[2].

Among of integrable systems, the generalized Landau-Lifshitz equation (GLLE) plays an important role in physics and mathematics [3]-[10]. In particular, it describes nonlinear dynamics of magnets. Also the GLLE can reproduce some integrable classes of curves and surfaces in differential geometry [11]-[14]. There are several types integrable GLLE. In this paper, we construct the DT for the GLLE with self-consistent potentials. Using the DT, we provide soliton solutions of the GLLE with self-consistent potentials.

The paper is organized as follows. In section 2, the GLLE self-consistent potential and its Lax representation are introduced. In section 3, we derived the DT of the GLLE equation. Using these Darboux transformations, one soliton solutions are derived in section 4. Section 5 is devoted to conclusion.

2. The Generalized Landau-Lifshitz equation with self-consistent potential

The Landau-Lifshitz equation is a fundamental equation in magnetism and describes the dynamics of magnetic magnetization in ferromagnetic materials [10]. This equation was proposed by famous Soviet physicists L. Landau and E. Lifshitz in 1935. In this paper we consider generalized Landau-Lifshitz equation in the presence of consistent potential.

The Generalized Landau-Lifshitz equation (GLLE) with self-consistent potential [3] reads as

$$iS_t + \frac{1}{2}[S, S_{xx}] + \frac{1}{a}[S, W] = 0, \qquad (1)$$

$$iW_x + a[S, W] = 0, \qquad (2)$$

where $a = const$, $S = \Sigma_{j=1}^{3} S_j(x, y, t)\sigma_j$ is a matrix analogue of the spin vector, W - potential with the matrix form $W = \Sigma_{j=1}^{3} W_j(x, y, t)\sigma_j$, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are Pauli matries. Spin matrix components are given as $\mathbf{S} = (S_x, S_y, S_z)$, and

$$S(x, y, t) = \begin{pmatrix} S_z & S_x + iS_y \\ S_x - iS_y & -S_z \end{pmatrix} = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix},$$

$$W(x, y, t) = \begin{pmatrix} W_z & W_x + iW_y \\ W_x - iW_y & -W_z \end{pmatrix} = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}.$$

In the absence of a potential $W = 0$, system (1)-(2) becomes well known Heisenberg ferromagnet equation.

2.1. The Lax representation

The GLL equation with self-consistent potential is integrable in terms of inverse scattering transformation and its Lax representation [15] can be written in the form

$$\Phi_x = U\Phi, \qquad (3)$$

$$\Phi_t = V\Phi, \qquad (4)$$

where the matrix operators U and V have the form

$$U = -i\lambda S, \qquad (5)$$

$$V = \lambda^2 V_2 + \lambda V_1 + \left(\frac{i}{\lambda + a} - \frac{i}{a}\right) W. \qquad (6)$$

Here

$$V_2 = -2iS, \quad V_1 = SS_x. \qquad (7)$$

In mathematics, in the theory of integrable systems, a Lax pair is a pair of time-dependent matrices or operators that satisfy a corresponding differential equation, called the Lax equation. Lax pairs were introduced by Peter Lax to discuss solitons in continuous media [15].

3. The Darboux transformation

DT is a powerful method for finding a variety of interesting soliton and soliton-like solutions in integrable systems. The essence of this method is to construct the desired solution based on a known trivial solution, which is sometimes called a seed. This means that we start with a specific solution, which is then used in the process of building more complex solutions [1].

Next, we construct a single Darboux transformation for the generalized Landau-Lifshitz equation with a self-consistent potential [16]. To do this, consider the following transformation of the solutions of the system (3)-(4)

$$\Phi' = L\Phi, \tag{8}$$

here

$$L = \lambda N - I \tag{9}$$

and

$$N = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}. \tag{10}$$

Let Φ' also satisfied Lax representation (3)-(4) so that

$$\Phi'_x = U'\Phi', \tag{11}$$

$$\Phi'_t = V'\Phi', \tag{12}$$

where $U' - V'$ depends on S' and W' the same as $U - V$ on S and W. This equations give rise to next expressions N

$$N_x = iS' - iS, \tag{13}$$

$$N_t = -S'S'_x - \frac{i}{a}W'N + \frac{i}{a}NW + SS_x \tag{14}$$

and

$$S' = NSN^{-1}. \tag{15}$$

$$W' = (I + aN)W(I + aN)^{-1}. \tag{16}$$

And another form of Darboux transformation S:

$$S' = S - iN_x. \tag{17}$$

Let $(\psi_1, \psi_2)^T$ are the solutions of (3)-(4) with λ. Then $(-\psi_2^*, \psi_1^*)^T$ are solutions of (3)-(4) with λ^*. Let us consider the solutions of matrix

$$H = \begin{matrix} \psi_1(\lambda_1; t, x, y) & \psi_1(\lambda_2; t, x, y) \\ \psi_2(\lambda_1; t, x, y) & \psi_2(\lambda_2; t, x, y) \end{matrix}. \tag{18}$$

aldo we can rewrtie Darboux transformations in term of eigen function

$$S^{+\prime} = S^+ + i\left(\frac{(\lambda_1^{-1} - \lambda_2^{-1})\psi_1^*\psi_2}{\Delta}\right)_x, \tag{19}$$

$$S^{-\prime} = S^- - i\left(\frac{(\lambda_1^{-1} - \lambda_2^{-1})\psi_1\psi_2^*}{\Delta}\right)_x, \tag{20}$$

$$S'_3 = S_3 - i\left(\frac{\lambda_1^{-1}|\psi_1|^2 + \lambda_2^{-1}|\psi_2|^2}{\Delta}\right)_x, \tag{21}$$

similarly for W.

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4. Solution of the generalized Landau-Lifshitz equation with self-consistent potential

To construct a solution based on the Darboux transformation, we take a seed solution as plane nonzero wave solution in the form

$$S^{+} = 2\alpha e^{-i\theta}, \tag{22}$$

$$S^{-} = 2\alpha e^{i\theta}, \tag{23}$$

$$S_3 = \beta, \tag{24}$$

$$W^{+} = \frac{2\alpha\mu^2}{a}(a^2\beta + b)e^{-i\theta}, \tag{25}$$

$$W^{-} = \frac{2\alpha\mu^2}{a}(a^2\beta + b)e^{i\theta}, \tag{26}$$

$$W_3 = \frac{\mu}{2}\left(\frac{2\mu\beta}{a} - 1\right)\left(a^2\beta + b\right). \tag{27}$$

where α, β, μ, a, b constants, $4\alpha^2 + \beta^2 = 1, \theta = ax + bt$.

Solution in the components of the spin matrix and vector potential, respectively

$$S^{+\prime} = \left(6\,\alpha + \frac{64\,i\alpha\,{Z_1}^2 K_1\,(K_1\,x + P_1\,t)}{a\Delta^2} - 16\,\frac{\alpha\,{Z_1}^2}{\Delta}\right)\mathrm{e}^{-i\gamma} \tag{28}$$

$$S^{-\prime} = \left(6\,\alpha - \frac{64\,i\alpha\,{Z_1}^2 K_1\,(K_1\,x + P_1\,t)}{a\Delta^2} - 16\,\frac{\alpha\,{Z_1}^2}{\Delta}\right)\mathrm{e}^{i\gamma} \tag{29}$$

$$S_3' = -16\,\frac{\alpha\,K_1}{a\Delta} + 64\,\frac{\alpha\,K_1\,\,(K_1\,x + P_1\,t)^2}{a\Delta^2} \tag{30}$$

$$\begin{aligned} W^{+\prime} &= -\frac{8\mu\delta}{a\Omega}e^{i\gamma}\left[\alpha + \frac{4(\alpha Z_1^2 + i\mu Z_1(K_1\,x + P_1\,t))}{\Delta} + \frac{16iZ_1^3(\mu K_1\,x + P_1\,t))}{\Delta^2}\right] + \\ &+ \frac{4\mu^2 v\delta}{a\Omega}e^{-i\gamma}\left[\frac{3}{2}\alpha - \frac{2i\mu Z_1(K_1\,x + P_1\,t)}{\Delta} - \frac{2\alpha\mu^2}{a}\left(1 - \frac{8Z_1^2}{a\Delta}\left(1 + \frac{2Z_1^2}{\Delta}\right)\right)\right] \end{aligned} \tag{31}$$

$$\begin{aligned} W^{-\prime} &= \frac{8\mu\delta}{a\Omega}e^{i\gamma}\left[\alpha + \frac{4Z_1}{\Delta}\left[i\mu(P_1 t + K_1 x)(1 - 4Z_1^4) - \alpha Z_1\right]\right] - \frac{32\alpha\mu^4 v}{a^3\Omega}e^{i\gamma}\left[1 - \frac{Z_1^2}{\Delta}\left(2 + \frac{Z_1^2}{\Delta}\right)\right] + \\ &+ \frac{64\alpha\mu^3 v e^{i\gamma}}{a\Delta\Omega}\left[(P_1 t + K_1 x)\left(i\alpha Z_1 - \frac{\mu Z_1}{\Delta}(P_1 t + K_1 x)\right) + 1\right], \end{aligned} \tag{32}$$

$$\begin{aligned} W_3' &= \delta + \frac{32\mu^2\delta Z_1^2}{a^2\Delta}\left[\frac{2a^2}{\Delta}(P_1 t + K_1 x)^2 - \frac{2Z_1^2}{\Delta} + 1\right] + \\ &+ \frac{4\mu^3 v\cos i\gamma}{a^2}\left[\frac{16\alpha i\mu Z_1}{\Delta}(P_1 t + K_1 x) + 1\right]\left[1 - \frac{4Z_1^2}{\Delta}\right], \end{aligned} \tag{33}$$

$\Omega = det(I + \omega N) = \omega^2|n_{11}|^2 + \omega(n_{11} + n_{11}^*) + 1 + \omega^2|n_{12}|^2$

Equations (28)-(33) are the desired solutions of spin system and potential.

Visualization of the solutions obtained is presented in Figures 1-3 for the components of the spin matrix. Figures 1-3 clearly show the movement of the soliton in the negative direction of the Ox axis, at time values from 0 to 10. At different time values, the spin wave maintains the height unchanged - i.e. is a soliton.

Figures 4-6 show the movement of the vector potential. The movement of the potential is consistent with the movement of the spin vector. The graphs were plotted with the following parameter values:$a = 0.2, \alpha = 0.1, \mu = 3, \delta = 0.5, K = 3, Z = 2$.

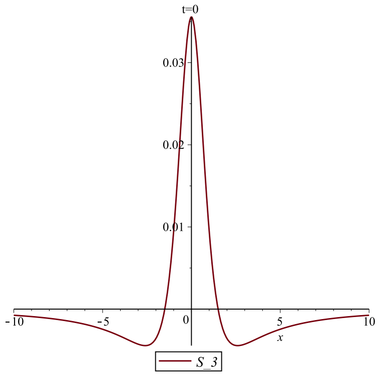

Figure 1. (a) $t = 0$

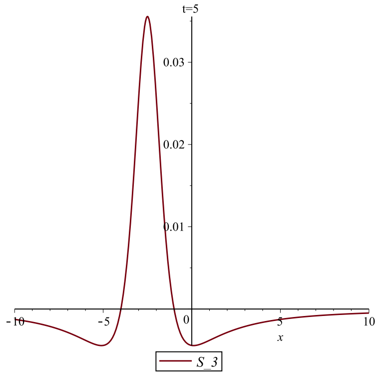

Figure 2. (b) $t = 5$

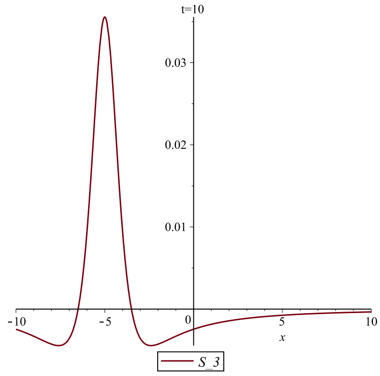

Figure 3. (c) $t = 10$

The one-soliton solution (28)-(33) at different moment of time with parameters
$a = 0.2, \alpha = 0.1, \mu = 3, \delta = 0.5, K = 3, Z = 2$

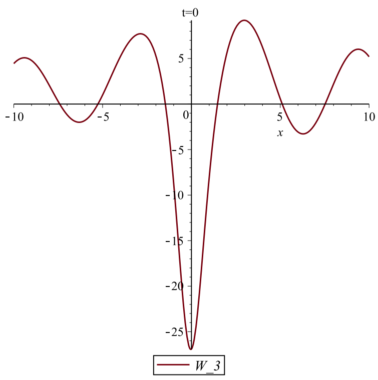

Figure 4. (a) $t = 0$

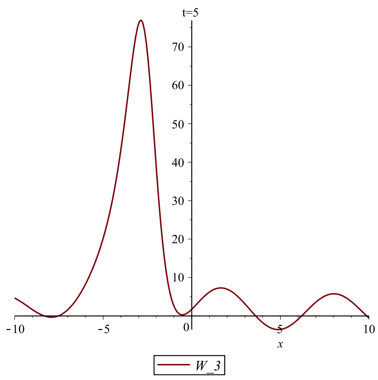

Figure 5. (b) $t = 5$

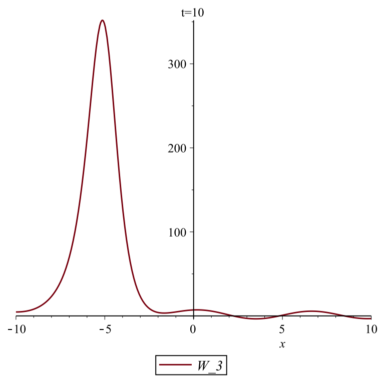

Figure 6. (c) $t = 10$

The solution of potential (33) at different moment of time with parameters
$a = 0.2, \alpha = 0.1, \mu = 3, \delta = 0.5, K = 3, Z = 2$

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5. Conclusion

Spin systems that include potentials belong to a large class of integrable nonlinear evolution equations with additional fields. In the paper it was demonstrated that the equations under consideration have Lax representations, which indicates their integrability. GLLE is a soliton equation and describes nonlinear magnetization processes in single- and double-layer ferromagnets. An important result of this study was the derivation of a solutions for a spin system with a self-consistent potential. The appearance of solitons in one-dimensional magnets is theoretically proven, using the example of the generalized Landau-Lifshitz equation with a self-consistent potential.

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